ADJOINTS OF COMPOSITION OPERATORS ON HILBERT SPACES OF ANALYTIC FUNCTIONS

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Abstract. We observe that a formula for the adjoint of a composition operator, known only for special symbols in some spaces of analytic functions, actually holds for every admissible symbol and in any Hilbert space of analytic functions with reproducing kernels. Along with some new results, all known formulas for the adjoint obtained so far follow easily as a consequence, some in an improved form.

INTRODUCTION

Let \( \varphi \) be a non-trivial analytic self-map of the unit disk \( \mathbb{D} \) (i.e., \( \varphi(\mathbb{D}) \subset \mathbb{D} \) and \( \varphi \neq \text{const} \)); the composition operator with symbol \( \varphi \) is defined by \( C_\varphi f = f \circ \varphi \). The modern theory of such operators (on Hardy spaces) began with [21] and [19]. An elegant account of their theory on \( H^2 \) is given in Shapiro’s monograph [22] and a comprehensive treatment of composition operators acting on the spaces from a large family can be found in the encyclopedia of the subject [5].

The spectral theory naturally occupies a central place in the study of any operators acting on Hilbert spaces of analytic functions (including the Bergman, Hardy, and Dirichlet space). In particular, in order to get a better understanding of the spectrum of a composition operator, it is important to be able to compute its adjoint. However, it is often stated in the literature that an explicit formula for \( C_\varphi^* \) exists only in some special cases and that no general formula is known (see p. 422 of [5], p. 19 of [6], or [24]). The purpose of this paper is to show that such a formula exists and has interesting applications. We begin by reviewing the history of this question.

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Generalizing Cowen’s result to finite Blaschke products $B$, McDonald [18] obtained a representation of $C_B^*f(w)$ as the value of a linear combination of products of two Toeplitz operators (not explicitly determined) and an averaging evaluation operator depending on $w$. A computation for yet another specific rational symbol was given in Wahl’s thesis [26].

Inner functions $\varphi$ such that $\varphi(0) = 0$ are of particular importance [6, 19, 23]. For these symbols, it is well known that

$$C_{\varphi}^*f(w) = \int_{\mathbb{T}} \frac{f(z)}{1 - \varphi(z)w} dm(z),$$

where $m$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. This has been known for quite some time; a proof based on orthogonal projections is given in [5], pp. 321-322. Lotto and McCarthy [14] also studied such symbols and obtained a generalization for other $H^p$ spaces via a disintegration of measures, representing $C_{\varphi}^*$ as a weighted expectation operator on these more general spaces.

Peña [20] has recently proved a general version of formula (1) for weighted Bergman spaces $A^2_{\alpha}$:

$$C_{\varphi}^*f(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \varphi(z)w)^{2+\alpha}} dA_{\alpha}(z),$$

valid for all admissible symbols, along with an analogue for weighted Dirichlet spaces. This generalization does not seem to have been noticed by the experts and surely deserves to be better known. However, his proofs are also based on rather lengthy computations.

Both (1) and (2) can actually be obtained in a straightforward manner, because it turns out that a general formula exists. It is rather simple (barely more than the definition) but it works for all possible symbols and for all Hilbert spaces and it seems quite useful. Traces of it appear implicitly in [3] and [25]; however, so far it does not seem to have been derived explicitly in full generality in the literature.
Let $\mathcal{H}$ be a Hilbert space of analytic functions in the unit disk $\mathbb{D}$ on which all point-evaluations are bounded. Then $\mathcal{H}$ possesses a family of reproducing kernels \( \{K_w : w \in \mathbb{D}\} \subset \mathcal{H} \); that is, \( \langle f, K_w \rangle = f(w) \) for every $f \in \mathcal{H}$. Then for each $w$ in $\mathbb{D}$ and all $f$ in $\mathcal{H}$ the adjoint formula is as follows:

\[
C_\phi^* f(w) = \langle f, K_w \circ \phi \rangle,
\]

which includes both (1) and (2) as special cases. It appears that formula (3) is not known to a wide group of experts in spite of its simplicity. The usual approach to the adjoint question one finds in the literature focuses on the formula $C_\phi^* K_w = K_{\phi(w)}$, which seems to lead only to a partial success [4, 5, 10]. The above statement became apparent to us while performing some manipulations of the adjoint in our work on isometric composition operators on some function spaces (see [17], for example) in order to show that the symbol of any such operator must fix the origin.

The paper is organized as follows. Basic definitions and facts will be reviewed in Section 1. The general formula is deduced in Section 2, along with the obvious applications to the Hardy, Bergman, and Dirichlet space.

In the case of linear fractional symbols, (3) can easily be transformed to yield as simple corollaries the results of Cowen [4], Hurst [13], and Gallardo-Montes [9]. It seems more natural to proceed in this way, getting all special results from one single general formula, rather than working hard towards each individual special result as was done so far. We write down these new proofs in Section 3.

