Variable $\mathbb{C}$ompleja, e$\mathcal{S}p$acios de $\mathcal{F}un$ciones y $\odot$peradores en$\mathbb{T}$re ellos

Problemas de Investigación $\mathbb{C}^{\mathcal{F}un}_{\mathcal{S}p\odot\mathbb{T}}$ Research Problems Section

$\mathbb{C}$omplex variables, $\mathcal{F}un$ction $\mathcal{S}p$aces and $\odot$perators between $\mathbb{T}$hem

GROMOV HYPERBOLICITY OF INVARIANT METRICS

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Abstract. We present some open problems concerning Gromov hyperbolicity of the Kobayashi and related metrics.

Introduction

A geodesic space (X, d) is said to be Gromov hyperbolic if all its geodesic triangles T are δ-thin for some $\delta \geq 0$: this means that each side of T is contained in the union of the δ-neighborhoods of the other two sides.

The importance of this condition is well appreciated in analysis and the question of whether various metrics are Gromov hyperbolic or not has been the focus of much study in recent years. Note that, although the condition is very simply stated, it is often quite difficult to verify because of the difficulty in getting sufficiently precise information about the distance function and about the geodesics.

I believe the following problem is open, but probably very difficult.

Problem 1. *Show that the Kobayashi metric fails to be Gromov hyperbolic on any domain $G \subset \mathbb{C}^n$ whose boundary contains a part of a hyperplane of positive area.*

Any progress in the direction of this problem, such as Problem 4 below, would also be of interest.

Closely related to Problem 1 is the following, which would likely be solved by a solution to Problem 4 below.

Problem 2. *Give an example of a C^2-smooth pseudoconvex domain $G \subset \mathbb{C}^n$ for which the Kobayashi metric fails to be Gromov hyperbolic.*

Key words and phrases. Gromov hyperbolicity, Kobayashi metric, Carathéodory metric, Bergman metric.

Research supported in part by Science Foundation Ireland.

I also believe that the analogous problems for other invariant metrics such as the Carathéodory, inner Carathéodory, and Bergman metrics are open. Again they are probably very difficult, and any progress would be of interest.

1. Background to the problems

In an *infinitesimal sense*, a lot is known about invariant metrics on strictly pseudoconvex domains in $\mathbb{C}^n$; for instance, see [4, 6, 7]. For a general survey of invariant metrics, see [5].

Various partial results about the global metric, i.e. distance function, have been found, but most of these cover rather special situations. The exceptions are the results of Balogh and Bonk in [1, 4.1; 2; 3] which in particular give the following result.

Theorem 3. *Suppose $\Omega \subset \mathbb{C}^n$, $n \geq 2$, is a bounded strictly pseudoconvex domain, with a C^2-smooth boundary. Equip Ω with the Kobayashi, Carathéodory, inner Carathéodory, or Bergman metric d. Then (Ω, d) is Gromov hyperbolic, and the Euclidean boundary of Ω can be identified with its Gromov boundary. Under this identification, the sub-Riemannian metric induced on $\partial\Omega$ by d is in the canonical quasisymmetric gauge of metrics associated with the Gromov boundary.*

Not all pseudoconvex domains are Gromov hyperbolic with respect to the Kobayashi metric. For instance, it is known that the Kobayashi metric on the bidisk is not Gromov hyperbolic. For this discussion, it is useful to sketch the proof.

We first recall that the Kobayashi metric on a domain of product type $G = G_1 \times G_2$ is the L^∞ combination of the Kobayashi metrics on G_1 and G_2. In particular the Kobayashi metric on the bidisk $G = D \times D$ is given by

$$k_G((z_1, w_1), (z_2, w_2)) = \max\{\rho(z_1, z_2), \rho(w_1, w_2)\},$$

where ρ is the Poincaré metric on the disk. Writing $Z = (z, 0)$ and $W = (0, z)$, it readily follows from the formula for k_D that there is a geodesic from Z to $-Z$ that passes through W (indeed this is easily specified). A second geodesic from Z to $-Z$ is given by the Euclidean segment between them. When $|z|$ is very close to 1, W is very far in the Kobayashi metric from all points on the straight line geodesic, contradicting Gromov hyperbolicity (since bigons are special cases of triangles).

The bidisk lacks two hypotheses of Theorem 3: it is not smooth, and its boundary has flat parts. I have spoken about this problem with some experts in Gromov hyperbolicity, and we all feel that it is the flatness of the boundary rather than the lack of smoothness that leads to Gromov hyperbolicity failing. Indeed a similar belief was stated by Balogh in [1, 4.6], where he essentially asks for a solution to Problem 2.

One could define a domain H which is very close (in the Hausdorff metric) to G, with the only difference being that we smoothen ∂G in an ϵ-neighborhood of its edges, for some small $\epsilon > 0$. Such a domain still has large flat parts, and is very similar to the bidisk. Moreover, the points Z, $-Z$, W, all lie far from the parts of the boundary that we need to alter (but unfortunately the same is not true of our two k_G-geodesic paths from Z to $-Z$). For these reasons, it is tempting to exclaim "Of course H cannot be Gromov hyperbolic!". However I cannot see how to get good estimates for the k_H-distances between these points, or the location of k_H-(quasi)geodesic segments, so resolving even this seemingly simple example seems to an open problem.

Problem 4. *Show that the Kobayashi metric fails to be Gromov hyperbolic on the "bidisk with smoothened edges" H.*

This problem might look very difficult to experts in several complex variables because there is no real hope of getting precise information about the geodesics. However, using the theory of Gromov hyperbolicity, there appears to be real hope of proving this. In place of geodesic paths, we can work with quasigeodesic paths: a path is *quasigeodesic* if the length of any of its subpaths is uniformly comparable to $1 + D$, where D is the distance between the endpoints of the subpath. It suffices to construct a piecewise quasigeodesic path $\gamma = \gamma_z$ from Z to $-Z$ via W such that the origin, for instance, does not remain within a bounded distance of γ_z as $|z|$ tends to 1.

Note also that the smoothening process used to construct H was not specified above. A careful choice of smoothening function may be crucial to the proof. Any choice that works is likely to give a pseudoconvex domain, and so a solution to Problem 2.

Finally we note that the above domain H is also likely not to be Gromov hyperbolic for the other invariant metrics mentioned above. We concentrated on the Kobayashi metric because the formula for product domains is likely to be helpful.

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