A SIMPLE CHARACTERIZATION OF WEIGHTED SOBOLEV SPACES WITH BOUNDED MULTIPLICATION OPERATOR

JOSE M. RODRIGUEZ(1)

Departamento de Matemáticas
Escuela Politécnica Superior
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés (Madrid), SPAIN
Telephone number: 34 91 6249098
Fax number: 34 91 6249151
email: jomaro@math.uc3m.es

Abstract. In this paper we give a simple characterization of weighted Sobolev spaces (with piecewise monotonous weights) such that the multiplication operator is bounded: it is bounded if and only if the support of μ_0 is large enough. We also prove some basic properties of the appropriate weighted Sobolev spaces. To have bounded multiplication operator has important consequences in Approximation Theory: it implies the uniform bound of the zeros of the corresponding Sobolev orthogonal polynomials, and this fact allows to obtain the asymptotic behavior of Sobolev orthogonal polynomials.

Key words and phrases: Multiplication operator; location of zeros; Sobolev orthogonal polynomials; weight; Sobolev spaces; weighted Sobolev spaces.

1. Introduction.

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics. In the classical books [11], [13], we can find the point of view of Partial Differential Equations (see also [26] and [7]). We are interested in the relationship between this topic and Approximation Theory in general, and Sobolev Orthogonal Polynomials in particular.

Sobolev orthogonal polynomials are becoming more and more interesting in recent years. In particular, in [8] and [9], the authors showed that the expansions with Sobolev orthogonal polynomials can avoid the Gibbs phenomenon which appears with classical orthogonal series in L^2.

In [20], [21], [22], [23], [24] and [25] the authors solved the following specific problems:

1) Find hypotheses on general measures $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathbb{R}$, as general as possible, so that we can define a Sobolev space $W^{k,p}(\mu)$ whose elements are functions. These measures are called p-admissible.

2) If a Sobolev norm with general measures $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathbb{R}$ is finite for any polynomial, what is the completion, $\mathbb{P}^{k,p}(\mu)$, of the space of polynomials with respect to the norm in $W^{k,p}(\mu)$? This problem has been studied previously in some particular cases (see e.g. [4], [3], [5]).

We think that this definition of weighted Sobolev space $W^{k,p}(\mu)$ with p-admissible measures is the best context in order to develop our work. However, the definition of these spaces is large and technical, and we have chosen in this work a definition of weighted Sobolev space inspired in the paper [12] by Kufner and

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Opic. Our definition generalizes the Kufner-Opic's definition, keeping its simplicity and obtaining a wide enough measure type as to include the usual examples in applications.

Our definition makes easy the reading of the paper to those people mainly interested in Sobolev orthogonal polynomials. We think that this is a good choice although we must pay with some loss of generality.

One of the central problems in the theory of Sobolev orthogonal polynomials is to determine its asymptotic behavior. In [14] the authors show how to obtain the n-th root asymptotic of Sobolev orthogonal polynomials if the zeros of these polynomials are contained in a compact set of the complex plane. Although the uniform bound of the zeros of orthogonal polynomials holds for every measure with compact support in the case without derivatives ($k = 0$), it is an open problem to bound the zeros of Sobolev orthogonal polynomials. The boundedness of the zeros is a consequence of the boundedness of the multiplication operator $\mathcal{M}f(x) = x\,f(x)$ in the corresponding space $\mathbb{P}^{k,2}(\mu)$: in fact, the zeros of the Sobolev orthogonal polynomials are contained in the disk $\{z : |z| \le \|\mathcal{M}\|\}$ (see [15]).

In [21], [23] and [1], there are some answers to the question stated in [14] about some conditions for $\mathcal{M}$ to be bounded.

The main aim of this paper is to find conditions (which should be very easy to check in practical cases) implying the boundedness of these zeros, when the measures are supported in the real line. In particular, Theorem 4.3 (the main result of this paper) states the following characterization: If $d\mu_j = w_j dx$ and w_j is piecewise monotonous for $1 \le j \le k$, then $\mathcal{M}$ is bounded if and only if the support of μ_0 is big enough (see the precise statement of Theorem 4.3). The hypothesis about the monotony of w_j is a weak one, since it is verified in almost every example (for instance, every Jacobi weights hold it). In order to work with these Sobolev spaces we need to develop the theory of such spaces: its completeness (see Theorem 3.1) and a strong version of the continuity of the evaluation operator (see Theorem 2.1), which can be viewed as an embeding theorem in weighted Sobolev spaces.

The outline of the paper is as follows. In Section 2 we introduce the weighted Sobolev spaces and prove some basic facts about them, which will be useful tools. In Section 3 we prove the completeness of the Sobolev spaces. After developing the basic theory of the weighted Sobolev spaces, Section 4 contains the results on the multiplication operator. There are some examples in Section 5.

Now we introduce the notation we use.

Notation. If A is a Borel set in $\mathbb{R}$, χ_A, $|A|$, $\sharp A$ and $\overline{A}$ denote, respectively, the characteristic function, the Lebesgue measure, the cardinal and the closure of A. By $f^{(j)}$ we mean the j-th distributional derivative of f. $\mathbb{P}$ denotes the set of polynomials and $\mathbb{P}_n$ the set of polynomials of degree least or equal than n. $\|\cdot\|_{L^p(A)}$ will denote the usual L^p-norm (without weights) on A. We say that an n-dimensional vector satisfies a one-dimensional property if each coordinate satisfies this property.

2. Background and previous results on Sobolev spaces.

The main concepts that we need to understand the statement of our results are contained in the following definitions.

Definition 2.1. *Given $1 \le p < \infty$ and a set A which is a union of intervals, we say that a weight w in A belongs to $B_p(A)$ if $w^{-1} \in L^{1/(p-1)}_{loc}(A)$.*

It is possible to construct a similar theory with $p = \infty$. We refer to [1], [17], [18] and [19] for the case $p = \infty$.

$B_p(\mathbb{R})$ contains, as a very particular case, the classical $A_p(\mathbb{R})$ weights appearing in Harmonic Analysis (see [16] or [6]). The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$, and $A_p(\mathbb{R}^n)$ $(1 < p < \infty)$ have been used in other definitions of weighted Sobolev spaces on $\mathbb{R}^n$ in [12] and [10] respectively.

In [12], Kufner and Opic define the following sets:

Definition 2.2. *Let us consider $1 \le p < \infty$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $\mathbb{R}$. For $0 \le j \le k$ we define the open set*

$$\Omega_j := \big\{ x \in \mathbb{R} \,:\, \exists \text{ an open neighbourhood } V \text{ of } x \text{ with } w_j \in B_p(V) \big\} ,$$

where $w_j = d\mu_j/dx$.

Notice that we always have $w_j \in B_p(\Omega_j)$ for any $0 \le j \le k$. In fact, Ω_j is the largest open set U with $w_j \in B_p(U)$. It is easy to check that if $f^{(j)} \in L^p(w_j)$ with $0 \le j \le k$, then $f^{(j)} \in L^1_{loc}(\Omega_j)$:

Given any compact interval $I \subset \Omega_j$, by Hölder's inequality

$$\begin{aligned} \int_I |f^{(j)}| &= \big\| f^{(j)} w_j^{1/p} w_j^{-1/p} \big\|_{L^1(I)} \le \big\| f^{(j)} w_j^{1/p} \big\|_{L^p(I)} \big\| w_j^{-1/p} \big\|_{L^{p/(p-1)}(I)} \\ &= \big\| f^{(j)} \big\|_{L^p(I,w_j)} \big\| w_j^{-1} \big\|^{1/p}_{L^{1/(p-1)}(I)} < \infty . \end{aligned}$$

Therefore $f^{(j-1)} \in AC_{loc}(\Omega_j)$ if $1 \le j \le k$ ($f^{(j-1)}$ is locally absolutely continuous in Ω_j).

In fact, this argument proves the following:

Lemma 2.1. *Let us consider $1 \le p < \infty$ and a weight w_j. The convergence in $L^p(w_j)$ implies the convergence in $L^1_{loc}(\Omega_j)$. In fact,*

$$\int_I |f^{(j)}| \le \big\| f^{(j)} \big\|_{L^p(I,w_j)} \big\| w_j^{-1} \big\|^{1/p}_{L^{1/(p-1)}(I)},$$

for every $f^{(j)} \in L^1_{loc}(\Omega_j)$ and every compact interval $I \subset \Omega_j$.

Definition 2.3. *Given $1 \le p < \infty$, we say that a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ belongs to the class $\mathcal{S}_p$ if it is a measure in $\mathbb{R}$ verifying the following properties:*

(i) We can make the decomposition $d\mu_0 = d(\mu_0)_s + w_0 dx$ (by Radon-Nikodym's Theorem, we can make this decomposition if μ_0 is σ-finite).