In Section 4 we first consider the monomial symbols (thus generalizing an example from [6]) and then consider the operators induced by arbitrary rational self-maps of $\mathbb{D}$. We derive formulas that appears to be more practical and explicit than the one of McDonald [18] and explain a practical algorithm which allows for a reasonably direct adjoint computation in $H^2$ for any rational self-map of $\mathbb{D}$, the finite Blaschke products included.

The paper ends (Section 5) with some remarks on obvious generalizations. We also get a characterization of the composition operators whose adjoint is a composition operator and of the self-adjoint composition operators; these statements (well known in the case of the Hardy space $H^2$) are valid in a large family of spaces of Hardy type, including the Bergman and the Dirichlet space, and seem new in some cases.
1. Background

1.1. Linear fractional self-maps of $\mathbb{D}$. It is easy to tell when a linear fractional transformation $\varphi$, written in the form $\varphi(z) = (az + b)/(cz + d)$ ($ad - bc \neq 0$), maps $\mathbb{D}$ into itself: this happens if and only if

$$|b\bar{d} - a\bar{c}| + |ad - bc| \leq |d|^2 - |c|^2.$$ 

For the lack of a specific reference, we refer the reader to the semi-expository paper [16].

The following important lemma due to Cowen played an essential role in the computations of adjoints in $H^2$, $A^2_\alpha$, and $\mathcal{D}$; see [4, 13, 9] for some of its applications.

**Lemma 1.** Let $\varphi(z) = \frac{az + b}{cz + d}$ be a linear fractional map and define the associated linear fractional transformation $\varphi^*$ by

$$\varphi^*(z) = \frac{1}{\varphi^{-1}(\frac{1}{z})} = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$ 

Then $\varphi$ is a self-map of the disk if and only if $\varphi^*$ is also a self-map of the disk.

A proof can be found in [4] and [5].

1.2. Spaces with reproducing kernels. In all Hilbert spaces of analytic functions in the disk where point evaluations are bounded functionals there exists a family of reproducing kernels $\{K_w : w \in \mathbb{D}\}$. Such kernels have the reproducing property: $f(w) = \langle f, K_w \rangle$, for all $f$ in $\mathcal{H}$. Since each $K_w \in \mathcal{H}$, it is an analytic function in the $z$ variable, $z \in \mathbb{D}$. It is also an anti-analytic function of $w$ due to the following standard property:

$$\overline{K_w(z)} = \langle K_w, K_z \rangle = \langle K_z, K_w \rangle = K_z(w).$$

1.3. Hardy and Bergman spaces. Let $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ denote the unit circle and $dm(z) = (2\pi)^{-1}d\theta = (2\pi iz)^{-1}dz$ the normalized arc length measure on it. The Hardy space $H^2$ of the disk is defined as the set of all functions analytic in $\mathbb{D}$ for which

$$\|f\|_{H^2} = \sup_{0 \leq r < 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2} < \infty.$$
All such functions possess radial limits $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$ for almost all $\theta$ in $[0, 2\pi)$ and $H^2$ can be identified with a closed subspace of $L^2(\mathbb{T}, dm)$. Thus, it is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{H^2} = \int_{\mathbb{T}} f\overline{g} dm = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} a_n\overline{b_n}$$

(here $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are the Taylor series in the unit disk of $f$ and $g$, respectively). It also has a reproducing kernel:

$$K_w(z) = \frac{1}{1 - wz}, \quad z, w \in \mathbb{D},$$

called the Szegő (Riesz) kernel. (Note that in the reproducing formula for this kernel we usually only work with the boundary values of $K_w(z)$.) See [7] for further details.

The weighted Bergman space $A^2_\alpha$ is the set of all analytic functions in the disk that are square integrable with respect to the weighted area measure $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, $-1 < \alpha < \infty$. Here $dA(z) = \pi^{-1}rdrd\theta$, the normalized Lebesgue area measure on $\mathbb{D}$. The inner product in this space is given by

$$\langle f, g \rangle_{A^2_\alpha} = \int_\mathbb{D} f(z)\overline{g(z)} dA_\alpha(z).$$

The standard (unweighted) Bergman space is obtained as a special case: $A^2 = A^2_0$.

The reproducing kernel in $A^2_\alpha$ for the point $w$ in the disk is given by

$$K_w(z) = \frac{1}{(1 - wz)^{2+\alpha}},$$

with a suitably chosen analytic branch of the power.

The space $H^2$ can be considered as the limit case of $A^2_\alpha$ as $\alpha \to -1^+$, in the sense that $\lim_{\alpha \to -1^+} \|f\|_{A^2_\alpha} = \|f\|_{H^2}$. See [27] for a detailed proof.

