Variable $\mathbb{C}$ompleja, e$\mathcal{S}p$acios de $\mathcal{F}un$ciones y $\odot$peradores en$\mathbb{T}$re ellos

Problemas de Investigación | $\mathbb{C}^{\mathcal{F}un}_{Sp\odot\mathbb{T}}$ | Research Problems Section

$\mathbb{C}$omplex variables, $\mathcal{F}un$ction $\mathcal{S}p$aces and $\odot$perators between $\mathbb{T}$hem

BERGMAN-TYPE REPRODUCING KERNELS: OPERATOR-THEORETICAL CHARACTERIZATION AND EXAMPLES

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The class of Bergman-type reproducing kernels was introduced by Stefan Richter and Scott McCullough [3]. A positive-definite kernel k defined in some domain Ω of the complex plane $\mathbb{C}$ and sesquianalytic there is called a B-kernel if it has a representation

$$k(z,w) = \frac{k(z,\alpha)k(\alpha,w)}{k(\alpha,\alpha)} \cdot \frac{1}{1-\phi(z)\overline{\phi(w)}(1-u(z,w))}, \tag{1}$$

where α is some point in Ω, ϕ is some function analytic in Ω and u is a positive-semidefinite function. If, in addition, Ω is the unit disk $\mathbb{D}$, $k(z,0) = 1$ for any z, and $k(z,z) \to \infty$ as $|z| \to 1-0$, then k is called a Bergman-type reproducing kernel. Many results on factorization of functions and shift invariant subspaces which hold in the classical Bergman space in the unit disk can be generalized to abstract Hilbert spaces of analytic functions with Bergman-type kernels (see [3]).

The definition of Bergman-type kernels can be compared with the description of complete Nevanlinna-Pick kernels. The latter kernels have a representation

$$k(z,w) = \frac{k(z,\alpha)k(\alpha,w)}{k(\alpha,\alpha)} \cdot \frac{1}{1-B(z,w)}, \tag{2}$$

where B is some positive-semidefinite kernel. An equivalent description in the case of zero-free kernels is that the function $1/k$ considered as an infinite-dimensional Hermitian matrix has exactly one positive square. On the other hand, the representation (1) is equivalent to the requirement that $1/k$ has exactly one negative square.

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An operator-theoretical characterization of complete Nevanlinna-Pick kernels was obtained by J. Agler and J. McCarthy [1]. They proved that a positive-definite kernel k is a complete Nevanlinna-Pick kernel if and only if the Nevanlinna-Pick problem $\phi(z_i) = w_i$, $i = 1, \dots, n$ for matrix-valued multipliers $\phi \in \mathrm{Mult}(\mathcal{H}(k) \otimes \mathbb{C}^N)$ with the norm control $\|\phi\|_{\mathrm{Mult}} \le 1$ is solvable whenever the block matrix $\big[(1 - w_i w_j^*) k(z_i, z_j)\big]_{1 \le i,j \le n}$ is positive-semidefinite. It would be quite interesting to obtain a similar characterization of Bergman-type reproducing kernels.

Problem 1. *To find an operator-theoretical characterization of Bergman-type reproducing kernels, probably, in terms of some interpolation problem.*

An easiest example of a Bergman-type kernel is the classical Bergman kernel $k(z, w) = (1 - z\bar{w})^{-2}$, $z, w \in \mathbb{D}$ which is the reproducing kernel for the Bergman space $L_a^2(\mathbb{D})$ of functions analytic and square area integrable in the unit disk. It is known that many important results on the space $L_a^2(\mathbb{D})$ can be generalized to weighted Bergman spaces $L_a^2(\mathbb{D}, w)$ with logarithmically subharmonic weight functions w (see, e.g. [2]). The condition of logarithmical subharmonicity of the weight has an alternative geometrical interpretation, it is equivalent to the negative curvature of the Riemannian metric given as $ds^2 = w(z)|dz|^2$. The following conjecture seems to be quite natural.

Conjecture 2. *Given an integrable and logarithmically subharmonic weight function w in the unit disk, the reproducing kernel for the weighted Bergman space $L_a^2(\mathbb{D}, w)$ is a Bergman-type reproducing kernel.*

Problem 3. *To prove (or, possibly, disprove) Conjecture* 2.

There are several partial results which support Conjecture 2. First, it follows from results of [3] that it holds for weight functions of the form $w(z) = |f(z)|^2$, where f is an arbitrary function from $L_a^2(\mathbb{D})$. Second, Conjecture 2 was proved in [4] for radial weight functions.

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