Variable $\mathbb{C}$ompleja, e$\mathcal{S}p$acios de $\mathcal{F}un$ciones y $\odot$peradores en$\mathbb{T}$re ellos

Problemas de Investigación | $\mathbb{C}^{\mathcal{F}un}_{\mathcal{S}p\odot\mathbb{T}}$ | Research Problems Section

$\mathbb{C}$omplex variables, $\mathcal{F}un$ction $\mathcal{S}p$aces and $\odot$perators between $\mathbb{T}$hem

SOLVING THE FUNCTIONAL EQUATION $f(z) + f(2z) + \cdots + f(nz) = 0$

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The search of the solutions, continuous and non-continuous, of the functional equation

$$f(x) + f(2x) + f(3x) + \cdots + f(nx) = 0, \quad x > 0, \tag{1}$$

is interesting from an applied and theoretical point of view. Indeed, for small values of n, the continuous solutions of (1) have been used to model the trajectory of certain particles because their graphs fill some special compacts of the plane with arbitrary small density [2]. Moreover, the functional equation (1) is essentially different from the classical Cauchy's functional equation as its non-continuous solutions show, thus its theoretical interest. They have peculiar properties connected with arithmetical questions linked to the prime numbers.

The solutions of (1) we have found, have been obtained by using complex analysis techniques. In fact, we have proved in [1] a result connecting those solutions with the zeros of the entire function

$$G_n(z) = 1 + 2^z + \cdots + n^z,$$

for each fixed integer $n \geq 2$.

The function $G_n(z)$ itself has very interesting properties because it is an approximation of the Riemann zeta function for $\operatorname{Re} z < -1$. About these properties, we have focused our interest in knowing the multiplicity of its zeros, so we put the following problem:

Problem 1. *For $n \geq 2$, are all the zeros of $G_n(z)$ simple?.*

We have given a positive answer to the previous question for the cases $n = 2, 3, 4$. In general, it is clear that $n - 2$ is the best upper bound of the multiplicity of the zeros of $G_n(z)$, for $n > 2$.

Related to the non-continuous solutions of (1) we are led to define a crucial set, namely

$$\Pi_n = \{x \in \mathbb{Q} : x = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots p_{m(n)}^{\alpha_{m(n)}}\}, \tag{2}$$

where $p_1, p_2, \ldots, p_{m(n)}$ are all the prime numbers not greater than n and $\alpha_1, \alpha_2, \ldots \alpha_{m(n)} \in \mathbb{Z}$.

About this set (2) we propose the following question

Problem 2. *Is there a mapping, say g_n, defined on Π_n and valued in the finite set $\{1, 2, \ldots, n\}$ such that for any $x \in \Pi_n$, $g_n(kx) \neq g_n(lx)$ for $1 \leq k, l \leq n$, with $k \neq l$?*

We have proved the existence of g_n for small values of n.

References

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