1.4. Toeplitz operators. Let $u \in L^\infty(\mathbb{T}, dm)$. The Toeplitz operator $T_u$ with symbol $u$ on the Hardy space $H^2$ is defined as $T_u f = Pf(u)$, where $P$ denotes the orthogonal (Szegő) projection from $L^2(\mathbb{T}, dm)$ onto its closed subspace $H^2$.

The weighted Bergman projection $P_\alpha$, defined by $P_\alpha f(w) = \langle f, K_w \rangle_{A^2_\alpha}$, is a bounded operator from $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha$ that fixes all analytic functions: $P_\alpha f = f$ for all $f \in A^2_\alpha$. The reader may consult Section 1.1 of [11] for further details. The special case $P = P_0$ yields the standard Bergman projection. The generalised Toeplitz operator $T_{u,\alpha}$ on $A^2_\alpha$ can be defined for any essentially bounded symbol in the disk, $u \in L^\infty(\mathbb{D})$, replacing the Riesz projection by the weighted Bergman projection from
$L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha$: $T_{u,\alpha} f = P_\alpha(uf)$. When $\alpha = 0$, we obtain the standard Toeplitz operator $T_u$.

One basic property of Toeplitz operators, valid in all of the spaces mentioned so far, is that $T_h^* = T_h$ for any $h \in H^\infty$. More generally, $T_{h,\alpha}^* = T_{h,\alpha}$, $\alpha > -1$. This is easily proved by standard manipulations of the defining formula for $T_{h,\alpha}^*$ and properties of orthogonal projections and inner products.

1.5. The Dirichlet space. The Dirichlet space $D$ is the space of all functions $f$ analytic in $\mathbb{D}$ with the finite Dirichlet integral: $f' \in A^2$. It is a Hilbert space when equipped with the inner product
\[
\langle f, g \rangle_D = f(0)\overline{g(0)} + \int_\mathbb{D} f'(z)\overline{g'(z)}dA(z) = \sum_{n=1}^\infty na_n\overline{b_n}.
\]

The reproducing kernel consistent with the above inner product is given by
\[
K_w(z) = 1 + \log \frac{1}{1 - wz},
\]
with the suitably chosen analytic branch of the logarithm so that property (5) can be satisfied.

A standard computation involving the Taylor coefficients shows that $\|f\|_{A^2} \leq \|f\|_{H^2} \leq \|f\|_D$ and so $D \subset H^2 \subset A^2$, a property that will be useful later.

1.6. A generalization of Green’s formula. If $f$ and $g$ are analytic functions such that $f', g' \in C(\overline{\mathbb{D}})$, the standard formula for the $\overline{\Gamma}$-derivative and the Cauchy-Green’s identity (see [8], p. 17, for example) can be applied to the function $F = \overline{f'}g$ to obtain
\[
\int_\mathbb{D} \overline{F'(z)}g(z)dA(z) = \frac{1}{2\pi i} \int_\Gamma \overline{f(z)}g(z)dz = \int_\mathbb{D} \overline{f(z)}g(z)dm(z).
\]
(Recall that our area measure is normalized.) After taking the complex conjugates we get $\langle f', g \rangle_{A^2} = \langle f, zg \rangle_{H^2}$. The following lemma asserts that much weaker smoothness assumptions still suffice.

**Lemma 2.** Let $f \in D$ and $g \in H^2$. Then
\[
\langle f', g \rangle_{A^2} = \langle f, zg \rangle_{H^2}.
\]

**Proof.** The finiteness of the integral $\langle f', g \rangle_{A^2} = \int_\mathbb{D} f'\overline{g}dA$ follows easily from Hölder’s inequality since $f \in D$ and $g \in H^2 \subset A^2$. The integral over the circle on the right in (10) is finite because both $f$ and $g$ are $H^2$ functions. Now if we denote
by \((a_n)\) and \((b_n)\) the sequences of Taylor coefficients of \(f\) and \(g\) respectively, a straightforward computation shows that both inner products equal \(\sum_{n=0}^{\infty} a_{n+1} b_n\), and (10) is proved. \(\square\)

2. The general formula and some classical spaces

As mentioned earlier, the formulas (1) for \(H^2\) and (2) for \(A^2_\alpha\) are special cases of a more general statement. Recall that a self-map of the disk need not induce a bounded composition operator on the Dirichlet space \(D\) (see [1] or [5] for a necessary and sufficient condition in terms of Carleson’s measures). However, for those symbols that do (including all univalent self-maps of \(\mathbb{D}\)), there is a simple adjoint formula as well.

