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AN OPEN PROBLEM FOR TOEPLITZ OPERATORS

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Let φ be a complex L^∞ function defined on the unit circle $\mathbb{T}$. The *Toeplitz operator* T_φ with symbol φ on the Hardy space H^2 of the disk is defined via the multiplication operator by φ, followed by the orthogonal projection P from $L^2(\mathbb{T})$ onto H^2; that is, $T_\varphi(f) = P(\varphi f)$. Toeplitz operators represent a natural generalization of the familiar Toeplitz matrices from linear algebra. For the basic theory of Toeplitz operators on the Hardy space, the reader may consult [1, 3, 6, 7].

It is not difficult to see that T_φ is the zero operator if and only if the symbol φ is zero almost everywhere. In an important paper [2] in the 60's, A. Brown and P. Halmos proved that the "product" (that is, the composition in the usual sense) of two Toeplitz operators on the Hardy space is again a Toeplitz operator only in the obvious cases: when either each symbol is analytic, or both symbols are co-analytic (anti-analytic), or one of them is constant. From there they derived the following theorem: the product of two Toeplitz operators is zero if and only if at least one of the two symbols is zero almost everywhere. Later on, Halmos asked the following question:

Problem (The zero product problem). *If the product of n Toeplitz operators is the zero operator, does at least one of the operators have to be the zero operator as well?*

This problem has remained unanswered in the full generality for about 40 years. S. Axler (and possibly also other analysts) answered it in the affirmative for $n = 3$; however, this work was not published and most of the

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research on this question stopped there until about 10 years ago. Note that the affirmative answer for n operators implies the same conclusion for any smaller number of operators since the identity operator is also a Toeplitz operator.

In 1995, K.Y. Guo [5] settled the question in the affirmative for five operators by finding some interesting relationships between the ranges and kernels of different operators. In the year 2000 C. Gu [4] published an analogous result for six Toeplitz operators. In the same paper, Gu answered the question on when is the product of several Toeplitz operators on the Hardy space again a Toeplitz operator whose symbol is the product of the symbols of these operators. To the best of our knowledge, this appears to be all the relevant progress published to this date on the zero product problem. It should also be remarked that several papers have appeared regarding the analogous questions on Bergman spaces, where definitions are similar but different techniques must be used.

References

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