(ii) $d\mu_j = w_j dx$ and $w_j = 0$ a.e. in $\mathbb{R} \setminus \Omega_j$ for $0 < j \le k$.

Remarks.

1. Hypothesis "$w_j = 0$ a.e. in $\mathbb{R} \setminus \Omega_j$ for $0 < j \le k$" is natural: if we do not require it, the corresponding weighted Sobolev space is not a Banach space (see [12]). Furthermore, it is not easy to construct a weight which does not satisfy this hypothesis.

2. We just consider vectorial measures in $\mathcal{S}_p$ in the definition of the Sobolev.

3. The class $\mathcal{S}_p$ depends on p since the sets Ω_j depend on p.

Definition 2.4. *Let us consider $1 \le p < \infty$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $\mathcal{S}_p$. We define the* Sobolev space $W^{k,p}_{ko}(\mu)$ *as the space of equivalence classes of*

$$\begin{aligned} V^{k,p}_{ko}(\mu) := \Big\{ f :\, \mathbb{R} \to \mathbb{C} \;/\; \|f\|_{W^{k,p}(\mu)} := \Big(\sum_{j=0}^{k} \big\| f^{(j)} \big\|^p_{L^p(\mu_j)} \Big)^{1/p} < \infty , \\ \text{and } f^{(j)} \in AC_{loc}(\Omega_{j+1} \cup \dots \cup \Omega_k) \text{ for } 0 \le j < k \Big\} , \end{aligned}$$

with respect to the seminorm $\| \cdot \|_{W^{k,p}(\mu)}$.

Let us notice that in [12], Kufner and Opic require the equalities $\Omega_0 = \Omega_1 = \dots = \Omega_k$ in their definition. Our definition is inspired in [12], is as simple, and allows to deal with a wider set of vectorial measures.

It is possible to define Sobolev spaces, which we call $W^{k,p}(\mu)$, for a wider class of measures (see e.g. [20], [21], [1]), but they need a big amount of technical background. For the sake of simplicity we have chosen the current definition in this paper. Since there is just a way to define the Sobolev norm, we use the notation $\| \cdot \|_{W^{k,p}(\mu)}$ instead of $\| \cdot \|_{W^{k,p}_{ko}(\mu)}$.

Now, we are going to develop the basic results about these weighted Sobolev spaces.

Since, for the sake of generality, we allow $\|\cdot\|_{W^{k,p}(\mu)}$ to be a seminorm, it is natural to introduce the following concept.

Definition 2.5. *Let us consider* $1 \le p < \infty$ *and* $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ *a vectorial measure in* $\mathcal{S}_p$. *Let us define the space* $\mathcal{K}_{ko}(\mu)$ *as*

$$\mathcal{K}_{ko}(\mu) := \Big\{ g : \Omega_1 \cup \cdots \cup \Omega_k \longrightarrow \mathbb{C} / \ g \in V_{ko}^{k,p}(\mu|_{\Omega_1 \cup \cdots \cup \Omega_k}),\ \|g\|_{W^{k,p}(\mu|_{\Omega_1 \cup \cdots \cup \Omega_k})} = 0 \Big\} .$$

$\mathcal{K}_{ko}(\mu)$ is the equivalence class of 0 in $W_{ko}^{k,p}(\mu|_{\Omega_1 \cup \cdots \cup \Omega_k})$. Therefore, $\|\cdot\|_{W^{k,p}(\mu|_{\Omega_1 \cup \cdots \cup \Omega_k})}$ is a norm if and only if $\mathcal{K}_{ko}(\mu) = 0$. This concept plays an important role in the study of the multiplication operator in Sobolev spaces (see Theorem 4.3 below).

Remark. Since the values of any $f \in V_{ko}^{k,p}(\mu)$ in two different connected components of $\Omega_1 \cup \cdots \cup \Omega_k$ are independent, it is direct to check that $\mathcal{K}_{ko}(\mu) = 0$ if and only if $\mathcal{K}_{ko}(\mu|_A) = 0$ for every connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$. Furthermore, if we consider the functions in $\mathcal{K}_{ko}(\mu|_A)$ defined as 0 in $\mathbb{R} \setminus A$, we have that

$$\mathcal{K}_{ko}(\mu) = \bigoplus_i \mathcal{K}_{ko}(\mu|_{A_i}),$$

where $\{A_i\}_i$ are the connected components of $\Omega_1 \cup \cdots \cup \Omega_k$.

Proposition 2.1. *Let us consider* $1 \le p < \infty$ *and a vectorial measure* $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ *in* $\mathcal{S}_p$. *If a connected component* A *of* $\Omega_1 \cup \cdots \cup \Omega_k$ *intersects* Ω_i *for some* $0 \le i \le k$, *then every function* $f \in \mathcal{K}_{ko}(\mu)$ *verifies* $f|_A \in \mathbb{P}_{i-1}$.

Remark. We use the convention $\mathbb{P}_{-1} = 0$.

Proof. Let us fix $0 \le j \le k$ and a function $f \in \mathcal{K}_{ko}(\mu)$. We prove first that $f^{(j)} = 0$ a.e. in $\Omega_0 \cup \cdots \cup \Omega_j$:

Let us consider a compact interval $I \subset \Omega_i$, with $0 \le i \le j$. By Lemma 2.1,

$$\int_I |f^{(i)}| \le \|f^{(i)}\|_{L^p(I,w_i)} \|w_i^{-1}\|_{L^{1/(p-1)}(I)}^{1/p} = 0 \cdot \|w_i^{-1}\|_{L^{1/(p-1)}(I)}^{1/p} = 0 .$$

Hence, $f^{(i)} = 0$ a.e. in Ω_i and $f^{(j)} = 0$ a.e. in Ω_i. Consequently, $f^{(j)} = 0$ a.e. in $\Omega_0 \cup \cdots \cup \Omega_j$.

Furthermore, the restriction of f to some connected component of Ω_i $(0 \le i \le j)$ belongs to $\mathbb{P}_{j-1}$ (recall that $f^{(i-1)} \in AC_{loc}(\Omega_i)$).

We prove now that if the restriction of f to some open interval $J \subseteq \Omega_m$ for some $0 \le m \le k$ belongs to $\mathbb{P}_{j-1}$, and H is an open interval $H \subseteq \Omega_n$ for some $0 \le n \le k$, with $J \cap H \ne \varnothing$, then $f|_H \in \mathbb{P}_{j-1}$ also:

Using the previous argument, we obtain that $f|_H \in \mathbb{P}_{n-1}$. Since, by hypothesis, $f|_J \in \mathbb{P}_{j-1}$, and J and H are open intervals with $J \cap H \ne \varnothing$, then $f|_{J \cup H} \in \mathbb{P}_{j-1}$ and $f|_H \in \mathbb{P}_{j-1}$.

If a connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$ intersects Ω_i for some $0 \le i \le k$, let us fix $x_0 \in \Omega_i \cap A$. Given $x \in A$, it is enough to prove that we can go from x_0 to x by crossing just a finite number of open intervals in some Ω_m, with $0 < m \le k$. (This also has sense for the case $i = 0$, since if $x_0 \in \Omega_0 \cap A$, then there exists $0 < m \le k$ with $x_0 \in \Omega_m$.)

Given $x \in A$, consider the compact interval $I_x \subseteq A$ with endpoints x and x_0. Since Ω_m is an open set for each m, it is a disjoint union of open intervals. Then A has an open covering of open intervals in some Ω_m, with $0 < m \le k$. Since I_x is a compact subset of A there exists a finite subcovering of open intervals in some Ω_m, with $0 < m \le k$. Then we can go from x_0 to x by crossing just a finite number of open intervals in some Ω_m, with $0 < m \le k$, and the proof is finished. ☐

Definition 2.6. *Given* $1 \le p < \infty$, *a vectorial measure* $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ *in* $\mathcal{S}_p$ *and a connected component* A *of* $\Omega_1 \cup \cdots \cup \Omega_k$, *the* defects *of* A *are*

$$d_0(A) := \mathrm{defect}_0(A) := \min \big\{ 0 \le j \le k : \ \Omega_j \cap A \ne \varnothing \big\} ,$$

$$d_1(A) := \mathrm{defect}_1(A) := \min \big\{ 1 \le j \le k : \ \Omega_j \cap A \ne \varnothing \big\} .$$

In general, it is easy to compute $\mathcal{K}_{ko}(\mu)$, as show the following results.