**Theorem 1.** Let \(\varphi\) be any self-map of \(\mathbb{D}\) and let \(K_w\) be the reproducing kernel for the point evaluation at \(w\) on a Hilbert space \(\mathcal{H}\) of analytic functions in \(\mathbb{D}\). Then, whenever \(C_\varphi\) is a bounded operator on \(\mathcal{H}\), formula (3) stated in the Introduction holds:

\[
C_\varphi^* f(w) = \langle f, K_w \circ \varphi \rangle, \quad f \in \mathcal{H}, \quad w \in \mathbb{D}.
\]

In the most significant special cases, it assumes the following specific forms.

(i) When \(\mathcal{H} = H^2\), we get (1) for all self-maps \(\varphi\) of \(\mathbb{D}\). The formula can also be written as a power series:

\[
C_\varphi^* f(w) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{T}} f \overline{\varphi}^n dm \right) w^n.
\]

Whenever \(\varphi\) is an inner function such that \(\varphi(0) = 0\), the coefficients of the above series are actually the Fourier coefficients of \(f\) with respect to an orthonormal system.

(ii) when \(\mathcal{H} = A^2_\alpha\), formula (2) holds for all possible symbols \(\varphi\). The expression for the adjoint can again be written as a power series:

\[
C_\varphi^* f(w) = \sum_{n=0}^{\infty} \int_{\mathbb{D}} f \overline{\varphi}^n dA \cdot (n+1) w^n.
\]

(iii) For the Dirichlet space \(D\), whenever \(C_\varphi\) is bounded on \(D\), it is the case that

\[
C_\varphi^* f(w) = f(0) K_w(\varphi(0)) + \int_{\mathbb{D}} \frac{w f'(z) \overline{\varphi'(z)}}{1 - w \overline{\varphi(z)}} dA(z).
\]
The adjoint can also be written as

\[
C^*\phi f(w) = f(0)K_\phi(0)(w) + \int_T f(z) \frac{w \overline{\varphi'(z)}}{1 - w \varphi(z)} dm(z)
\]

Each of the integrals in (13) and (14) can be written as a power series:

\[
\sum_{n=0}^{\infty} \left( \int_D f' \varphi^n dA \right) w^{n+1} = \sum_{n=0}^{\infty} \left( \int_T f(z) \overline{\varphi'(z)} \varphi(z)^n dm(z) \right) w^{n+1}.
\]

**Proof.** For each fixed \( w \in \mathbb{D} \) the kernel \( K_w(z) \) is an analytic function of the variable \( z \), hence it is legitimate to apply \( C_\phi \) to it: \( C_\phi K_w = K_w \circ \varphi \). Thus, by the basic property of reproducing kernels and the definition of the adjoint, we have

\[
C^*\phi f(w) = \langle C^*\phi f, K_w \rangle = \langle f, C_\phi K_w \rangle = \langle f, K_w \circ \varphi \rangle.
\]

This proves the general formula.

(i) The basic geometric series argument shows that

\[
C^*\phi f(w) = \int_T \frac{f(z)}{1 - \varphi(z)w} dm(z) = \sum_{n=0}^{\infty} \left( \int_T f \varphi^n dm \right) w^n,
\]

which is (11). Interchanging the integral and the sum above is justified for arbitrary \( w \) in \( \mathbb{D} \) by the Lebesgue Dominated Convergence Theorem since

\[
\left| \sum_{n=0}^{N} f(z) \varphi(z)^n w^n \right| \leq \sum_{n=0}^{N} |f(z)||w|^n \leq \frac{|f(z)|}{1 - |w|}
\]

for all \( N \) and for almost every \( z \) on \( T \).

As shown by Nordgren [19], \( C_\phi \) is a partial isometry of \( H^2 \) if \( \varphi \) is inner and \( \varphi(0) = 0 \); see also [21]. In this case, \( \{\varphi^n\}_{n=0}^{\infty} \) is an orthonormal system in \( H^2 \). The Fourier coefficients of \( f \) with respect to this orthonormal system are precisely \( \hat{f}_\varphi(n) = \int_T f \varphi^n dm \). This completes the proof of (i).

(ii) The proof is completely analogous to that of (i) and uses the standard formula

\[
\frac{1}{(1 - \varphi w)^2} = \sum_{n=0}^{\infty} (n + 1) \varphi^n w^n.
\]
(iii) Using formula (13), Lemma 2, and property (5), we obtain
\[
C^*_\varphi f(w) = \langle f, K_w \circ \varphi \rangle_D
\]
\[
= f(0)K_w(\varphi(0)) + \langle f', (K_w \circ \varphi)' \rangle_{A^2}
\]
\[
= f(0)K_{\varphi(0)}(w) + \langle f(z), z(K_w \circ \varphi)'(z) \rangle_{H^2}
\]
(16)
\[
= f(0)K_{\varphi(0)}(w) + \int_T f(z) \frac{wz\varphi'(z)}{1-w\varphi(z)}dm(z)
\]
The rest follows by arguments similar to the one above. Some simple additional considerations are needed; for example, the integral \( \int_D f' \varphi^n dA \) is convergent by the Cauchy-Schwarz inequality since \( f \in D \) and so does the function \( \varphi' \varphi^n \), which is a constant multiple of \( (\varphi^{n+1})' \) (here is where one uses that \( C_\varphi \) acts boundedly on \( D \)), etc. \( \square \)

