

## WEIERSTRASS' THEOREM IN WEIGHTED SOBOLEV SPACES WITH $K$ DERIVATIVES: ANNOUNCEMENT OF RESULTS\*

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**Abstract.** We characterize the set of functions which can be approximated by smooth functions and by polynomials with the norm

$$\|f\|_{W^{k,\infty}(w)} := \sum_{j=0}^k \|f^{(j)}\|_{L^\infty(w_j)},$$

for a wide range of (even non-bounded) weights  $w_j$ 's. We allow a great deal of independence among the weights  $w_j$ 's.

**Key words.** Weierstrass' theorem, weight, Sobolev spaces, weighted Sobolev spaces

**AMS subject classifications.** 41A10, 46E35, 46G10

**1. Introduction.** If  $I$  is any compact interval, Weierstrass' Theorem says that  $C(I)$  is the largest set of functions which can be approximated by polynomials in the norm  $L^\infty(I)$ , if we identify, as usual, functions which are equal almost everywhere.

In [29] and [24] we study the same problem with the norm  $L^\infty(w)$  defined by

$$\|f\|_{L^\infty(w)} := \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)|w(x),$$

where  $w$  is a (bounded or unbounded) weight, i.e. a non-negative measurable function.

Considering weighted norms  $L^\infty(w)$ , has been proved to be interesting mainly because of two reasons: on the one hand, it allows to wider the set of approximable functions (since the functions in  $L^\infty(w)$  can have singularities where the weight tends to zero); and, on the other hand, it is possible to find functions which approximate  $f$  whose qualitative behaviour is similar to the one of  $f$  at those points where the weight tends to infinity.

If  $w = (w_0, \dots, w_k)$  is a vectorial weight, we study this approximation problem with the Sobolev norm  $W^{k,\infty}(w)$  defined by

$$\|f\|_{W^{k,\infty}(w)} := \sum_{j=0}^k \|f^{(j)}\|_{L^\infty(w_j)}.$$

The papers [27], [28], [29], [30], [31], [1], [32], [24] and [25] are the beginning of a theory of Sobolev spaces with respect to general measures for  $1 \leq p \leq \infty$ . This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see [19], [20], [28] and [30]). The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [19]).

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**2. Main results.** The fundamental results of this paper guarantee that a function  $f$  belongs to the closure of the space of polynomials (respectively, smooth functions) in the norm  $W^{k,\infty}(w)$ , if and only if,  $f^{(j)}$  belongs to the closure of polynomials (respectively, smooth functions) in the norm  $L^\infty(w_j)$ , for every  $0 \leq j \leq k$ .

This article is an abridged version of the paper [26].

**THEOREM 2.1.** *Let be a vectorial weight  $w = (w_0, \dots, w_k)$  on  $[\alpha, \beta]$  satisfying:*

- (i)  $\int_\alpha^\beta 1/w_k < \infty$ .
- (ii)  $w_j \in L^\infty_{loc}([\alpha, \beta] \setminus \{a_1^j, \dots, a_{m_j}^j\})$ , for every  $0 \leq j < k$ .
- (iii)  $w_j(x) \left| \int_{a_i^j}^x 1/(1+w_{j+1}) \right| \leq c$ , a.e. in some neighborhood of  $a_i^j$ , for every  $1 \leq i \leq m_j$ ,  $0 \leq j \leq k-2$ , and  $w_{k-1}(x) \left| \int_{a_i^{k-1}}^x 1/w_k \right| \leq c$ , a.e. in some neighborhood of  $a_i^{k-1}$ , for every  $1 \leq i \leq m_{k-1}$ .

*Then the closure of the space of polynomials in  $W^{k,\infty}(w)$  is*

$$H_1 := \left\{ f \in W^{k,\infty}(w) : f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(w_k)}^{L^\infty(w_k)} \right\}.$$

**REMARK 1.**

(i) *We observe that this theorem characterizes the closure of  $\mathbb{P} \cap W^{k,\infty}(w)$  in  $W^{k,\infty}(w)$ , in terms of the similar problem in  $L^\infty(w_k)$ . This question of approximation in  $L^\infty(w_k)$  is solved in [24].*

(ii) *The hypothesis (ii) is not restrictive at all, since if  $\text{ess lim sup}_{x \rightarrow a} w_j(x) = \infty$  for an infinite number of points  $a \in \mathbb{R}$ , for some  $0 \leq j < k$ , then 0 is the only polynomial in  $L^\infty(w_j)$ , and it is trivial to find the closure of the space of polynomials in  $W^{k,\infty}(w)$ .*

(iii) *Notice that hypothesis,  $w_j(x) \left| \int_{a_i^j}^x 1/(1+w_{j+1}) \right| \leq c$ , is much weaker than  $w_j(x) \left| \int_{a_i^j}^x 1/w_{j+1} \right| \leq c$ , since some  $w_{j+1}$  are allowed to be 0.*

(iv) *The possibility of some  $w_j$  to be bounded is, naturally, allowed. That is to say,  $\{a_1^j, \dots, a_{m_j}^j\}$  might be the empty set.*

*Sketch of the proof.* It is obvious that the closure of the space of polynomials in  $W^{k,\infty}(w)$  is contained in  $H_1$ .