PROPOSITION 2.2. *Let us consider $1 \le p < \infty$, a vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathcal{S}_p$, and a connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$. Then*

$$\mathcal{K}_{ko}(\mu|_A) = \Big\{ f \in \mathbb{P}_{d_0(A)-1} : \int_A |f|^p \, d\mu_0 = 0 \Big\} = \Big\{ f \in \mathbb{P}_{d_1(A)-1} : \int_A |f|^p \, d\mu_0 = 0 \Big\},$$

and

$$\dim \mathcal{K}_{ko}(\mu|_A) = \big(d_0(A) - \sharp \operatorname{supp}(\mu_0|_A) \big)_+ = \big(d_1(A) - \sharp \operatorname{supp}(\mu_0|_A) \big)_+ ,$$

where, as usual, $x_+ := \max\{x, 0\}$.

Proof. Let us fix a connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$. We prove first the results with $d_0(A)$.

By definition of $d_0(A)$, we have that $\Omega_{d_0(A)}$ intersects A and $\Omega_j = \varnothing$ for any $0 \le j < d_0(A)$ (if $d_0(A) > 0$). Then $\mu_j = 0$ for any $0 < j < d_0(A)$ (if $d_0(A) > 1$). By Proposition 2.1 we obtain $f|_A \in \mathbb{P}_{d_0(A)-1}$ (recall that $\mathbb{P}_{-1} = 0$), for every function $f \in \mathcal{K}_{ko}(\mu|_A)$.

We have that

$$\mathcal{K}_{ko}(\mu|_A) = \Big\{ f \in \mathbb{P}_{d_0(A)-1} : \int_A |f|^p \, d\mu_0 = 0 \Big\},$$

since $\mu_j = 0$ for any $0 < j < d_0(A)$, if $d_0(A) > 1$ (conditions $\int_A |f^{(j)}|^p \, d\mu_j = 0$ for $d_0(A) \le j \le k$ are not relevant since we know that $f|_A \in \mathbb{P}_{d_0(A)-1}$).

If $d_0(A) = 0$, there is nothing to prove, since $\mathcal{K}_{ko}(\mu|_A) \subseteq \mathbb{P}_{-1} = 0$ and $\sharp \operatorname{supp}(\mu_0|_A) \ge 0$.

If $d_0(A) > 0$, then we denote $r := \sharp \operatorname{supp}(\mu_0|_A)$.

If $r = \infty$, then $\mathcal{K}_{ko}(\mu|_A) = 0$ and $d_0(A) \le k$; hence, there is nothing to prove.

If $r < \infty$, then $\mu_0|_A = c_1 \delta_{x_1} + \cdots + c_r \delta_{x_r}$, with $c_i > 0$ and $x_i \in A$. Therefore, any $f \in \mathcal{K}_{ko}(\mu|_A)$ can be written as

$$f(x) = \alpha_{d_0(A)} x^{d_0(A)-1} + \cdots + \alpha_2 x + \alpha_1,$$

with the restrictions

$$0 = f(x_i) = \alpha_{d_0(A)} x_i^{d_0(A)-1} + \cdots + \alpha_2 x_i + \alpha_1, \qquad 1 \le i \le r.$$

This is a homogeneous linear system of r equations with the $d_0(A)$ unknowns $\alpha_{d_0(A)}, \dots, \alpha_2, \alpha_1$.

If $d_0(A) \le r < \infty$, then $\mathcal{K}_{ko}(\mu|_A) = 0$ and $(d_0(A) - r)_+ = 0$.

If $r < d_0(A)$, then the r equations are linearly independent and $\dim \mathcal{K}_{ko}(\mu|_A) = d_0(A) - r = (d_0(A) - r)_+$.

We prove now the results with $d_1(A)$.

If $\Omega_0 \cap A = \varnothing$, then $d_0(A) = d_1(A)$ and there is nothing to prove. If $\Omega_0 \cap A \ne \varnothing$, then $d_0(A) = 0 < d_1(A)$, but in this case $\sharp \operatorname{supp}(\mu_0|_A) = \infty$; consequently, $\mathcal{K}_{ko}(\mu|_A) = 0$ and $0 = \big(d_0(A) - \sharp \operatorname{supp}(\mu_0|_A) \big)_+ = \big(d_1(A) - \sharp \operatorname{supp}(\mu_0|_A) \big)_+$. □

COROLLARY 2.1. *Let us consider $1 \le p < \infty$ and a vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathcal{S}_p$. Then the following conditions are equivalents:*

(A) $\mathcal{K}_{ko}(\mu) = 0$.

(B) $\sharp \operatorname{supp}(\mu_0|_A) \ge d_0(A)$ for every connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$.

(C) $\sharp \operatorname{supp}(\mu_0|_A) \ge d_1(A)$ for every connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$.

Proof. By the Remark after Definition 2.5, $\mathcal{K}_{ko}(\mu) = 0$ if and only if $\mathcal{K}_{ko}(\mu|_A) = 0$ for every connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$. Then we just need to apply Proposition 2.2. □

We need two technical results from [20]:

Lemma A. ([20, Theorem 4.1]) *Let us consider $1 \le p < \infty$ and a measure μ_0 on $[a,b]$ such that* supp μ_0 *has at least k points. Let w_k be a weight in $B_p([a,b])$. Then there exists a positive constant c such that*

$$c \sum_{j=0}^{k-1} \|f^{(j)}\|_{L^\infty([a,b])} \le \|f\|_{L^p([a,b],\mu_0)} + \|f^{(k)}\|_{L^p([a,b],w_k)}, \qquad \text{for all } f \text{ with } f^{(k-1)} \in AC([a,b]).$$

Lemma B. ([20, Lemma 4.2]) *Let us suppose that $1 \le p < \infty$ and $w = (w_0, \dots, w_k)$ is a vectorial weight in $\mathcal{S}_p$. If I is a compact interval contained in $\Omega_{j+1} \cup \cdots \cup \Omega_k$ for some $0 \le j < k$, and $I \cap \Omega_0 \cup \cdots \cup \Omega_j \ne \varnothing$, then there exists a positive constant c such that*

$$c\,\|f^{(j)}\|_{L^\infty(I)} \le \|f\|_{W^{k,p}(w)}, \qquad \text{for every } \ f \in V_{ko}^{k,p}(w).$$

We introduce now a technical concept which we need in order to state Theorem 2.1.

Given $1 \le p < \infty$, a vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathcal{S}_p$, $0 \le j < k$ and $b \in \mathbb{R}$, we say that $b^- \in \Omega(j)$ (respectively, $b^+ \in \Omega(j)$), if there exist $\varepsilon > 0$ and $j < i \le k$ with $w_i \in B_p([b-\varepsilon, b])$ (respectively, $w_i \in B_p([b, b+\varepsilon])$). Also, we say that $b \in \Omega(j)$ if $b^- \in \Omega(j)$ and $b^+ \in \Omega(j)$.

Remark. Let us notice that if $b^- \in \Omega(j)$, then $b^- \in \Omega(i)$ for each $0 \le i \le j$. Hence, $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega(j)$. Furthermore, $\Omega(j) \subseteq \overline{\Omega_{j+1} \cup \cdots \cup \Omega_k}$, and if I is a compact interval contained in $\Omega(j)$, then $\Omega(j) \setminus \big(\Omega_{j+1} \cup \cdots \cup \Omega_k\big)$ is a finite set.

When we use this definition we think of a point $\{b\}$ as the union of two half-points $\{b^+\}$ and $\{b^-\}$. With this convention, each one of the following sets

$$(a,b) \cup (b,c) \cup \{b^+\} = (a,b) \cup [b^+, c) \ne (a,c)\,,$$
$$(a,b) \cup (b,c) \cup \{b^-\} = (a,b^-] \cup (b,c) \ne (a,c)\,,$$

has two connected components, and the set $(a,b) \cup (b,c) \cup \{b^-\} \cup \{b^+\} = (a,b) \cup (b,c) \cup \{b\} = (a,c)$ is connected.

We just use this convention in order to study the sets of absolute continuity of functions: we want that if $f \in AC(A)$ and $f \in AC(B)$, where A and B are union of intervals, then $f \in AC(A \cup B)$. With the usual definition of absolute continuity in an interval, if $f \in AC([a,b)) \cap AC([b,c])$ then we do not have $f \in AC([a,c])$. Of course, we have $f \in AC([a,c])$ if and only if $f \in AC([a,b^-]) \cap AC([b^+,c])$, where, by definition, $AC([b^+,c]) = AC([b,c])$ and $AC([a,b^-]) = AC([a,b])$. This idea can be formalized with a suitable topological space.

The following Theorem is a basic tool in the theory of weighted Sobolev spaces and, in particular, in the study of the multiplication operator (see the proofs of Theorems 3.1 and 4.3). It allows us to control the L^∞-norm (in appropriate sets) of a function and its derivatives in terms of its Sobolev norm (it is also a version of an embeding theorem in weighted Sobolev spaces). Furthermore, it is important by itself, since it answers to the following main question: when the evaluation functional of f (or $f^{(j)}$) in a point is a bounded operator in $W_{ko}^{k,p}(\mu)$?

