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SOME PROBLEMS ON CARLESON MEASURES FOR BESOV-SOBOLEV SPACES

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Abstract. We present some open problems concerning Carleson measures for Besov-Sobolev spaces. This is not a survey, and the choice of the references is only functional to the contextualization of the problems themselves.

The *Besov-Sobolev space* (or *analytic Besov space*) $B_p^\sigma(\mathbb{B}_n)$, $1 < p < \infty$, $\sigma \geq 0$, is the space of the functions f which are holomorphic in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ for which the seminorm $\|f\|^*_{n,\sigma,p}$ is finite,

$$\|f\|^*_{n,\sigma,p} = \left(\int_{\mathbb{B}_n} |(1-|z|^2)^{m+\sigma} f^{(m)}(z)|^p \frac{dA(z)}{(1-|z|^2)^{n+1}}\right)^{1/p}.$$

Here m is any positive integer satisfying $p(m+\sigma) - n > 0$ and different values of m give equivalent norms. $f^{(m)}$ is the tensor of the complex partial derivatives of order m of f and dA is area measure. The space $B_p^\sigma(\mathbb{B}_n)$ is a Banach space under the norm

$$\|f\|_{n,\sigma,p} = \left(\sum_{k=0}^{m-1} |f^{(k)}(0)|^p + \|f\|^{*p}_{n,\sigma,p}\right)^{1/p}.$$

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The basic theory of the Besov-Sobolev spaces is treated in [21], when $\sigma = 0$ and, e.g., in [20] when $n = 1$.

The case $p = 2$ is especially interesting, since many important Hilbert spaces of analytic functions are Besov-Sobolev spaces: the Dirichlet space ($\sigma = 0$), the Hardy space $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$, the weighted Bergman spaces ($\sigma > n/2$). In higher dimensions, we have the Hardy-Sobolev spaces (see, e.g., [11] and the references therein),

$$H^2_{\frac{n}{2}-\sigma}(\mathbb{B}_n) = B_2^\sigma(\mathbb{B}_n), \qquad 0 \le \sigma \le \frac{n}{2},$$

which are intermediate between Dirichlet and Hardy. Of special interest is the case $n \ge 2$, $\sigma = \frac{1}{2}$, corresponding to the *Drury-Arveson space*:

$$DA_n = B_2^{1/2}(\mathbb{B}_n), \qquad \left\| \sum_{k \in \mathbb{N}^n} a_k z^k \right\|^2_{DA_n} = \sum_{k \in \mathbb{N}^n} |a_k|^2 \frac{k!}{|k|!}.$$

Note that the DA_n norm is essentially a-dimensional, while the $B_2^{1/2}(\mathbb{B}_n)$ norm is not. We used here multi-index notation. The Drury-Arveson space is an extension of the Hardy space $H^2(\mathbb{B}_1) = DA_1$ and it was introduced in [13] in connection with the multi-variable Von Neumann inequality (see also [8]). It turns out to be universal, in a precise sense, among spaces with the complete Nevanlinna-Pick property. See [2] for a wholly accessible, in-depth exposition of the theory.

Let μ be a positive, Borel measure on $\mathbb{B}_n$. We say that μ is a *Carleson measure for* $B_p^\sigma(\mathbb{B}_n)$, $\mu \in CM(B_p^\sigma(\mathbb{B}_n))$, if $B_p^\sigma(\mathbb{B}_n)$ is continuously immersed in $L^p(\mu)$,

$$Id : B_p^\sigma(\mathbb{B}_n) \to L^p(\mu), \qquad \|\mu\|_{CM(B_p^\sigma(\mathbb{B}_n))} = |||Id|||_{(B_p^\sigma(\mathbb{B}_n), L^p(\mu))}.$$

Here, $|||\cdot|||$ is the operator norm and Id is the identity map, $Id(f) = f$. Knowledge about Carleson measures is often crucial in the analysis of $B_p^\sigma(\mathbb{B}_n)$: pointwise multipliers, interpolating sequences, exceptional sets, just to mention a few topics. See [16, 18] for applications to multipliers and interpolating sequences for the Dirichlet space.

