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BOUNDEDNESS OF THE BILINEAR HILBERT TRANSFORM ON BERGMAN SPACES

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1. The problem

Let $0 < p < \infty$ and $A^p(\mathbb{D})$ denote the Bergman space of analytic functions on the unit disc such that $\|f\|_{A^p} = (\int_{\mathbb{D}} |f(z)|^p dA(z))^{1/p} < \infty$. Consider the bilinear operator, defined on polynomials $f(z) = \sum_{n=0}^{N} a_n z^n$ and $g(z) = \sum_{n=0}^{M} b_n z^n$, by the formula

$$B(f,g)(z) = \sum_{n=0}^{N+M} (\sum_{k+j=n} a_k b_j sig(k-j)) z^n.$$

Problem. *Find the values* $0 < p_1, p_2, p_3 < \infty$ *with* $1/p_3 = 1/p_1 + 1/p_2$ *for which* B *continuously extend to a bounded operator* $A^{p_1}(\mathbb{D}) \times A^{p_2}(\mathbb{D}) \to A^{p_2}(\mathbb{D})$.

I know that the result holds true for $1 < p_1, p_2 < \infty$ and $p_3 > 2/3$ (see the proof below) . This follows using the boundedness of the bilinear Hilbert transform on L^p-spaces, but I believe that a much simpler proof and covering even more cases should be found for Bergman spaces.

2. What I know

In the last decade and after the solution of the Calderón conjecture on the bilinear Hilbert transform by M. Lacey and C. Thiele (see [6,7]), multilinear

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operators have become a matter of great interest in Harmonic Analysis. The following result contains the work in the mentioned papers.

Theorem 2.1. *Suppose that*

$$1 < p_1, p_2 < \infty; \tag{2.1}$$

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}; \tag{2.2}$$

$$\frac{2}{3} < p_3 < \infty. \tag{2.3}$$

Then for each $f \in L^{p_1}(\mathbb{R}) \bigcap L^2(\mathbb{R})$, *and each* $g \in L^{p_2}(\mathbb{R}) \bigcap L^2(\mathbb{R})$,

$$H(f,g)(x) \equiv \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{f(x+y)\, g(x-y)}{y}\, dy \tag{2.4}$$

exists for almost all $x \in \mathbb{R}$, *and*

$$\|H(f,g)\|_{L^{p_3}(\mathbb{R})} \le B_{p_1,p_2} \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}, \tag{2.5}$$

where B_{p_1,p_2} *is a constant depending only on* p_1 *and* p_2.

This result and other bilinear multipliers have been transferred to different settings by using different techniques. First, D. Fan and S. Sato (see [5]) used DeLeeuw approach to get the boundedness of the analogue to (2.4) in $\mathbb{T}$ (see also [2,4] for further extensions). Later in [3] (see also [1]) another proof using the extension of Coiffman-Weiss transference method to the bilinear situation was achieved.

Note that another possible way to write (2.4) is:

$$H(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta) sig(\xi - \eta) e^{ix(\xi+\eta)}\, d\xi d\eta \tag{2.6}$$

Now the transferred operator to $\mathbb{T}$ looks as follows: If f, g are trigonometric polynomials on $\mathbb{T}$ then

$$\tilde{B}(f,g)(\theta) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \hat{f}(n)\hat{g}(m) sig(n-m) e^{i\theta(n+m)}, \tag{2.7}$$

or equivalently

$$\tilde{B}(f,g)(\theta) = \sum_{n\in\mathbb{Z}} \Big(\sum_{j+k=n} \hat{f}(j)\hat{g}(k) sig(j-k) \Big) e^{in\theta} \tag{2.8}$$

and the previously mentioned transferred result establishes that

$$\tilde{B} : L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$$

whenever (2.1), (2.2) and (2.3) holds.

Let us denote, for $0 < p < \infty$, $H^p(\mathbb{T})$ the corresponding Hardy space. Using (2.8) and the just mentioned result one obtains the following corollary.

Corollary 2.2. *If* (2.1), (2.2) *and* (2.3) *holds then* $\tilde{B} : H^{p_1}(\mathbb{T}) \times H^{p_2}(\mathbb{T}) \to H^{p_3}(\mathbb{T})$ *is bounded and*

$$\left\| \tilde{B}\,(f,g) \right\|_{H^{p_3}(\mathbb{T})} \leq A_{p_1,p_2} \left\| f \right\|_{H^{p_1}(\mathbb{T})} \left\| g \right\|_{H^{p_2}(\mathbb{T})}, \tag{2.9}$$

where A_{p_1,p_2} *is a constant depending only on* p_1 *and* p_2*.*

Corollary 2.3. *If* (2.1), (2.2) *and* (2.3) *holds then* $B : A^{p_1}(\mathbb{D}) \times A^{p_2}(\mathbb{D}) \to A^{p_3}(\mathbb{D})$ *is bounded and*

$$\left\| B\,(f,g) \right\|_{A^{p_3}(\mathbb{D})} \leq A_{p_1,p_2} \left\| f \right\|_{A^{p_1}(\mathbb{D})} \left\| g \right\|_{A^{p_2}(\mathbb{D})}, \tag{2.10}$$

where A_{p_1,p_2} *is a constant depending only on* p_1 *and* p_2*.*

Proof. Let f, g be analytic polynomials and denote by $f_r(e^{i\theta}) = f(re^{i\theta})$. It is elementary to see that

$$B(f,g)(re^{i\theta}) = \tilde{B}(f_r, g_r)(\theta).$$

Therefore

$$\begin{aligned}
\|B(f,g)\|^{p_3}_{A^{p_3}(\mathbb{D})} &\leq C \int_0^1 \|\tilde{B}(f_r,g_r)\|^{p_3}_{L^{p_3}(\mathbb{T})} dr \\
&\leq C \int_0^1 \|f_r\|^{p_3}_{L^{p_1}(\mathbb{T})} \|g_r\|^{p_3}_{L^{p_2}(\mathbb{T})} dr \\
&\leq C (\int_0^1 \|f_r\|^{p_1}_{L^{p_1}(\mathbb{T})} dr)^{p_3/p_1} (\int_0^1 \|\tilde{g}_r\|^{p_2}_{L^{p_2}(\mathbb{T})} dr)^{p_3/p_2} \\
&\leq C \left\| f \right\|^{p_3}_{A^{p_1}(\mathbb{D})} \left\| g \right\|^{p_3}_{A^{p_2}(\mathbb{D})}.
\end{aligned}$$

□

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