Theorem 2.1. *Let us consider $1 \le p < \infty$ and a vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathcal{S}_p$. Let K_j be a finite union of compact intervals contained in $\Omega(j)$, for $0 \le j < k$. Then there exists a positive constant $c_1 = c_1(\mu, K_0, \dots, K_{k-1})$ such that for any $f \in V_{ko}^{k,p}(\mu)$ there exists $f_0 \in V_{ko}^{k,p}(\mu)$, independent of $K_0, \dots, K_{k-1}$ and c_1, with*

$$\|f_0 - f\|_{W^{k,p}(\mu)} = 0\,,$$
$$\sum_{j=0}^{k-1} \|f_0^{(j)}\|_{L^\infty(K_j)} + \sum_{j=0}^{k-1} \|f_0^{(j+1)}\|_{L^1(K_j)} \le c_1\,\|f_0\|_{W^{k,p}(\mu)} = c_1\,\|f\|_{W^{k,p}(\mu)}\,.$$

Furthermore, if g_0, f_0 are these representatives of g, f respectively, we have for the same constant c_1

$$\sum_{j=0}^{k-1} \|g_0^{(j)} - f_0^{(j)}\|_{L^\infty(K_j)} + \sum_{j=0}^{k-1} \|g_0^{(j+1)} - f_0^{(j+1)}\|_{L^1(K_j)} \le c_1 \, \|g - f\|_{W^{k,p}(\mu)} \, .$$

Besides, if $\mathcal{K}_{ko}(\mu) = 0$, then there exists a positive constant $c_2 = c_2(\mu, K_0, \dots, K_{k-1})$ such that

$$\sum_{j=0}^{k-1} \|f^{(j)}\|_{L^\infty(K_j)} + \sum_{j=0}^{k-1} \|f^{(j+1)}\|_{L^1(K_j)} \le c_2 \, \|f\|_{W^{k,p}(\mu)},$$

for every $f \in V_{ko}^{k,p}(\mu)$. In particular, if $b^- \in \Omega(j)$ or $b^+ \in \Omega(j)$, we have respectively,

$$|f^{(j)}(b^-)| \le c_2 \, \|f\|_{W^{k,p}(\mu)}, \qquad |f^{(j)}(b^+)| \le c_2 \, \|f\|_{W^{k,p}(\mu)},$$

for every $f \in V_{ko}^{k,p}(\mu)$.

Remark. If (a, b) is a connected component of $\Omega_1 \cup \dots \cup \Omega_k$, and $b^- \in \Omega(j)$, then there exists the limit $f^{(j)}(b^-)$ for every $f \in V_{ko}^{k,p}(\mu)$, since there exist $\varepsilon > 0$ and $j < i \le k$ with $f^{(i-1)} \in AC([b - \varepsilon, b])$. A similar remark holds for b^+.

Proof. By the Remark after Definition 2.5, without loss of generality we can assume that $\Omega_1 \cup \dots \cup \Omega_k$ is connected. We can assume also that $\Omega_k \ne \varnothing$, since in other case we can consider $\max\{1 \le j \le k : \Omega_j \ne \varnothing\}$ instead of k.

Since K_j is a finite union of compact intervals, without loss of generality we can assume that K_j is a single compact interval.

We prove first the inequalities concerning the L^∞-norm, with the following additional hypothesis: K_j is a compact interval contained in $\Omega_{j+1} \cup \dots \cup \Omega_k$ (which is a subset of $\Omega(j)$), for $0 \le j < k$.

Let us define $k_1 := d_1(\Omega_1 \cup \dots \cup \Omega_k)$. Let us fix $k_1 \le j < k$. Since $\Omega_1 \cup \dots \cup \Omega_k$ is connected, we can find a compact interval I_j such that $K_j \subseteq I_j \subset \Omega_{j+1} \cup \dots \cup \Omega_k$ and $I_j \cap (\Omega_0 \cup \dots \cup \Omega_j) \ne \varnothing$. By Lemma B we deduce that there exists a constant c_3 with

$$\sum_{j=k_1}^{k-1} \|f^{(j)}\|_{L^\infty(K_j)} \le \sum_{j=k_1}^{k-1} \|f^{(j)}\|_{L^\infty(I_j)} \le c_3 \|f\|_{W^{k,p}(\mu)} \, , \tag{1}$$

for every $f \in V_{ko}^{k,p}(\mu)$.

Since $k_1 > 0$, then $\Omega_{k_1} \cup \dots \cup \Omega_k = \Omega_1 \cup \dots \cup \Omega_k$. Furthermore, if $k_1 > 1$, then $\Omega_1 \cup \dots \cup \Omega_{k_1 - 1} = \varnothing$.

If $\mathcal{K}_{ko}(\mu) = 0$, then Corollary 2.1 gives $\sharp \operatorname{supp}\big(\mu_0|_{\Omega_1 \cup \dots \cup \Omega_k}\big) \ge k_1$. Without loss of generality we can assume that $K_0 = K_1 = \dots = K_{k_1 - 1} \subset \Omega_{k_1} \cup \dots \cup \Omega_k$ and $\sharp \operatorname{supp}\big(\mu_0|_{K_0}\big) \ge k_1$, since in other case we can enlarge K_j $(j = 0, 1, \dots, k_1 - 1)$.

Since K_0 is a compact interval contained in $\Omega_{k_1} \cup \big(\Omega_{k_1+1} \cup \dots \cup \Omega_k\big)$, then there are a finite number of compact intervals $J_1, \dots, J_{m_1} \subset \Omega_{k_1}$, $J^1, \dots, J^{m_2} \subset \Omega_{k_1+1} \cup \dots \cup \Omega_k$, with $J_1 \cup \dots \cup J_{m_1} \cup J^1 \cup \dots \cup J^{m_2} = K_0$. Let us define $w_{k_1}' := w_{k_1} + \chi_{\cup_i J^i}$; it belongs to $B_p(K_0)$ since $w_{k_1} \in B_p\big(J_1 \cup \dots \cup J_{m_1}\big)$ and $1 \in B_p\big(J^1 \cup \dots \cup J^{m_2}\big)$.

By (1) we have that

$$\|f^{(k_1)}\|_{L^p(w_{k_1}'|_{K_0})} \le c_4 \big(\|f^{(k_1)}\|_{L^p(w_{k_1}|_{K_0})} + \|f^{(k_1)}\|_{L^\infty(\cup_i J^i)}\big) \le c_5 \|f\|_{W^{k,p}(\mu)} \, , \tag{2}$$

for every $f \in V_{ko}^{k,p}(\mu)$, since $J^1, \dots, J^{m_2} \subset \Omega_{k_1+1} \cup \dots \cup \Omega_k$.

Since $\sharp \operatorname{supp}\big(\mu_0|_{K_0}\big) \ge k_1$ and $w_{k_1}' \in B_p(K_0)$, Lemma A and (2) give that

$$\sum_{j=0}^{k_1-1} \|f^{(j)}\|_{L^\infty(K_0)} \le c_6 \big(\|f\|_{L^p(\mu_0|_{K_0})} + \|f^{(k_1)}\|_{L^p(w_{k_1}'|_{K_0})}\big) \le c_7 \|f\|_{W^{k,p}(\mu)} \, ,$$

for every $f \in V_{ko}^{k,p}(\mu)$. Therefore,

$$\sum_{j=0}^{k-1} \|f^{(j)}\|_{L^\infty(K_j)} \le (c_3 + c_7)\|f\|_{W^{k,p}(\mu)} \,,$$

for every $f \in V_{ko}^{k,p}(\mu)$.

If $\mathcal{K}_{ko}(\mu) \neq 0$, let us define $r := \sharp\,\mathrm{supp}\big(\mu_0|_{\Omega_1\cup\cdots\cup\Omega_k}\big)$. By Proposition 2.2, $\dim \mathcal{K}_{ko}(\mu) = k_1 - r > 0$. Then without loss of generality we can assume that $K_0 = K_1 = \cdots = K_{k_1-1} \subset \Omega_{k_1} \cup \cdots \cup \Omega_k$ and $\mathrm{supp}\big(\mu_0|_{\Omega_1\cup\cdots\cup\Omega_k}\big) \subset K_0$.