Meta-Problem. *Find a geometric or potential theoretic characterization of the measures in* $CM(B_p^\sigma(\mathbb{B}_n))$ *for different values of* n, p, σ.

Indeed, the characterization is known for several ranges of (n, p, σ). Below we consider different subproblems which are interesting in their own and we will say what is known at present, to the best of our knowledge. When $\sigma < 0$, $B_p^\sigma(\mathbb{B}_n) \subset C(\overline{\mathbb{B}_n})$ is a space of functions which are continuous on the closure of the unit ball, hence the Carleson measures are exactly the bounded measures. The case $p \leq 1$ is also of interest, but we do not consider it here.

Carleson's original theorem characterized $CM(H^2(\mathbb{B}_1)) = CM(B_2^{1/2}(\mathbb{B}_1))$. He showed that a measure μ is Carleson for $H^2(\mathbb{B}_1)$ if and only if

$$\text{(SC)} \qquad \mu(S(I)) \leq C(\mu)|I|, \ \forall I \subset \partial\mathbb{B}_1,$$

for all arcs I, where $S(I) = \{z : \ z/|z| \in I, \ 0 < 1 - |z| \leq |I|\}$ is the *Carleson box* based on I. Later on, simple condition of the same kind were proven to characterize the Carleson measures in a variety of other cases. For instance, condition (SC) characterizes the Carleson measures for $B_p^{1/p}(\mathbb{B}_1)$ when $1 < p \leq 2$ [20].

Problem 5. *Characterize the measures in* $CM(B_p^{1/p}(\mathbb{B}_1))$ *for* $p > 2$.

It was recently proved in [14] that condition (SC), although necessary, is not sufficient for membership in $CM(B_p^{1/p}(\mathbb{B}_1))$ when $p > 2$.

Carleson's result can be extended in another direction. For instance, it is known [18] that $\mu \in CM(B_2^\sigma(\mathbb{B}_1))$, $\sigma \geq 1/2$ (Hardy and weighted Bergman cases), if and only if

$$\text{(SC)}_\sigma \qquad \mu(S(I)) \leq C(\mu)|I|^{2\sigma}, \qquad \forall I \subset \partial\mathbb{B}_1.$$

This result can be extended to the higher dimensional case. A measure μ is Carleson for $B_2^\sigma(\mathbb{B}_n)$, $\sigma \geq n/2$, if and only if

$$\text{(SC)}_{n,\sigma} \qquad \mu(S(Q)) \leq C(\mu)|Q|^{\frac{2\sigma}{n}}, \qquad \forall Q \subset \partial\mathbb{B}_n,$$

where Q ranges over a class of anisotropic regions on $\partial\mathbb{B}_n$,

$$Q = Q(z) = \left\{ w \in \partial\mathbb{B}_n : \ \left|1 - \overline{w} \cdot \frac{z}{|z|}\right| \leq 1 - |z| \right\}, \ z \in \mathbb{B}_n.$$

We denote by $|Q|$ the usual area measure of Q. It is known that $\text{(SC)}_{n,\sigma}$ is necessary, but not sufficient anymore, at the Drury-Arveson endpoint $\sigma = 1/2$ (see [6] and below).

Problem 6. *Characterize the measures in* $CM(B_2^\sigma(\mathbb{B}_n))$ *for* $1/2 < \sigma < n/2$.

The ranges of (n, p, σ) considered above are extensions of the Hardy case $\sigma = n/2$ originally considered by Carleson. The *bona fide* Besov-Sobolev spaces, those behaving closer to the Sobolev spaces, correspond to $\sigma = 0$ or anyway σ close to 0. In 1980, Stegenga [18] showed that Carleson measures for the classical Dirichlet space $B_2^0(\mathbb{B}_1)$ where characterized by the capacitary condition