3. Linear fractional symbols, revisited

3.1. Cowen’s formula. Cowen’s proof of the adjoint formula (1) for a composition operator with a linear fractional symbol \([4, 5]\) goes as follows: he exhibits the final form of the adjoint as a product of three operators and then verifies that it satisfies the defining property of the adjoint when acting on the family of reproducing kernels which span a dense subset of \( H^2 \). Unfortunately, this presentation does not reveal the intuitive reasons that led to the formula.

Although it is not clear whether the formulas of Cowen \([4]\) and Hurst \([13]\) in \( H^2 \) and \( A^2_\alpha \) respectively are really simpler than the formula from Theorem 1, they can be recovered from it in a straightforward manner.

**Theorem 2.** Let \( \varphi \) be a linear fractional symbol. Then \( C^*_\varphi \) acting on \( A^2_\alpha \) is given by the formula
\[
C^*_\varphi = T_{g,\alpha}C_{\varphi'} T_{h,\alpha},
\]
where \( g(w) = (\overline{d} - \overline{b}w)^{-2+\alpha} \) and \( h(z) = (cz + d)^{2+\alpha} \). Here \( \alpha \geq -1 \), including the Hardy space case: \( H^2 = A^2_{-1} \).

**Proof.** We give a proof for the true weighted Bergman space \( A^2_\alpha, -1 < \alpha < \infty \). First of all, because of the assumptions on \( \varphi \) none of the functions \( \overline{d} - \overline{b}w, cz + d \) vanishes in \( D \), so we can choose appropriate analytic branches of the powers and define well
the analytic functions $g$ and $h$ mentioned above. By applying Theorem 1, grouping the terms, and using the definition (4) of $\phi^*$ from Lemma 1 we get

$$C^*_\phi f(w) = \int_{\mathbb{D}} \frac{f(z)}{\left(1 - \frac{wz}{\phi(z)}\right)^{2+\alpha}} dA_\alpha(z)$$

$$= \int_{\mathbb{D}} \left(\frac{w\phi(z)}{d - bw + \phi(z)}\right)^{2+\alpha} f(z) dA_\alpha(z)$$

$$= \frac{1}{(d - bw)^{2+\alpha}} \int_{\mathbb{T}} \left(1 - \frac{\phi^*(w)z}{\phi(z)}\right)^{2+\alpha} dA_\alpha(z)$$

$$= T_{g,\alpha} P(h^*(w))$$

$$= T_{g,\alpha} C_{\phi^*} T_{h,\alpha} f(w).$$

The proof for $H^2$ is completely analogous, replacing the integration over $\mathbb{D}$ by that over $\mathbb{T}$ and considering the appropriate kernel. □

3.2. The formula of Gallardo and Montes. We now turn to the result for composition operators induced by linear fractional symbols and acting on the Dirichlet space proved recently by Gallardo-Gutiérrez and Montes-Rodríguez [9]. Its proof seems more involved and it also allows for a significant simplification of the statement of Theorem 1. Even in this case the computations become rather simple.

The original proof of Theorems 3.2 and 3.3 of [9] was first worked out for the subspace $D_0 = \{f \in D : f(0) = 0\}$ by operator-theoretic properties and by the Cauchy integral formula (note that in this case the actual operator $C_\phi$ has to be modified slightly) and then by transferring the setting to the true Dirichlet space using a property from [12]. The direct proof given below works directly for $D$ without having to analyze the operator on $D_0$.

**Theorem 3.** Let $\phi$ be a linear fractional symbol. Then $C^*_\phi$ acting on $D$ is given by the formula

$$C^*_\phi f = f(0)K_{\phi(0)} - (C_{\phi^*} f)(0) + C_{\phi^*} f,$$

where $\phi^*$ is again as in Lemma 1.