Then $\mu_0|_A = b_1\delta_{x_1} + \cdots + b_r\delta_{x_r}$, with $b_i > 0$ and $x_i \in \Omega_1 \cup \cdots \cup \Omega_k$. By Proposition 2.2, any $f \in \mathcal{K}_{ko}(\mu) = 0$ can be written as

$$f(x) = \alpha_{k_1} x^{k_1-1} + \cdots + \alpha_2 x + \alpha_2 \,,$$

with the restrictions

$$0 = f(x_i) = \alpha_{k_1} x_i^{k_1-1} + \cdots + \alpha_2 x_i + \alpha_1 \,, \qquad 1 \le i \le r \,.$$

This is a homogeneous linear system of r equations with the k_1 unknowns $\alpha_{k_1}, \dots, \alpha_2, \alpha_1$, and $\dim \mathcal{K}_{ko}(\mu) = k_1 - r$. Let us choose points $y_1, \dots, y_{k_1-r} \in \Omega_1 \cup \cdots \cup \Omega_k$ with $x_i \neq y_j$ for every i, j.

Fix $f \in V_{ko}^{k,p}(\mu)$. We define $q_f = 0$ in $\mathbb{R} \setminus \big(\Omega_1 \cup \cdots \cup \Omega_k\big)$ and $q_f|_{\Omega_1\cup\cdots\cup\Omega_k} \in \mathbb{P}_{k_1-1} \cap \mathcal{K}_{ko}(\mu)$ as the unique polynomial in $\mathbb{P}_{k_1-1}$ verifying $q_f(x_i) = 0$ for every $1 \le i \le r$ and $q_f(y_i) = f(y_i)$ for every $1 \le i \le k_1 - r$.

The function $f_0 := f - q_f$ satisfies $\|f_0 - f\|_{W^{k,p}(\mu)} = \|q_f\|_{W^{k,p}(\mu)} = 0$. If we define a vectorial measure μ^* in $\mathcal{S}_p$ by $\mu_0^* = \mu_0 + \delta_{y_1} + \cdots + \delta_{y_{k_1-r}}$, and $w_j^* = w_j$ for $1 \le j \le k$, then $\mathcal{K}_{ko}(\mu^*) = 0$, by Proposition 2.2, since $\sharp\,\mathrm{supp}\big(\mu_0^*|_{\Omega_1\cup\cdots\cup\Omega_k}\big) = k_1$. Consequently,

$$\sum_{j=0}^{k-1} \|f_0^{(j)}\|_{L^\infty(K_j)} \le c_8\|f_0\|_{W^{k,p}(\mu^*)} = c_8\|f_0\|_{W^{k,p}(\mu)} = c_8\|f\|_{W^{k,p}(\mu)} \,,$$

since $f_0(y_i) = 0$ for every $1 \le i \le k_1 - r$.

We have the same inequality for $g_0 - f_0$ instead of f_0, since $q_{g-f} = q_g - q_f$.

Then, we have proved the inequalities concerning the L^∞-norm, if K_j is a finite union of compact intervals contained in $\Omega_{j+1} \cup \cdots \cup \Omega_k$, for $0 \le j < k$. We finish now the proof just in the case $\mathcal{K}_{ko}(\mu) = 0$, since the other case is similar (using the same measure μ^* and the function $f_0 = f - q_f$).

By the Remark after Lemma B, we have that if $K_j \subseteq \Omega(j)$, then $K_j \setminus \big(\Omega_{j+1} \cup \cdots \cup \Omega_k\big)$ is a finite set. Hence, without loss of generality we can assume that $K_j = [a,b] \subseteq \Omega(j)$, with $(a,b) \subseteq \Omega_{j+1} \cup \cdots \cup \Omega_k$.

In order to finish the proof of the L^∞-inequalities, it is enough to show that if $a \notin \Omega_{j+1} \cup \cdots \cup \Omega_k$, then

$$\|f^{(j)}\|_{L^\infty([a,a+\varepsilon])} \le c\,\|f\|_{W^{k,p}(\mu)} \,,$$

for every $f \in V_{ko}^{k,p}(\mu)$, since the case $b \notin \Omega_{j+1} \cup \cdots \cup \Omega_k$ is similar.

Since $a^+ \in \Omega(j)$, there exist $\varepsilon > 0$ and $j < i \le k$ with $w_i \in B_p([a, a+2\varepsilon])$ (then $(a, a+2\varepsilon) \subseteq \Omega_i$). Therefore,

$$f^{(j)}(x) = f^{(j)}(a+\varepsilon) + f^{(j+1)}(a+\varepsilon)(x-a-\varepsilon) + \cdots + f^{(i-1)}(a+\varepsilon)\frac{(x-a-\varepsilon)^{i-j-1}}{(i-j-1)!} + \int_{a+\varepsilon}^{x} f^{(i)}(t)\,\frac{(x-t)^{i-j-1}}{(i-j-1)!}\,dt \,,$$

for every $x \in [a, a+2\varepsilon]$. Lemma 2.1 gives

$$\begin{aligned}\Big|\int_{a+\varepsilon}^{x} f^{(i)}(t)\,\frac{(x-t)^{i-j-1}}{(i-j-1)!}\,dt\Big| &\le c_9\big\|f^{(i)}\big\|_{L^1([a,a+2\varepsilon])} \le c_9\big\|f^{(i)}\big\|_{L^p([a,a+2\varepsilon],w_i)}\big\|w_i^{-1}\big\|_{L^{1/(p-1)}([a,a+2\varepsilon])}^{1/p} \\ &\le c_{10}\|f\|_{W^{k,p}(\mu)} \,,\end{aligned}$$

and then

$$\begin{aligned}\big|f^{(j)}(x)\big| &\le \big|f^{(j)}(a+\varepsilon)\big| + \varepsilon\,\big|f^{(j+1)}(a+\varepsilon)\big| + \cdots + \frac{\varepsilon^{i-j-1}}{(i-j-1)!}\,\big|f^{(i-1)}(a+\varepsilon)\big| + c_{10}\|f\|_{W^{k,p}(\mu)} \\ &\le c_{11}\sum_{m=j}^{i-1}\big|f^{(m)}(a+\varepsilon)\big| + c_{10}\big\|f\big\|_{W^{k,p}(\mu)}\,.\end{aligned}$$

Since $a+\varepsilon\in\Omega_i$, then $a+\varepsilon\in\Omega_{m+1}\cup\cdots\cup\Omega_k$ for every $m<i$, and the proved part of this Theorem gives

$$\big\|f^{(j)}\big\|_{L^\infty([a,a+2\varepsilon])} \le c_{11}\sum_{m=j}^{i-1}\big|f^{(m)}(a+\varepsilon)\big| + c_{10}\big\|f\big\|_{W^{k,p}(\mu)} \le c_{12}\big\|f\big\|_{W^{k,p}(\mu)}\,.$$

We prove now the L^1-inequalities. For each $0\le j<k$, we can write $K_j = K_j^{j+1}\cup\cdots\cup K_j^k$, where K_j^i is a finite union of compact intervals with $w_i\in B_p(K_j^i)$. Let us define $w'_{j+1} := w_{j+1} + \chi_{K_j^{j+2}\cup\cdots\cup K_j^k}$; it belongs to $B_p(K_j)$ since $w_{j+1}\in B_p\big(K_j^{j+1}\big)$ and $1\in B_p\big(K_j^{j+2}\cup\cdots\cup K_j^k\big)$. By Lemma 2.1,

$$\big\|f^{(j+1)}\big\|_{L^1(K_j)} \le c_{13}\big\|f^{(j+1)}\big\|_{L^p(K_j,w'_{j+1})} \le c_{14}\big\|f^{(j+1)}\big\|_{L^p(w_{j+1})} + c_{14}\big\|f^{(j+1)}\big\|_{L^\infty(K_j^{j+2}\cup\cdots\cup K_j^k)}\,.$$

Since $K_j^{j+2}\cup\cdots\cup K_j^k\subseteq\Omega(j+1)$, the proved part of this Theorem gives

$$\big\|f^{(j+1)}\big\|_{L^\infty(K_j^{j+2}\cup\cdots\cup K_j^k)} \le c_{15}\big\|f\big\|_{W^{k,p}(\mu)}\,,$$

and consequently

$$\big\|f^{(j+1)}\big\|_{L^1(K_j)} \le c_{16}\big\|f\big\|_{W^{k,p}(\mu)}\,,$$

for every $f\in V_{ko}^{k,p}(\mu)$. □

3. Completeness of the Sobolev space.

The following Theorem is a central fact in the theory of Sobolev spaces.

Theorem 3.1. *Given $1\le p<\infty$ and a vectorial measure $\mu=(\mu_0,\dots,\mu_k)$ in $\mathcal{S}_p$, the Sobolev space $W_{ko}^{k,p}(\mu)$ is a Banach space.*

Proof. Given a Cauchy sequence $\{f_n\}_n$ in $W_{ko}^{k,p}(\mu)$, for each $0\le j\le k$, $\{f_n^{(j)}\}_n$ is a Cauchy sequence in $L^p(\mu_j)$, and then $\{f_n^{(j)}\}_n$ converges to some g_j in $L^p(\mu_j)$.