$$\text{(Cap)} \qquad \mu(\cup_j S(I_j)) \le C(\mu)\, Cap(\cup_j I_j),$$

to be verified over all finite, disjoint unions $\cup_j I_j$ of arcs on the boundary of the disc. In (Cap), the capacity $Cap(E)$ of a subset of the unit disc is comparable with $\left(\log(cap(E))^{-1}\right)^{-1}$, where cap is logarithmic capacity in the plane. Stegenga's result was extended by several authors ([19, 20]) to a variety of contexts, with conditions involving the right capacities. Capacitary characterizations of $CM(B_p^\sigma(\mathbb{B}_1))$, $1 < p < \infty$, are known if $1/p > \sigma$. In higher dimension, [12] characterize the Carleson measures for the Hardy-Sobolev spaces $H^p_{\frac{n}{2}-\sigma}(\mathbb{B}_n)$ exactly in the range $0 \le \sigma < 1/p$ and they observe at the end of their article that this result can be easily extended to the Besov-Sobolev spaces. See also [11].

Alternative characterizations in terms of different testing conditions, which involve single intervals (or anisotropic regions) instead of unions thereof, have been proved in recent years. Consider $n = 1$ for simplicity. Let $T = \{I \subseteq \partial\mathbb{B}_1 : \ I \text{ is a dyadic arc}\}$ be the (dyadic) tree of the dyadic subarcs of $\partial\mathbb{B}_1$: if $I \in T$, the children of I are its left and right halves. For $1 < p < \infty$, $\sigma \ge 0$, consider the *tree condition*

$$\text{(Tree)}_{p,\sigma} \qquad \sum_{I \subseteq J \in T} |I|^{-\sigma p'} \mu(S(I))^{p'} \le C(\mu)\mu(S(J)), \qquad \forall J \in T.$$

Condition $\text{(Tree)}_{p,\sigma}$ characterizes $CM(B_p^\sigma)(\mathbb{B}_1)$ whenever $0 \le \sigma < 1/p$ [4]. In particular, tree conditions and capacitary conditions are equivalent for this range of (p, σ), since they both characterize Carleson measures for the same spaces.

Problem 7. *Give a direct proof that* $\text{(Tree)}_{2,0}$ *is equivalent to* (Cap).

In higher dimension, the unit ball $\mathbb{B}_n$ can be discretized in a similar fashion, obtaining an approximatively 2^n-adic tree T_n, whose elements are anisotropic regions $Q(z)$. The condition extending $\text{(Tree)}_{p,\sigma}$ to higher dimension is

$$\text{(Tree)}_{n,p,\sigma} \qquad \sum_{Q \subseteq R \in T_n} |Q|^{-\frac{\sigma p'}{n}} \mu(S(Q))^{p'} \le C(\mu)\mu(S(R)), \qquad \forall R \in T_n.$$

Condition $(\mathrm{Tree})_{n,p,\sigma}$ is sufficient for membership of μ in $CM(B_p^\sigma(\mathbb{B}_n))$ whenever $1 < p < \infty$ and $\sigma \geq 0$. It is proved to be necessary, however, only in the (overlapping) ranges

$$\mathcal{R}_1 = \{(n,p,\sigma) : \ \frac{n-1+\sigma}{n} < \frac{1}{p} < 1\},$$

and

$$\mathcal{R}_2 = \{(n,p,\sigma) : \ \frac{n-1+\sigma}{2n-1} < \frac{1}{p} < 1-\sigma\}.$$

We know that $(\mathrm{Tree})_{n,p,\sigma}$ is not a necessary condition for membership of μ in $CM(B_2^{1/2}(\mathbb{B}_n)) = CM(DA_n)$.

Problem 8. *Characterize the Carleson measures for* $B_p^0(\mathbb{B}_n)$ *when* $p \geq 2 + \frac{1}{n-1}$.

More generally, we can ask the following.

Problem 9. *Find all* (n,p,σ) *such that* $(\mathrm{Tree})_{n,p,\sigma}$ *characterizes* $CM(B_p^\sigma(\mathbb{B}_n))$.

Problem 10. *By comparison with the* $H^p_{\frac{n}{2}-\sigma}$ *case, we expect* $(\mathrm{Tree})_{n,p,\sigma}$ *to be necessary and sufficient at least when* $0 \leq \sigma < 1/p$.