**Proof.** According to formula (13) of Theorem 1, we have

$$C^*_\phi f(w) = f(0)K_{\phi(0)}(w) + \int_{\mathbb{T}} f(z) \frac{wz\phi'(z)}{1 - w\phi(z)} dm(z).$$
We may certainly assume that \( d \neq 0 \), for otherwise either \( \varphi \) is not analytic in \( \mathbb{D} \) (contrary to our assumptions) or it is constant (a case not of interest). Also, there is no loss of generality in assuming that \( ad - bc = 1 \), hence \( \varphi'(z) = (cz + d)^{-2} \). For the same reason, we readily compute

\[
\overline{\varphi + d \varphi^*(w)} = \frac{w}{\overline{d - bw}}.
\]

We can work with the denominator containing the term \( 1 - w\varphi(z) \) in (14) just as we did in the proof of Theorem 2. After an application of (17) we get

\[
C_\varphi^* f(w) = f(0) K_{\varphi(0)}(w) + \frac{w}{\overline{d - bw}} \int_T \frac{f(z)}{(\overline{\varphi' + d\overline{\varphi}})(1 - \varphi^*(w)\overline{z})} dm(z).
\]

Taking into account that \(-\overline{c/d} = \varphi^*(0)\), an application of elementary partial fractions and the reproducing property of the Szegő kernel for \( H^2 \) yield

\[
C_\varphi^* f(w) = f(0) K_{\varphi(0)}(w) + \int_T \frac{f(z)}{1 - \varphi^*(w)\overline{z}} dm(z) - \int_T \frac{f(z)}{\overline{c + d\overline{z}}} dm(z)
\]

\[
= f(0) K_{\varphi(0)}(w) + f(\varphi^*(w)) - \int_T \frac{f(z)}{1 + (c/d)\overline{z}} dm(z)
\]

\[
= f(0) K_{\varphi(0)}(w) + f(\varphi^*(w)) - f(\varphi^*(0)),
\]

and we are done. \( \square \)

4. Composition operators induced by other rational symbols

4.1. An explicit formula for monomial symbols. The following formula for the case \( \varphi(z) = z^2 \) was derived in Cowen’s and MacCluer’s survey [6] from the final remark in the paper by Lotto and McCarthy [14]:

\[
C_\varphi^* f(w) = f(0) K_{\varphi(0)}(w) + \frac{w}{\overline{d - bw}} \int_T \frac{f(z)}{(\overline{\varphi' + d\overline{\varphi}})(1 - \varphi^*(w)\overline{z})} dm(z).
\]

In view of our formula (3), an analogous statement can be proved in a straightforward manner for arbitrary monomial symbol \( \varphi(z) = z^n \).

**Theorem 4.** Let \( \varphi(z) = z^n \). For an arbitrary point \( w = re^{i\theta} \) in \( \mathbb{D} \), writing its \( n \)-th roots as \( r_{k,w} = r^{1/n} e^{i(\theta + 2k\pi)/n} \), \( k = 0, 1, \ldots, n-1 \), the adjoint of \( C_\varphi \) (viewed as an operator on the Hardy space \( H^2 \)) is given by the formula

\[
C_\varphi^* f(w) = \frac{1}{n} \sum_{k=0}^{n-1} f(r_{k,w}).
\]
Proof. From (3), recalling that \( z \bar{z} = 1 \) on \( T \) and \( dm(z) = (2\pi i z)^{-1}dz \), we get in this case
\[
C^*_{\varphi} f(w) = \int_T \frac{f(z)}{1 - wz} dm(z) = \frac{1}{2\pi i} \int_T f(z) \frac{z^{n-1}}{z^n - w} dz.
\]
A simple algebra of partial fractions helps us determine the unknown coefficients \( C_k \) in the representation:
\[
\frac{z^{n-1}}{z^n - w} = \sum_{k=0}^{n-1} C_k \frac{1}{z - r_k,w}.
\]
Namely, some basic algebraic operations with cyclotomic polynomials (essentially the formula for factoring \( z^n - r_{k,w}^n \) and evaluations at \( r_{k,w} \)) easily yield \( C_k = 1/n \) for all \( k \). Thus, the Cauchy Integral Formula gives
\[
C^*_{\varphi} f(w) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_T \frac{f(z)}{z - r_{k,w}} dz = \frac{1}{n} \sum_{k=0}^{n-1} f(r_{k,w})
\]
for any polynomial \( f \).

Recalling that the polynomials are dense in \( H^2 \) and that norm convergence in this space implies the uniform convergence of compact sets, the formula follows for arbitrary \( f \) in \( H^2 \).

4.2. General rational symbols and Blaschke products. Besides the results already quoted on linear fractional symbols, there are only two results published for the adjoint of a composition operator with a rational symbol that we know of.

In her thesis, Wahl (cf. [26, 6]) obtained a representation for the adjoint of the composition operator with some special rational symbols, e.g., \( \varphi(z) = z^2/(2-z) \). She then used this representation in studying the spectrum of the induced composition operator.