Then, it suffices to prove that every function in  $H_1$  can be approximated by polynomials in the norm  $W^{k,\infty}(w)$ . Let us consider then,  $f \in H_1$  and  $\{p_n\}_n$  a sequence of polynomials converging to  $f^{(k)}$  in the norm  $L^\infty(w_k)$ . From the sequence  $\{p_n\}_n$  we will construct another one of polynomials converging to  $f$  in the norm  $W^{k,\infty}(w)$ .

The key idea in order to carry out such a process, is to find, from  $p_n$ , a polynomial  $q_{n,k}$  in  $M$ , where  $M$  is the space of polynomials which have a primitive of order  $k$  in  $W^{k,\infty}(w)$ . If  $\mathbb{P}$  were a Hilbert space and  $M$  a closed subspace, it would suffice to take as  $q_{n,k}$  the orthogonal projection of  $p_n$  on  $M$ . However, since our norms do not come from an inner product, the problem is much more complicated; fortunately, we can find a finite set of polynomials  $B$  in  $L^\infty(w_k)$ , such that  $q_{n,k}$  can be expressed as a linear combination of  $p_n$  and elements of  $B$ .

**DEFINITION 2.2.** *We say that a vectorial weight  $w = (w_0, \dots, w_k)$  in  $[a, b]$  is of type 1 if  $1/w_k \in L^1([a, b])$  and  $w_0, \dots, w_{k-1} \in L^\infty([a, b])$ .*

In the following theorems we describe the closure of smooth functions in Sobolev spaces with weights.

**THEOREM 2.3.** *Let us consider a vectorial weight  $w = (w_0, \dots, w_k)$  of type 1 in a compact interval  $I = [a, b]$ . Then the closure of  $\mathbb{P} \cap W^{k,\infty}(I, w)$ ,  $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I, w)$  and  $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$  in  $W^{k,\infty}(I, w)$  are, respectively,*

$$\begin{aligned}
 H_1 &:= \left\{ f \in W^{k,\infty}(I, w) : f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(I, w_k)}^{L^\infty(I, w_k)} \right\}, \\
 H_2 &:= \left\{ f \in W^{k,\infty}(I, w) : f^{(k)} \in \overline{C^\infty(I) \cap L^\infty(I, w_k)}^{L^\infty(I, w_k)} \right\}, \\
 H_3 &:= \left\{ f \in W^{k,\infty}(I, w) : f^{(k)} \in \overline{C(I) \cap L^\infty(I, w_k)}^{L^\infty(I, w_k)} \right\}.
 \end{aligned}$$

REMARK 2.

(i) We observe that Theorem 2.3 characterizes the closure of  $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$ ,  $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I, w)$  and  $\mathbb{P} \cap W^{k,\infty}(I, w)$  in  $W^{k,\infty}(I, w)$ , in terms of the similar problem in  $L^\infty(I, w_k)$ . This question of approximation in  $L^\infty(I, w_k)$  is solved in [24].

(ii) If  $w_k \in L^\infty(I)$ , then the closure of  $C^k(\mathbb{R})$ ,  $\mathbb{P}$  and  $C^\infty(\mathbb{R})$  are the same. This is a consequence of Bernstein's proof of Weierstrass' Theorem (see e.g. [5, p. 113]), which gives a sequence of polynomials converging uniformly up to the  $k$ -th derivative for any function in  $C^k(I)$ .

*Sketch of the proof.* The inclusion

$$\overline{C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)}^{W^{k,\infty}(I, w)} \subseteq H_3$$

is obvious. Let us now consider a function  $f \in H_3$ , and let  $g \in C(\mathbb{R})$ , be a function which approximates  $f^{(k)}$  in  $L^\infty(I, w_k)$  norm. If we consider the function

$$h(x) := \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \int_a^x g(t) \frac{(x-t)^{k-1}}{(k-1)!} dt,$$

it is possible to show that  $h$  approximates  $f$  in  $W^{k,\infty}(I, w)$  norm.