It is easy to show that $(\mathrm{Tree})_{n,p,\sigma}$ implies $(\mathrm{SC})_{n,p,\sigma}$. The opposite implication fails, at least in the interesting range $0 \leq \sigma \leq 1/p$. In [6] it is proved that for the ranges $\mathcal{R}_j$ the Carleson measures are characterized by the tree condition if $\sigma = 0$ (in this case $\mathcal{R}_1 \subset \mathcal{R}_2$: the two ranges correspond to an easy duality proof and to a more involved T^*T-type argument). For $\sigma > 0$, the proof is in [7].

When $n = 1$, a different testing condition $(\mathrm{KS})_{2,\sigma}$ for membership in $CM(B_2^\sigma(\mathbb{B}_1))$, $0 \leq \sigma \leq 1/2$, was given in [15].

$$(\mathrm{KS})_{p,\sigma} \qquad \int_J \sup_{\{I:\ e^{i\theta} \in I \subset J\}} \frac{\mu(S(I))^{p'}}{|I|^{\sigma p'+1}} d\theta \leq C(\mu)\mu(S(J)), \qquad \forall J \in T.$$

It is easy to show that $(\mathrm{KS})_{p,\sigma}$ is *a priori* weaker that $(\mathrm{Tree})_{p,\sigma}$ and stronger that $(SC)_{p,\sigma}$, although, from what we have been saying so far, we have equivalence with the tree condition for $p = 2$, $0 \leq \sigma < 1/2$, and with the simple condition when $p = 2$, $\sigma = 1/2$. In [3] it is proved that the equivalences of $(\mathrm{KS})_{p,\sigma}$ with tree-type and simple-type conditions continue to hold when $1 < p < \infty$ and, respectively, $0 \leq \sigma < 1/p$ or $\sigma = 1/p$. It might be interesting to have the following.

Problem 11. *Find a direct proof that* $(\mathrm{KS})_{p,\sigma}$ *characterizes the Carleson measures for* $B_p^\sigma(\mathbb{B}_1)$ *when* $p \neq 2$ *and* $0 \leq \sigma < 1/p$.

The measures for $DA_n = B_2^{1/2}(\mathbb{B}_n)$, $n \geq 2$, are characterized in [6] in terms of a *split-tree condition* which reflects the anisotropic features of the complex geometry of $\mathbb{B}_n$. In some applications it would be important to have a "dimension free" characterization of $CM(DA_n)$. By this, we mean finding a geometric or potential theoretic quantity $[\mu]_n$ such that

$$c[\mu]_n \leq \|\mu\|_{CM(DA_n)} \leq c[\mu]_n,$$

with $0 < c < C$ independent of n.

Problem 12. *Find a dimensionless characterization of* $CM(DA_n)$.

We end the problem set by advertising a very interesting conjecture by K. Seip [17]. The main characters are a *complete Nevanlinna-Pick kernel* $K(x, y) = K_y(x)$ on a space X, the associated Hilbert function space H_K [2] and a sequence Z in X. The sequence Z is *universally interpolating for* H_K if

$$f \mapsto \left\{K(z_i, z_i)^{1/2} f(z_i) : \ z_i \in Z\right\}$$

maps H_K boundedly *onto* ℓ^2.

Conjecture. *A sequence Z in X is universally interpolating for H_K if and only if (i) Z is separated,* $\exists \rho \in (0,1) \forall z_i \neq z_j \in Z : \ |K(z_i, z_j)| \leq \sigma |K(z_i, z_i)|^{1/2} |K(z_i, z_i)|^{1/2}$, *and (ii) the measure* $\mu_Z = \sum_{z \in Z} \|K_z\|_{H_K}^{-1}$ *is Carleson for* H_K.

B. Böe proved in [9] that the conjecture holds under the assumption that $Re(K(x, y)) \geq c|K(x, y)|$. The comparison between the real part and the absolute value of suitable complex kernels in also one of the main themes in the proofs (or lack thereof) of Carleson measure theorems for a variety of Besov-Sobolev spaces.

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