Topics in Complex Analysis and Operator Theory
Proceedings of the Winter School held in Antequera, Málaga, Spain (February 5–9, 2006)
Pages 155–158

A PROBLEM CONCERNING THE PERMISSIBLE RATES OF GROWTH OF FREQUENTLY HYPERCYCLIC ENTIRE FUNCTIONS

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The problem we present here consists in finding an optimal result on the possible rates of growth of entire functions that are frequently hypercyclic for the differentiation operator.

In order to motivate this problem we start by discussing the corresponding problem for hypercyclic entire functions. We recall that an operator T on a topological vector space X is called *hypercyclic* if there exists an element x in X whose orbit $\{T^n x : n \geq 1\}$ is dense in X, see [4]. The element x is then also called *hypercyclic*.

By a result of MacLane [5] the differentiation operator D is hypercyclic on the space $H(\mathbb{C})$ of entire functions, endowed with its usual compact-open topology. In other words, there exists an entire function f such that, to every entire function g, every compact set $K \subset \mathbb{C}$ and every $\varepsilon > 0$ there is some $n \geq 1$ such that

$$\sup_{z \in K} |f^{(n)}(z) - g(z)| < \varepsilon.$$

It is natural to ask how slowly a D-hypercyclic entire function can grow. The following optimal result was obtained by the second author [3].

Theorem 1. (a) *For any function $\varphi : (0,\infty) \to (0,\infty)$ with $\lim_{r\to\infty} \varphi(r) = \infty$ there is a D-hypercyclic entire function f with*

$$|f(z)| \leq \varphi(|z|)\frac{e^{|z|}}{\sqrt{|z|}}, \quad \textit{if } |z| \textit{ is sufficiently big.}$$

(b) *There is no D-hypercyclic entire function f such that, for some $M > 0$,*

$$|f(z)| \le M \frac{e^{|z|}}{\sqrt{|z|}}, \quad \textit{for all } z \in \mathbb{C}.$$

Recently, F. Bayart and S. Grivaux [1] have introduced a new, and stronger, notion of hypercyclicity. They call an operator T on X *frequently hypercyclic* if for any non-empty open subset U of X we have

$$\underline{\text{dens}}\{n \in \mathbb{N} : T^n x \in U\} > 0,$$

where $\underline{\text{dens}}$ denotes lower density, that is, $\underline{\text{dens}}\,(A) = \liminf_{N\to\infty} \frac{\#\{n\in A: n\le N\}}{N}$, for $A \subset \mathbb{N}$. In that case, the vector x is also called *frequently hypercyclic*.
Bayart and Grivaux have shown that the differentiation operator D is even frequently hypercyclic on the space of entire functions. Since an increasing sequence (n_k) of positive integers has positive lower density if and only if $n_k = O(k)$, the result of Bayart and Grivaux amounts to the following improvement of MacLane's theorem. There exists an entire function f such that, to every entire function g, every compact set $K \subset \mathbb{C}$ and every $\varepsilon > 0$ there is an increasing sequence (n_k) of positive integers with $n_k = O(k)$ such that, for every $k \ge 1$,

$$\sup_{z\in K} |f^{(n_k)}(z) - g(z)| < \varepsilon.$$

Our main problem consists in finding an analogue of Theorem 1 for frequent hypercyclicity.

Problem 13. *Find an optimal result on the possible rates of growth of entire functions that are frequently hypercyclic for the differentiation operator.*

A partial solution to this problem was recently given by the authors [2]. There we have obtained the following result.

Theorem 2. *For any function $\varphi : (0,\infty) \to (0,\infty)$ with $\lim_{r\to\infty} \varphi(r) = \infty$ there is a D-frequently hypercyclic entire function f with*

$$|f(z)| \le \varphi(|z|)e^{|z|}, \quad \textit{if } |z| \textit{ is sufficiently big.}$$

The best negative result so far is the one implied by Theorem 1. *There is no D-hypercyclic and hence also no D-frequently hypercyclic entire function*

f such that, for some $M > 0$,

$$|f(z)| \leq M \frac{e^{|z|}}{\sqrt{|z|}}, \quad \textit{for all } z \in \mathbb{C}.$$

We have not been able to close the gap, but we expect that the rate of growth given in Theorem 2 is optimal. This leads to the following.

Problem 14. *Show that there is no D-frequently hypercyclic entire function such that, for some $M > 0$,*

$$|f(z)| \leq M e^{|z|}, \quad \textit{for all } z \in \mathbb{C}.$$

In order to attack this problem one might try to mimic the proof of part (b) of Theorem 1. That proof runs as follows. If f is an entire function satisfying, for all $z \in \mathbb{C}$,

$$|f(z)| \leq M \frac{e^{|z|}}{\sqrt{|z|}},$$

then the Cauchy estimates imply, for any $n \geq 1$,

$$|f^{(n)}(0)| \leq M \frac{n! e^n}{n^{n+1/2}}.$$

Then Stirling's formula implies that $(f^{(n)}(0))$ is bounded, which contradicts the hypercyclicity of f.

In other words, hypercyclicity of f is already impossible because $(f^{(n)}(0))$ is not dense in $\mathbb{C}$. One may wonder if the same could be true for frequent hypercyclicity. This leads us to the following problem.

Problem 15. *Show that if there exists some $M > 0$ such that*

$$|f(z)| \leq M e^{|z|}, \quad \textit{for all } z \in \mathbb{C},$$

then $(f^{(n)}(0))$ is not frequently dense in $\mathbb{C}$, that is, there is a non-empty and open set $U \subset \mathbb{C}$ such that

$$\underline{\mathrm{dens}}\{n \in \mathbb{N} : f^{(n)}(0) \in U\} = 0.$$

Finally, one might hope that a suitable set

$$U = \{z \in \mathbb{C} : |z| > C\}$$

already suffices. Since $\underline{\mathrm{dens}}\{n \in \mathbb{N} : n \in A\} = 1 - \overline{\mathrm{dens}}\{n \in \mathbb{N} : n \notin A\}$, where $\overline{\mathrm{dens}}$ denotes upper density, we are now asking for the following.

Problem 16. *Show that if there exists some $M > 0$ such that*

$$|f(z)| \leq Me^{|z|}, \quad \text{for all } z \in \mathbb{C},$$

then there is $C > 0$ such that

$$\overline{\text{dens}}\{n \in \mathbb{N} : |f^{(n)}(0)| \leq C\} = 1.$$

In closing we want to emphasize that Problems 14, 15 and 16 are only meant as suggestions on how to tackle our main problem, Problem 13, of finding an optimal improvement of Theorem 2.

References

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