Variable $\mathbb{C}$ompleja, e$\mathcal{S}p$acios de $\mathcal{F}un$ciones y $\odot$peradores en$\mathbb{T}$re ellos

Problemas de Investigación | $\mathbb{C}^{\mathcal{F}un}_{\mathcal{S}p\odot\mathbb{T}}$ | Research Problems Section

$\mathbb{C}$omplex variables, $\mathcal{F}un$ction $\mathcal{S}p$aces and $\odot$perators between $\mathbb{T}$hem

APPROXIMATION BY EULER OPERATORS OUTSIDE THE ORIGIN

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Abstract. We present an open problem concerning the universality of a special class of diagonal operators acting on the space of entire functions.

Let G be a domain in the complex plane $\mathbb{C}$, with $0 \in G$, and X be a topological vector space such that $X \subset H(G)$, where $H(G)$ denotes the space of all holomorphic functions on G. We say that a (linear, continuous) operator $T : X \to X$ is a *diagonal operator* (or a *coefficient multiplier*) whenever there exists a sequence $\alpha = (a_n)_{n\geq 0} \subset \mathbb{C}$ such that for every $f \in X$ with

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \qquad \text{around the origin,}$$

we have

$$Tf(z) = \sum_{n=0}^{\infty} a_n f_n z^n \qquad \text{around the origin.}$$

We denote $T = T_\alpha$. For instance, if $G = \mathbb{C}$ and $\alpha = (a_n)_{n\geq 0} \subset \mathbb{C}$ with $\limsup_{n\to\infty} |a_n|^{1/n} < +\infty$, then T_α is a diagonal operator on $H(\mathbb{C})$. A remarkable instance of these operators is the class of Euler operators, whose definition will be recalled later.

Before going on, we recall three degrees of approximation by operators in a topological vector space X. An operator $T : X \to X$ is said to be *cyclic* (*supercyclic*, *hypercyclic*, resp.) if there is a vector $x_0 \in X$ such that the linear span *span*$\{T^n x_0\}_{n\geq 0}$ of the orbit of x_0 under T (the projective orbit $\{\lambda T^n x_0 : \ \lambda \in \mathbb{C}, n \geq 0\}$, the orbit itself $\{T^n x_0 : \ n \geq 0\}$ of x_0 under T,

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resp.) is dense in X. Hypercyclic operators, that is, those ones supporting some dense orbit, are also called *universal* operators.

Diagonal operators have an opportunity to be cyclic. There are several results related to this topic in the pertinent literature. A recent one is the following, due to Marín Jr. and Seubert ([3], see also [2]).

Theorem 1. *Let $\alpha = (a_n)_{n\geq 0} \subset \mathbb{C}$ be a sequence with $\limsup_{n\to\infty} |a_n|^{1/n} < +\infty$. Then $T_\alpha : H(\mathbb{C}) \to H(\mathbb{C})$ is cyclic if and only if $a_m \neq a_n$ for all m, n with $m \neq n$.*

The following assertion, which extends the last one, seems to be a rather general result. It can be found in [1].

Theorem 2. *Assume that $G \subset \mathbb{C}$ is a domain with $0 \in G$ and that $X \subset H(G)$ is an F-space satisfying:*

(a) *Convergence in X implies convergence on compacta in G.*
(b) *The set of polynomials is a dense subset of X.*

Let $\alpha = (a_n)_{n\geq 0} \subset \mathbb{C}$. If T_α is a diagonal operator on X, then T_α is cyclic if and only if $a_m \neq a_n$ for all m, n with $m \neq n$.

Nevertheless, T_α is never supercyclic, so never hypercyclic. In fact, it is easy to check directly that T_α never supports a dense orbit. By following the proof of this fact, the reason seems to be that not every function in X can be approximated *near the origin* by a T_α-orbit.

So we can wonder: What about *leaving* 0? Is there a chance for universal approximation?

In this context, V.A. Martirosian and A.Z. Martirosyan established in 2004 [5] the following theorem. Denote $(Lf)(z) := (zf(z))'$ and $A(K) := C(K) \cap H(K^\circ)$, where $K \subset \mathbb{C}$ is a compact set. Note that $A(K)$ is a Banach space under uniform convergence on K.

Theorem 3. *There exists a function $f \in H(\mathbb{C})$ with the following property: Given a compact subset $K \subset \mathbb{C}$ with $0 \notin K$ and connected complement, the sequence $\{L^n f\}_{n\geq 0}$ is dense in $A(K)$.*

Observe that $(Lf)(z) = \sum_{n=0}^{\infty}(1+n)f_n z^n$, so $L : H(\mathbb{C}) \to H(\mathbb{C})$ is a diagonal operator. In [4] K.-G. Grosse-Erdmann discovered a serious gap in the proof of [5]. In it, f is constructed as a telescopic series $f = f_0 + \sum_{n=0}^{\infty}(f_{n+1} - f_n)$, where $f_n \to 0$ compactly in $\mathbb{C}$. But $f = \lim_{n\to\infty} f_n$, so $f = 0$, which is absurd. Hence the corresponding problem remains open.

Now let us consider the simple *Euler operator* $E : H(\mathbb{C}) \to H(\mathbb{C})$ given by $(Ef)(z) = zf'(z)$. Note that $L = I + E$. Assume that Φ is an entire function of exponential type, that is, there are constants $A, B \in (0, +\infty)$ such that $|\Phi(z)| \leq Ae^{B|z|}$ $(z \in \mathbb{C})$. Then the general *Euler differential operator associated to* Φ is defined as

$$\Phi(E)f = \sum_{n=0}^{\infty} \varphi_n E^n f,$$

whenever $\Phi(z) = \sum_{n=0}^{\infty} \varphi_n z^n$. It is known that

$$(\Phi(E)f)(z) = \sum_{n=0}^{\infty} \Phi(n) f_n z^n.$$

Hence $\Phi(E)$ is a diagonal operator. We have the special case $L = \Phi(E)$ where $\Phi(z) = 1 + z$. This leads naturally to the following more general problem.

Problem. *Let $\Phi \in H(\mathbb{C})$ with exponential type. Does an $f \in H(\mathbb{C})$ exists such that, given a compact set $K \subset \mathbb{C} \setminus \{0\}$ with $\mathbb{C} \setminus K$ connected, the set $\{\Phi(E)^n f : \ n \geq 0\}$ is dense in $A(K)$?*

Of course, the answer might depend on Φ. Finally, observe that, at least, $\Phi(E)$ is cyclic on $H(\mathbb{C})$ if $\Phi|_{\mathbb{N}}$ is one-to-one.

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