First of all, let us show that g_j can be extended to a function in $C(\Omega(j))$ (if $0\le j<k$) and in $L^1_{loc}(\Omega(j-1))$ (if $0<j\le k$).

If $0\le j<k$, let us consider any compact interval $K\subseteq\Omega(j)$. Theorem 2.1 gives that there exists a representative (independent of K) of the class of $f_n\in W_{ko}^{k,p}(\mu)$ (which we also denote by f_n) and a positive constant c such that for every $n,m\in\mathbb{N}$

$$\|f_n^{(j)} - f_m^{(j)}\|_{L^\infty(K)} \le c\sum_{i=0}^{k}\|f_n^{(i)} - f_m^{(i)}\|_{L^p(\mu_i)}\,.$$

Then $\{f_n^{(j)}\}_n$ is a Cauchy sequence in $\big(C(K),\|\cdot\|_{L^\infty(K)}\big)$, and there exists a function $h_j\in C(K)$ such that $\{f_n^{(j)}\}_n$ converges to h_j in $L^\infty(K)$.

Consequently,

$$\|f_n^{(j)} - h_j\|_{L^\infty(K)} \le c\sum_{i=0}^{k}\|f_n^{(i)} - g_i\|_{L^p(\mu_i)}\,.$$

Since we can take as K any compact interval contained in $\Omega(j)$, we obtain that the function h_j can be extended to $\Omega(j)$ and we have in fact $h_j \in C(\Omega(j))$. It is obvious that $g_j = h_j$ in $\Omega(j)$ (except for at most a set of zero μ_j-measure), since $f_n^{(j)}$ converges to g_j in the norm of $L^p(\mu_j)$ and to h_j uniformly on each compact interval $K \subseteq \Omega(j)$. Therefore, without loss of generality we can assume that $g_j \in C(\Omega(j))$.

If $0 < j \le k$, let us consider any compact interval $J \subseteq \Omega(j-1)$. Now Theorem 2.1 gives

$$\|f_n^{(j)} - f_m^{(j)}\|_{L^1(J)} \le c \sum_{i=0}^{k} \|f_n^{(i)} - f_m^{(i)}\|_{L^p(\mu_i)} .$$

Then $\{f_n^{(j)}\}_n$ is a Cauchy sequence in $L^1(J)$, and there exists a function $u_j \in L^1(J)$ such that $\{f_n^{(j)}\}_n$ converges to u_j in $L^1(J)$.

Consequently,

$$\|f_n^{(j)} - u_j\|_{L^1(J)} \le c \sum_{i=0}^{k} \|f_n^{(i)} - g_i\|_{L^p(\mu_i)} .$$

Since we can take as J any compact interval contained in $\Omega(j-1)$, we obtain that the function u_j can be extended to $\Omega(j-1)$ and we have in fact $u_j \in L^1_{loc}(\Omega(j-1))$. It is obvious that $g_j = u_j$ in $\Omega(j)$ (except for at most a set of zero Lebesgue measure), since $f_n^{(j)}$ converges to u_j in $L^1_{loc}(\Omega(j)) \subseteq L^1_{loc}(\Omega(j-1))$ and to g_j locally uniformly in $\Omega(j)$. We just need to show $u_j = g_j$ in $\Omega_j \setminus \Omega(j)$ (recall that by hypothesis $w_j = 0$ a.e. in $\mathbb{R} \setminus \Omega_j$), but this is immediate since the convergence in $L^p(w_j)$ implies the convergence in $L^1_{loc}(\Omega_j)$ (see Lemma 2.1). Therefore, $g_j \in L^1_{loc}(\Omega(j-1))$.

In fact, we have seen that $\{f_n^{(j)}\}$ converges to g_j in $L^\infty_{loc}(\Omega(j))$ (if $0 \le j < k$) and in $L^1_{loc}(\Omega(j-1))$ (if $0 < j \le k$).

Let us see now that $g_j' = g_{j+1}$ in the interior of $\Omega(j)$ for $0 \le j < k$. Let us consider a connected component I of $int(\Omega(j))$. Given $\varphi \in C_c^\infty(I)$, let us consider the convex hull K of supp φ. We have that K is a compact interval contained in $I \subseteq \Omega(j)$. The uniform convergence of $\{f_n^{(j)}\}$ in K and the L^1 convergence of $\{f_n^{(j+1)}\}$ in K gives that

$$\int_K \varphi' \, g_j = \lim_{n\to\infty} \int_K \varphi' \, f_n^{(j)} = - \lim_{n\to\infty} \int_K \varphi \, f_n^{(j+1)} = - \int_K \varphi \, g_{j+1} .$$

Consequently, $g_j' = g_{j+1}$ in $int(\Omega(j))$. Then, $g_{j+1} = g_0^{(j+1)}$ in $int(\Omega(j))$ and $g_0^{(j)} \in AC_{loc}(int(\Omega(j)))$ for $0 \le j < k$. In particular, $g_0^{(j)} \in AC_{loc}(\Omega_{j+1} \cup \cdots \cup \Omega_k)$, and $g_0 \in V_{ko}^{k,p}(\mu)$. Consequently, $\{f_n\}_n$ converges to g_0 in $W_{ko}^{k,p}(\mu)$. □

4. Results on the multiplication operator.

In order to clarify the proof of Theorem 4.3 (the main result of this section), we have proved some technical results on weighted Sobolev spaces in the previous sections: Propositions 2.1 and 2.2, Corollary 2.1, and Theorems 2.1 and 3.1. We also need to prove two more previous results: Theorem 4.2 and Lemma 4.2.

We begin with some previous concepts.

Recall that when every polynomial has finite $W^{k,p}(\mu)$-norm, we denote by $\mathbb{P}^{k,p}(\mu)$ the completion of $\mathbb{P}$ with that norm. Since our aim is to bound the multiplication operator in $\mathbb{P}^{k,p}(\mu)$, in this section we just consider measures such that every polynomial has finite Sobolev norm. Hence, for any $0 \le j \le k$,

$$\mu_j(\mathbb{R})^{1/p} = \big\|1\big\|_{L^p(\mu_j)} \le \big\|x^j/j!\big\|_{W^{k,p}(\mu)} < \infty ,$$

and consequently, μ is finite.

M. Castro and A. Durán [2] proved that if the multiplication operator is bounded in $\mathbb{P}^{k,p}(\mu)$ then the support of μ is compact. Then, we just need to consider finite vectorial measures with compact support.

First of all, some remarks about the definition of the multiplication operator. We start with a definition which has sense for arbitrary vectorial measures (they do not need to belong to $\mathcal{S}_p$).

Definition 4.1. *If $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ is a vectorial measure in $\mathbb{R}$, we say that the multiplication operator is* well defined in $\mathbb{P}^{k,p}(\mu)$ *if given any sequence $\{s_n\}$ of polynomials converging to 0 in the $W^{k,p}(\mu)$-norm, then $\{xs_n\}$ also converges to 0 in the $W^{k,p}(\mu)$-norm. In this case, if $\{q_n\} \in \mathbb{P}^{k,p}(\mu)$, we define $\mathcal{M}(\{q_n\}) := \{xq_n\}$. If we choose another Cauchy sequence $\{r_n\}$ representing the same element in $\mathbb{P}^{k,p}(\mu)$ (i.e. $\{q_n - r_n\}$ converges to 0 in the $W^{k,p}(\mu)$-norm), then $\{xq_n\}$ and $\{xr_n\}$ represent the same element in $\mathbb{P}^{k,p}(\mu)$ (since $\{x(q_n - r_n)\}$ converges to 0 in the $W^{k,p}(\mu)$-norm).*

We can also think of another definition which is natural as the previous one (if $\mu \in \mathcal{S}_p$):

Definition 4.2. *If $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ is a vectorial measure in $\mathcal{S}_p$, we say that the multiplication operator is* well defined in $W^{k,p}_{ko}(\mu)$ *if given any function $h \in V^{k,p}_{ko}(\mu)$ with $\|h\|_{W^{k,p}(\mu)} = 0$, we have $\|xh\|_{W^{k,p}(\mu)} = 0$. In this case, if $[f]$ is an equivalence class in $W^{k,p}_{ko}(\mu)$, we define $\mathcal{M}([f]) := [xf]$. If we choose another representative g of $[f]$ (i.e. $\|f - g\|_{W^{k,p}(\mu)} = 0$) we have $[xf] = [xg]$, since $\|x(f - g)\|_{W^{k,p}(\mu)} = 0$.*

Although both definitions are natural, it is possible for a vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k) \in \mathcal{S}_p$ with $W^{k,p}_{ko}(\mu) = \mathbb{P}^{k,p}(\mu)$, that $\mathcal{M}$ is well defined in $W^{k,p}_{ko}(\mu)$ and not well defined in $\mathbb{P}^{k,p}(\mu)$ (see the example after Theorem 4.2). The following elementary lemma gives an unexpected characterization of the spaces $\mathbb{P}^{k,p}(\mu)$ with $\mathcal{M}$ well defined in them.