DEFINITION 2.4. We say that  $u, v$ , are comparable functions in the set  $A$  if there exists a positive constant  $c$  such that  $c^{-1}u \leq v \leq cu$  a.e. in  $A$ .

DEFINITION 2.5. We say that a vectorial weight  $w = (w_0, \dots, w_k)$  in  $[a, b]$  is of type 2 if there exist real numbers  $a \leq a_1 < a_2 < a_3 < a_4 \leq b$  such that

- (i)  $1/w_k \in L^1([a_1, a_4])$ , and  $w_0, \dots, w_{k-1} \in L^\infty([a, b])$ ,
- (ii) if  $a < a_1$ , then  $w_j$  is comparable to a finite non-decreasing weight in  $[a, a_2]$ , for  $0 \leq j \leq k$ ,
- (iii) if  $a_4 < b$ , then  $w_j$  is comparable to a finite non-increasing weight in  $[a_3, b]$ , for  $0 \leq j \leq k$ .

THEOREM 2.6. Let us consider a vectorial weight  $w = (w_0, \dots, w_k)$  of type 2 in a compact interval  $I = [a, b]$ . Then the closure of  $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$  in  $W^{k,\infty}(I, w)$  is

$$H_4 := \left\{ f \in W^{k,\infty}(I, w) : f^{(j)} \in \overline{C(I) \cap L^\infty(I, w_j)}^{L^\infty(I, w_j)} \text{ for } 0 \leq j \leq k \right\}.$$

*Sketch of the proof.* Given a function  $f \in H_4$ , we can split it as a sum of three functions by using an appropriate partition of unity. The function in  $[a_1, a_4]$  can be approximated with similar arguments than the measures of type 1. In the approximation of the functions in  $[a, a_2]$  and  $[a_3, b]$  we use shift arguments that allow us to construct a convolution with an approximation of the identity.

The next theorem makes the results of this paper more valuable since it allows to deal with weights which can be obtained by “gluing” simpler ones.

**THEOREM 2.7.** *Let us consider strictly increasing sequences of real numbers  $\{a_n\}$ ,  $\{b_n\}$  ( $n$  belonging to a finite set, to  $\mathbb{Z}$ ,  $\mathbb{Z}^+$  or  $\mathbb{Z}^-$ ) with  $b_{n-1} < a_{n+1} < b_n$  for every  $n$ . Let  $w = (w_0, \dots, w_k)$  be a vectorial weight in the interval  $I := \cup_n [a_n, b_n]$ . Assume that for each  $n$  there exists an interval  $I_n \subset [a_{n+1}, b_n]$  with  $w_1, \dots, w_k \in L^\infty(I_n)$ . Let us assume also that for each  $n$  we have either  $w$  is of type 1 in  $[a_n, b_n]$ , or  $1/w_k \in L^\infty([a_n, b_n])$ . Then the closure of  $C^k(I) \cap W^{k,\infty}(I, w)$  in  $W^{k,\infty}(I, w)$  is*

$$H_3 := \left\{ f \in W^{k,\infty}(I, w) : f^{(k)} \in \overline{C(I) \cap L^\infty(I, w_k)}^{L^\infty(I, w_k)} \right\}.$$

We can deduce the following consequence.

**THEOREM 2.8.** *Let us consider a vectorial weight  $w = (w_0, \dots, w_k)$  in the interval  $I$ , with  $w \in L^\infty_{loc}(I)$  and  $1/w_k \in L^1_{loc}(I)$ . Then the closure of  $C^k(I) \cap W^{k,\infty}(I, w)$  in  $W^{k,\infty}(I, w)$  is*

$$H_3 := \left\{ f \in W^{k,\infty}(I, w) : f^{(k)} \in \overline{C(I) \cap L^\infty(I, w_k)}^{L^\infty(I, w_k)} \right\}.$$

*Proof.* This theorem is a direct consequence of Theorems 2.3 and 2.7. It is enough to split  $I$  as a union of compact intervals  $[a_n, b_n]$  ( $n$  belonging to a finite set, to  $\mathbb{Z}$ ,  $\mathbb{Z}^+$  or  $\mathbb{Z}^-$ ), with  $b_{n-1} < a_{n+1} < b_n$  for every  $n$ . We have that  $w$  is of type 1 in each  $[a_n, b_n]$ , since  $w \in L^\infty([a_n, b_n])$  and  $1/w_k \in L^1([a_n, b_n])$  for every  $n$ . If we choose  $I_n := [a_{n+1}, b_n]$ , then we can apply Theorems 2.3 and 2.7.  $\square$

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