McDonald [18] recently obtained a representation of the adjoint of a composition operator whose symbol is a finite Blaschke product as a linear combination of finitely many operators which are products of two Toeplitz operators and an “average” of the same number of evaluation operators. His formula does not identify explicitly these Toeplitz operators and its proof is lengthy and relies on the Pick-Nevanlinna interpolation; moreover, the operators change as the point \( w \) in \( \mathbb{D} \) changes. We have actually been able to transform formula (3) into a form that allows us to identify explicitly the coefficients and the symbols of the Toeplitz operators in McDonald’s theorem. This is done by replacing \( \bar{z} \) by \( 1/z \) in integration over \( T \) and then applying partial fractions and the Residue Theorem. However, we do not consider presenting
the details of such computation here very useful because there is a general and effective algorithm for computing $C_B^*f(w)$ in this case for arbitrary $f$ and $w$, based on the same method. The method actually works for evaluating $C_R^*f(w)$ for any rational symbol $R$.

Note that any analytic self-map of $\mathbb{D}$ which is a rational function must have all poles in the complement of the closed unit disk. Thus, we may define the associated function $R^*$ (not necessarily a self-map of the disk) by the formula

$$R^*(z) = \overline{R\left(\frac{1}{z}\right)}, \quad z \in \mathbb{D}.$$ 

It follows by elementary algebra that $R^*$ is again a rational function.

**Theorem 5.** Let $R$ be any rational self-map of the disk. Define the associated rational function $R^*$ as above. We then have the following formula for the adjoint of $C_R$ on the Hardy space $H^2$:

$$C_R^*f(w) = \sum_k \text{Res} \left( \frac{f(z)}{z(1 - wR^*(z))}; z_k \right), \quad w \in \mathbb{D},$$

where the sum is taken over those zeros $z_k$ of the rational function $z(1 - wR^*(z))$ which belong to $\mathbb{D}$.

**Proof.** As before, let $f$ first be a polynomial. Then

$$C_R^*f(w) = \int_{\mathbb{T}} \frac{f(z)}{1 - wR(z)} dm(z) = \int_{\mathbb{T}} \frac{f(z)}{z(1 - wR(1/z))} dz,$$

and the statement follows from the Residue Theorem since the only poles in $\mathbb{D}$ of the integrand are the zeros of the denominator. The statement for arbitrary $f$ in $H^2$ follows from the density of the polynomials. $\square$

Since finite Blaschke products are very special and particularly comfortable to work with, we also give a statement for this special case where a slight modification is possible due to fact that $|B(z)| = 1$ at all points on $\mathbb{T}$. Recall that if $B$ is a finite Blaschke product of degree $n$ then, by Rouché’s theorem, for each $w$ in $\mathbb{D}$ the
equation \( B(z) = w \) has exactly \( n \) solutions in the unit disk (not necessarily all the same).

**Theorem 6.** Let \( B \) be a finite Blaschke product, say of degree \( n \), and given a point \( w \) in the disk, denote by \( z_{k,w} \) all the points in \( \mathbb{D} \) (taking multiplicities into account) for which \( B(z_{k,w}) = w \). We then have the following formula for the adjoint of \( C_B \) on the Hardy space \( H^2 \):

\[
C_B^* f(w) = \sum_{k=1}^{n} \text{Res} \left( \frac{f(z)B(z)}{z(B(z) - w)} ; z_{k,w} \right).
\]

**Proof.** It suffices again to prove the statement only when \( f \) is a polynomial. Each finite Blaschke product is analytic on a larger disk and \( |B(z)| = 1 \) on \( \mathbb{T} \), hence from (3) and the Residue Theorem we get

\[
C_B^* f(w) = \int_{\mathbb{T}} \frac{f(z)}{1 - wz(B(z))} dm(z) = \frac{1}{i2\pi} \int_{\mathbb{T}} \frac{f(z)B(z)}{z(B(z) - w)} dz,
\]

from where the statement follows immediately as \( z_{k,w} \) and the origin are the only poles of the function \( \frac{f(z)B(z)}{z(B(z) - w)} \) in \( \mathbb{D} \). \( \square \)

5. **Normal and self-adjoint composition operators and some closing remarks**

Formula (3) carries over without difficulties to the classical Hilbert spaces of functions holomorphic in the unit ball or polydisk of \( \mathbb{C}^n \). It also has an interpretation that works for the Banach spaces such as \( H^p \) or \( A^p \) of the disk \( (1 < p < \infty) \), among others. However, we have chosen to focus on the one-dimensional Hilbert space case only for the sake of avoiding repetitions and keeping the presentation shorter.

Everything seems to suggest that one should be able to use formula (3) to prove in a different manner at least some statements in the spectral theory of composition operators on Hilbert spaces of analytic functions presented, for example, in [5]. In order to illustrate what is meant by this, we present just one general application of the adjoint formula.
A great deal of the study of composition operators is devoted to characterizing those operators from this class that are self-adjoint, unitary, normal, subnormal, essentially normal, etc. (cf. Chapter 8 of [5] or the recent papers [15] and [2] for some further research on this topic).