Lemma 4.1. ([1, Lemma 8.1]) *Let us consider $1 \le p < \infty$ and $\mu = (\mu_0, \dots, \mu_k)$ a vectorial measure in $\mathbb{R}$. The following facts are equivalent:*
1. *The multiplication operator is well defined in $\mathbb{P}^{k,p}(\mu)$.*
2. *The multiplication operator is bounded in $\mathbb{P}^{k,p}(\mu)$.*
3. *There exists a positive constant c such that*

$$\|xq\|_{W^{k,p}(\mu)} \le c\, \|q\|_{W^{k,p}(\mu)}\,, \qquad \text{for every } q \in \mathbb{P}\,.$$

Definition 4.3. *A vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $\mathbb{R}$ is* extended sequentially dominated *(and we write $\mu \in ESD$) if there exists a positive constant c such that $\mu_{j+1} \le c\, \mu_j$ for $0 \le j < k$.*

This kind of measures plays a main role in the study of the multiplication operator:

Theorem 4.1. ([1, Theorem 8.1]) *Let us consider $1 \le p < \infty$ and $\mu = (\mu_0, \dots, \mu_k)$ a finite vectorial measure in $\mathbb{R}$ with compact support. Then, the multiplication operator is bounded in $\mathbb{P}^{k,p}(\mu)$ if and only if there exists a vectorial measure $\mu' \in ESD$ such that the Sobolev norms in $W^{k,p}(\mu)$ and $W^{k,p}(\mu')$ are comparable on $\mathbb{P}$. Furthermore, we can choose $\mu' = (\mu'_0, \dots, \mu'_k)$ with $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$ for $0 \le j \le k$.*

Although this result characterizes the measures with $\mathcal{M}$ bounded, it is convenient to obtain more practical criteria in order to guarantee the boundedness of $\mathcal{M}$. This is the goal of Theorem 4.3.

Let us notice that the multiplication operator $\mathcal{M}$ is bounded in $W^{k,p}_{ko}(\mu)$ if and only if there exists a positive constant c such that

$$\|xf\|_{W^{k,p}(\mu_j)} \le c\, \|f\|_{W^{k,p}(\mu)}\,,$$

for every $f \in V^{k,p}_{ko}(\mu)$. Consequently, if $\mathcal{M}$ is bounded in $W^{k,p}_{ko}(\mu)$ and $\mathbb{P} \subseteq W^{k,p}_{ko}(\mu)$, then it is bounded in $\mathbb{P}^{k,p}(\mu)$, since $W^{k,p}_{ko}(\mu)$ is a complete space by Theorem 3.1.

The following result characterizes when $\mathcal{M}$ is a well defined operator in $W^{k,p}_{ko}(\mu)$.

Theorem 4.2. *Let us consider $1 \le p < \infty$ and a vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in $\mathcal{S}_p$. Then the multiplication operator $\mathcal{M}$ is well defined in $W^{k,p}_{ko}(\mu)$ if and only if $\mathcal{K}_{ko}(\mu) = 0$.*

Proof. Let us suppose first that $\mathcal{K}_{ko}(\mu)=0$ and let us consider $f\in V_{ko}^{k,p}(\mu)$ with $\|f\|_{W^{k,p}(\mu)}=0$. On the one hand, $f\in\mathcal{K}_{ko}(\mu)=0$ implies that $f|_{\Omega_1\cup\cdots\cup\Omega_k}\equiv 0$, and so $\|xf\|_{W^{k,p}(\mu|_{\Omega_1\cup\cdots\cup\Omega_k})}=0$. On the other hand, we also have $\|f\|_{L^p(\mu_0)}=0$, and so $f(x)=0$ for μ_0-almost every $x\in\mathbb{R}$. Then $xf(x)=0$ for μ_0-almost every $x\in\mathbb{R}$ and $\|xf\|_{L^p(\mu_0)}=0$. Let us observe that μ_j is concentrated in $\Omega_j\subseteq\Omega_1\cup\cdots\cup\Omega_k$ for $1\le j\le k$. We deduce from these facts that

$$\|xf\|_{W^{k,p}(\mu)}^p\le\|xf\|_{L^p(\mu_0)}^p+\|xf\|_{W^{k,p}(\mu|_{\Omega_1\cup\cdots\cup\Omega_k})}^p=0\,,$$

and therefore the multiplication operator is well defined in $W_{ko}^{k,p}(\mu)$.

On the converse, let us suppose that $\mathcal{K}_{ko}(\mu)\ne 0$, and let us consider $f\in\mathcal{K}_{ko}(\mu)\setminus\{0\}$; then $f\in V_{ko}^{k,p}(\mu)$ and $\|f\|_{W^{k,p}(\mu|_{\Omega_1\cup\cdots\cup\Omega_k})}=0$, but f is not identically zero in $\Omega_1\cup\cdots\cup\Omega_k$. We know that there exists an interval $I_0\subseteq\Omega_1\cup\cdots\cup\Omega_k$ such that $f|_{I_0}\ne 0$ (since $f\in AC_{loc}(\Omega_1\cup\cdots\cup\Omega_k)$), and therefore there is another interval $I_1\subseteq I_0$ such that $I_1\subseteq\Omega_i$ for some $1\le i\le k$ and $f|_{I_1}\ne 0$. If g belongs to $\mathcal{K}_{ko}(\mu)$, Proposition 2.1 gives $g|_{I_1}\in P_{i-1}$. If $\deg q$ denotes the degree of the polynomial q, let us choose now $h\in\mathcal{K}_{ko}(\mu)$ such that $\deg h|_{I_1}\ge\deg g|_{I_1}$ for all $g\in\mathcal{K}_{ko}(\mu)$ (we have $\deg h|_{I_1}\ge 0$ since the function f is not identically zero in I_1); we define $h=0$ in $\mathbb{R}\setminus\big(\Omega_1\cup\cdots\cup\Omega_k\big)$, and then $\|h\|_{W^{k,p}(\mu)}=0$. Since $\deg xh|_{I_1}>\deg h|_{I_1}$, we deduce that $xh\notin\mathcal{K}_{ko}(\mu)$ and $\|xh\|_{W^{k,p}(\mu)}>0$; hence, $\mathcal{M}$ is not well defined in $W_{ko}^{k,p}(\mu)$. □

Remark. Let us notice that when $W_{ko}^{k,p}(\mu)$ and $\mathbb{P}^{k,p}(\mu)$ are the same, we have two different definitions of the well defined character of $\mathcal{M}$.

One can think that, in a similar way to Lemma 4.1, the multiplication operator $\mathcal{M}$ is well defined in $W_{ko}^{k,p}(\mu)$ if and only if it is bounded in $W_{ko}^{k,p}(\mu)$. However, this is not true, as shows the following:

Example. Let us consider the absolutely continuous finite vectorial measure $\mu=(\mu_0,\mu_1)$ given by $w_0(x):=\sum_{n\ge1}\frac{1}{n}\,\chi_{I_n}$ and $w_1(x):=\sum_{n\ge1}\chi_{I_n}$, where $I_n:=[2^{-2n-1},2^{-2n}]$. It is easy to see that $\mathcal{K}_{ko}(\mu)=0$; then $\mathcal{M}$ is well defined in $W_{ko}^{1,p}(\mu)$ by Theorem 4.2. We show now that $\mathcal{M}$ is not bounded in $W_{ko}^{1,p}(\mu)$: Let us consider $f_n\in C_c^\infty((2^{-2n-3/2},2^{-2n+1/2}))$ with $f_n=1$ in I_n; then $\|f_n\|_{W^{1,p}(\mu)}=\|1\|_{L^p(I_n,w_0)}=\big(\frac{1}{n}\,2^{-2n-1}\big)^{1/p}$ and $\|xf_n\|_{W^{1,p}(\mu)}\ge\|1\|_{L^p(I_n,w_1)}=\big(2^{-2n-1}\big)^{1/p}$. Consequently, $\|\mathcal{M}\|\ge n^{1/p}$ for every $n\ge1$, and $\mathcal{M}$ is not bounded in $W_{ko}^{1,p}(\mu)$. It is not difficult to prove that $W_{ko}^{1,p}(\mu)=\mathbb{P}^{1,p}(\mu)$, and then $\mathcal{M}$ is not bounded in $\mathbb{P}^{1,p}(\mu)$. Consequently, $\mathcal{M}$ is well defined in $W_{ko}^{1,p}(\mu)$ and it is not well defined in $\mathbb{P}^{1,p}(\mu)$ by Lemma 4.1.

