BOUNDDEDNESS OF THE BILINEAR HILBERT TRANSFORM ON BERGMAN SPACES

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1. The problem

Let $0 < p < \infty$ and $A^p(\mathbb{D})$ denote the Bergman space of analytic functions on the unit disc such that $\|f\|_{A^p} = \left( \int_\mathbb{D} |f(z)|^p dA(z) \right)^{1/p} < \infty$. Consider the bilinear operator, defined on polynomials $f(z) = \sum_{n=0}^{N} a_n z^n$ and $g(z) = \sum_{n=0}^{M} b_n z^n$, by the formula

$$B(f, g)(z) = \sum_{n=0}^{N+M} \left( \sum_{k+j=n} a_k b_j \text{sgn}(k-j) \right) z^n.$$ 

Problem. Find the values $0 < p_1, p_2, p_3 < \infty$ with $1/p_3 = 1/p_1 + 1/p_2$ for which $B$ continuously extend to a bounded operator $A^{p_1}(\mathbb{D}) \times A^{p_2}(\mathbb{D}) \to A^{p_3}(\mathbb{D})$.

I know that the result holds true for $1 < p_1, p_2 < \infty$ and $p_3 > 2/3$ (see the proof below). This follows using the boundedness of the bilinear Hilbert transform on $L^p$-spaces, but I believe that a much simpler proof and covering even more cases should be found for Bergman spaces.

2. What I know

In the last decade and after the solution of the Calderón conjecture on the bilinear Hilbert transform by M. Lacey and C. Thiele (see [6,7]), multilinear

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operators have become a matter of great interest in Harmonic Analysis. The following result contains the work in the mentioned papers.

**Theorem 2.1.** Suppose that

\[ 1 < p_1, p_2 < \infty; \]

\[ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}; \]

\[ \frac{2}{3} < p_3 < \infty. \]

Then for each \( f \in L^{p_1}(\mathbb{R}) \cap L^2(\mathbb{R}) \), and each \( g \in L^{p_2}(\mathbb{R}) \cap L^2(\mathbb{R}) \),

\[ H(f, g)(x) = \lim_{\varepsilon \to 0^+} \int_{|y| > \varepsilon} \frac{f(x+y)g(x-y)}{y} \, dy \]

exists for almost all \( x \in \mathbb{R} \), and

\[ \|H(f, g)\|_{L^{p_3}(\mathbb{R})} \leq B_{p_1, p_2} \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}, \]

where \( B_{p_1, p_2} \) is a constant depending only on \( p_1 \) and \( p_2 \).

This result and other bilinear multipliers have been transferred to different settings by using different techniques. First, D. Fan and S. Sato (see [5]) used DeLeeuw approach to get the boundedness of the analogue to (2.4) in \( \mathbb{T} \) (see also [2, 4] for further extensions). Later in [3] (see also [1]) another proof using the extension of Coifman-Weiss transference method to the bilinear situation was achieved.

Note that another possible way to write (2.4) is:

\[ H(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta) \text{sig}(\xi - \eta) e^{ix(\xi+\eta)} \, d\xi d\eta \]

Now the transferred operator to \( \mathbb{T} \) looks as follows: If \( f, g \) are trigonometric polynomials on \( \mathbb{T} \) then

\[ \hat{B}(f, g)(\theta) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{f}(n)\hat{g}(m) \text{sig}(n-m) e^{i\theta(n+m)}, \]

or equivalently

\[ \hat{B}(f, g)(\theta) = \sum_{n \in \mathbb{Z}} \sum_{j+k=n} \hat{f}(j)\hat{g}(k) \text{sig}(j-k) e^{in\theta} \]
and the previously mentioned transferred result establishes that
\[ \tilde{B} : L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T}) \]
whenever (2.1), (2.2) and (2.3) holds.
Let us denote, for \( 0 < p < \infty \), \( H^p(\mathbb{T}) \) the corresponding Hardy space.
Using (2.8) and the just mentioned result one obtains the following corollary.

**Corollary 2.2.** If (2.1), (2.2) and (2.3) holds then \( \tilde{B} : H^{p_1}(\mathbb{T}) \times H^{p_2}(\mathbb{T}) \to H^{p_3}(\mathbb{T}) \) is bounded and
\[
\left\| \tilde{B}(f, g) \right\|_{H^{p_3}(\mathbb{T})} \leq A_{p_1, p_2} \| f \|_{H^{p_1}(\mathbb{T})} \| g \|_{H^{p_2}(\mathbb{T})},
\]
where \( A_{p_1, p_2} \) is a constant depending only on \( p_1 \) and \( p_2 \).

**Corollary 2.3.** If (2.1), (2.2) and (2.3) holds then \( B : A^{p_1}(\mathbb{D}) \times A^{p_2}(\mathbb{D}) \to A^{p_3}(\mathbb{D}) \) is bounded and
\[
\| B(f, g) \|_{A^{p_3}(\mathbb{D})} \leq A_{p_1, p_2} \| f \|_{A^{p_1}(\mathbb{D})} \| g \|_{A^{p_2}(\mathbb{D})},
\]
where \( A_{p_1, p_2} \) is a constant depending only on \( p_1 \) and \( p_2 \).

**Proof.** Let \( f, g \) be analytic polynomials and denote by \( f_r(e^{i\theta}) = f(re^{i\theta}) \). It is elementary to see that
\[ B(f, g)(re^{i\theta}) = \tilde{B}(f_r, g_r)(\theta). \]
Therefore
\[
\| B(f, g) \|_{A^{p_3}(\mathbb{D})}^{p_3} \leq C \int_0^1 \| \tilde{B}(f_r, g_r) \|_{L^{p_3}(\mathbb{T})}^{p_3} dr
\leq C \int_0^1 \| f_r \|_{L^{p_1}(\mathbb{T})}^{p_3} \| g_r \|_{L^{p_2}(\mathbb{T})}^{p_3} dr
\leq C \left( \int_0^1 \| f_r \|_{L^{p_1}(\mathbb{T})}^{p_3/p_1} dr \right)^{p_3/p_1} \left( \int_0^1 \| g_r \|_{L^{p_2}(\mathbb{T})}^{p_3/p_2} dr \right)^{p_3/p_2}
\leq C \| f \|_{A^{p_1}(\mathbb{D})}^{p_3} \| g \|_{A^{p_2}(\mathbb{D})}^{p_3}.
\]

**References**