We begin by looking at when the operation of taking the adjoint preserves the property of being a composition operator. Intuitively, the adjoint of a composition operator should very seldom again be a composition operator. For example, it was shown in [16] that for any admissible $\varphi$ for the Dirichlet space $\mathcal{D}$, the operator $C^{\ast}_{\varphi}$ is again a composition operator if and only if $\varphi(z) = az$, where $a$ is a constant of modulus at most one. This is actually true for a large family of spaces that includes the Hardy, Bergman, and Dirichlet space.

**Theorem 7.** Let $\mathcal{H}$ be a Hilbert space of analytic functions in which $C_{\varphi}$ acts boundedly. Assume also that $K_w(z) = u(z\bar{w})$, where the function $u$ is one-to-one in $\mathbb{D}$. Then the adjoint $C^{\ast}_{\varphi}$ is again a composition operator if and only if $\varphi(z) = az$, where $0 < |a| \leq 1$. If this is the case, then $\psi(z) = \bar{a}z$.

**Proof.** If $C^{\ast}_{\varphi} = C_{\psi}$ for some analytic self-map $\psi$ of $\mathbb{D}$, then the identity $C^{\ast}_{\varphi}f(w) = f(\psi(w))$ for all $w$ in $\mathbb{D}$ and formula (3) together yield

$$(f, K_w \circ \varphi) = (f, K_{\psi(w)}), \quad f \in \mathcal{H}, \quad w \in \mathbb{D}.$$  

By the elementary Hilbert space properties it follows that $K_w(\varphi(z)) = K_{\psi(w)}(z)$ for all $z$ and $w$ in $\mathbb{D}$. Our assumptions on the kernels now implies that $\psi(\varphi(z)/z)$, analytic in $\mathbb{D} \setminus \{0\}$, must be constant there, so it extends to an analytic function in all of $\mathbb{D}$ and $\varphi(z) = az$. In view of the Schwarz lemma, we have $|a| \leq 1$, and the trivial case $a = 0$ is to be excluded. The equality $\psi(w) = \bar{a}w$ is immediate.

By retracing backwards all steps in the proof, it is easy to check that the condition obtained for $\varphi$ is also sufficient. \qed

It is elementary to verify that each of the reproducing kernels (6), (8) with $\alpha = 0$, and (9) for the Hardy, Bergman, and the Dirichlet space respectively, verifies the property $K_w(z) = u(\bar{w}z)$, where the corresponding function $u$ is one-to-one in $\mathbb{D}$.
Indeed, in these cases we have
\[
\begin{align*}
u(\zeta) &= \frac{1}{1-\zeta}, & u(\zeta) &= \frac{1}{(1-\zeta)^2}, & u(\zeta) &= 1 + \log \frac{1}{1-\zeta}
\end{align*}
\]
respectively.

It is well known that a composition operator $C_\varphi$ on the Hardy space $H^2$ is normal (that is, $C_\varphi^* C_\varphi = C_\varphi C_\varphi^*$) if and only if $\varphi(z) = az$, $|a| \leq 1$ (see [5], p. 309). We, thus, have the following immediate consequence.

**Corollary 1.** The following statements are equivalent for a composition operator $C_\varphi$ acting on $H^2$:

(a) $C_\varphi$ is normal;

(b) $C_\varphi^* = C_\psi$, also a composition operator;

(c) $\varphi(z) = az$, $0 < |a| \leq 1$.

Whenever one of the three conditions is fulfilled (and therefore all of them are), we have $\psi(z) = \overline{az}$.

A characterization of self-adjoint composition operators on $H^2$ is well known; for example, this result is left as an exercise after the characterization of normal operators in [5], Section 8.1. We have not been able to find a characterization of the self-adjoint composition operators on $D$ in the literature, although it should be pointed out that Corollary 3.4 in [9] characterizes the unitary composition operators on $D$, while the isometries among the composition operators were characterized in our most recent paper [17]. The above Theorem 7 actually yields for free a characterization of the self-adjoint operators on the larger family of spaces mentioned above.

**Corollary 2.** If $H$ is a Hilbert space of analytic functions as in Theorem 7 and $C_\varphi$ a bounded operator on $H$, then this operator is self-adjoint if and only if $\varphi(z) = az$, where $a$ is a real number such that $0 < |a| \leq 1$.

It is clear that any serious attempt of rewriting a significant portion of the spectral theory by using the adjoint formula from Theorem 1 (if such an enterprise were indeed possible) would require a considerable effort and time. We have limited ourselves only to working out the most direct applications of the formula.
References


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