Lemma 4.2. *Let us consider $1\le p<\infty$ and a vectorial measure $\mu=(\mu_0,\mu_1,\dots,\mu_k)$ in $\mathcal{S}_p$ with compact support. Then, the multiplication operator $\mathcal{M}$ is bounded in $W_{ko}^{k,p}(\mu)$ if and only if there exists a positive constant c such that*

$$\|f^{(j-1)}\|_{L^p(\mu_j)}\le c\,\|f\|_{W^{k,p}(\mu)}\,,$$

for every $1\le j\le k$ and $f\in V_{ko}^{k,p}(\mu)$.

Proof. If $\mathcal{M}$ is bounded in $W_{ko}^{k,p}(\mu)$, we have that

$$\|(xf)^{(j)}\|_{L^p(\mu_j)}\le\|\mathcal{M}\|\,\|f\|_{W^{k,p}(\mu)}\,,$$

for every $1\le j\le k$ and $f\in V_{ko}^{k,p}(\mu)$. Since

$$\|(xf)^{(j)}\|_{L^p(\mu_j)}=\|xf^{(j)}+jf^{(j-1)}\|_{L^p(\mu_j)}\ge\|f^{(j-1)}\|_{L^p(\mu_j)}-K\,\|f^{(j)}\|_{L^p(\mu_j)}\,,$$

with $K:=\max\{|x|:\,x\in\cup_{j=0}^k\operatorname{supp}\mu_j\}$, we have

$$\|f^{(j-1)}\|_{L^p(\mu_j)}\le K\,\|f^{(j)}\|_{L^p(\mu_j)}+\|\mathcal{M}\|\,\|f\|_{W^{k,p}(\mu)}\le(K+\|\mathcal{M}\|)\,\|f\|_{W^{k,p}(\mu)}\,,$$

for every $1\le j\le k$ and $f\in V_{ko}^{k,p}(\mu)$.

We now prove the converse implication. Notice that

$$\|(xf)^{(j)}\|_{L^p(\mu_j)}=\|xf^{(j)}+jf^{(j-1)}\|_{L^p(\mu_j)}\le j\,\|f^{(j-1)}\|_{L^p(\mu_j)}+K\,\|f^{(j)}\|_{L^p(\mu_j)}\,,$$

with K as before, for every $1 \le j \le k$ and $f \in V_{ko}^{k,p}(\mu)$. Then

$$\begin{aligned}\|(xf)^{(j)}\|_{L^p(\mu_j)}^p &\le 2^{p-1}\Big(j^p\|f^{(j-1)}\|_{L^p(\mu_j)}^p + K^p\|f^{(j)}\|_{L^p(\mu_j)}^p\Big)\\ &\le 2^{p-1}\Big(j^p c^p\|f\|_{W^{k,p}(\mu)}^p + K^p\|f^{(j)}\|_{L^p(\mu_j)}^p\Big),\end{aligned}$$

for every $0 \le j \le k$ and $f \in V_{ko}^{k,p}(\mu)$ (if $j=0$ the inequality is trivial). Consequently, since $\sum_{j=0}^k j^p \le k^{p+1}$,

$$\|xf\|_{W^{k,p}(\mu)}^p \le 2^{p-1}\Big(k^{p+1}c^p\|f\|_{W^{k,p}(\mu)}^p + K^p\|f\|_{W^{k,p}(\mu)}^p\Big)$$

and

$$\|xf\|_{W^{k,p}(\mu)} \le 2^{(p-1)/p}\big(k^{p+1}c^p + K^p\big)^{1/p}\|f\|_{W^{k,p}(\mu)},$$

for every $f \in V_{ko}^{k,p}(\mu)$. Hence, $\mathcal{M}$ is bounded in $W_{ko}^{k,p}(\mu)$. □

In order to state the main result of this section we need two definitions.

Definition 4.4. *A function u in a compact interval $[\alpha,\beta]$ is* piecewise monotonous *if there exist points $b_1 = \alpha < b_2 < \cdots < b_{m-1} < b_m = \beta$ such that u is a monotonous function in $[b_i, b_{i+1}]$ for each $1 \le i < m$.*

Definition 4.5. *We say that two functions u, v are* comparable *on the set $F \subseteq \mathbb{R}$ if there are positive constants c_1, c_2 such that $c_1 v(x) \le u(x) \le c_2 v(x)$ for almost every $x \in F$.*

Theorem 4.3. *Let us consider $1 \le p < \infty$ and a finite vectorial measure $\mu = (\mu_0, \mu_1, \dots, \mu_k)$ in a compact interval $[\alpha,\beta]$, such that $d\mu_j = w_j dx$ and w_j is comparable to a piecewise monotonous function for any $1 \le j \le k$. Then $\mu \in \mathcal{S}_p$, and the following conditions are equivalents:*
(A) The multiplication operator $\mathcal{M}$ is bounded in $W_{ko}^{k,p}(\mu)$.
(B) $\mathcal{K}_{ko}(\mu) = 0$.
(C) $\sharp\operatorname{supp}(\mu_0|_A) \ge d_0(A)$ for every connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$.
(D) $\sharp\operatorname{supp}(\mu_0|_A) \ge d_1(A)$ for every connected component A of $\Omega_1 \cup \cdots \cup \Omega_k$.

Remarks.
1. Let us notice that when Theorem 4.3 holds, it implies that $\mathcal{M}$ is bounded in $\mathbb{P}^{k,p}(\mu)$.
2. By monotonous we mean non-strictly monotonous; hence, it is possible to have $w_j = 0$ in some interval.
3. The partition in intervals can be different for each w_j.

Proof. Since μ_0 is finite, Radon-Nikodym's Theorem gives that $d\mu_0 = d(\mu_0)_s + w_0 dx$; then, in order to prove that $\mu \in \mathcal{S}_p$, it suffices to show that if w_j is comparable to a monotonous function u in $[a,b]$, then $u = 0$ a.e. in $[a,b] \setminus \Omega_j$. If $u = 0$ in $[a,b]$, then $w_j = 0$ a.e. in $[a,b]$, and there is nothing to prove. Then, we can assume that $u(x) > 0$ for some $x \in [a,b]$.

By symmetry, we can assume that u is a non-decreasing function in $[a,b]$. Let us define $a_0 := \inf\{x \in [a,b] : u(x) > 0\}$. Consequently, $u = 0$ in $[a, a_0)$, since u is a non-decreasing function.

If $a_0 = b$, then $u = 0$ in $[a,b)$ and $w = 0$ a.e. in $[a,b]$.

If $a_0 \in [a,b)$, then $(a_0,b) \subseteq \Omega_j$: Given any $0 < \varepsilon < b - a_0$, then $u(x) \ge u(a_0+\varepsilon) > 0$ for every $x \in [a_0+\varepsilon, b]$, and hence $\int_{a_0+\varepsilon}^b w_j^{-1/(p-1)} < \infty$ and $w_j \in B_p([a_0+\varepsilon, b])$ for every $0 < \varepsilon < b - a_0$. Consequently, $(a_0,b) \subseteq \Omega_j$. Since $u = 0$ in $[a,a_0)$, we deduce that $w_j = 0$ a.e. in $[a,b] \setminus \Omega_j$. Therefore, $\mu \in \mathcal{S}_p$.

We prove now the equivalence of the two first conditions. The other conditions are equivalent to (B) by Corollary 2.1.

If $\mathcal{M}$ is bounded in $W_{ko}^{k,p}(\mu)$, then it is well defined in $W_{ko}^{k,p}(\mu)$: if $\|f\|_{W^{k,p}(\mu)} = 0$, then $\|xf\|_{W^{k,p}(\mu)} = 0$, since $\|xf\|_{W^{k,p}(\mu)} \le \|\mathcal{M}\|\|f\|_{W^{k,p}(\mu)} = 0$. By Theorem 4.2 we deduce that $\mathcal{K}_{ko}(\mu) = 0$.

Let us assume now that $\mathcal{K}_{ko}(\mu) = 0$.

For each $1 \le j \le k$, there exist points $b_1^j = \alpha < b_2^j < \cdots < b_{m^j-1}^j < b_{m^j}^j = \beta$ such that w_j is comparable to a monotonous function in $[b_i^j, b_{i+1}^j]$ for each $1 \le i < m^j$. Splitting in two subintervals some intervals if it is necessary, without loss of generality we can assume also that in each interval $[b_i^j, b_{i+1}^j]$, we have either $w_j = 0$ a.e. or $w_j > 0$ a.e.

We consider the points $a_1 = \alpha < a_2 < \cdots < a_{n-1} < a_n = \beta$, which are the ordered points in the set $\{b_i^j\}_{1\le i<m^j,\,1\le j\le k}$. Consequently, for any fixed $1 \le i < n$ and $1 \le j \le k$, w_j is comparable to a monotonous function in $[a_i, a_{i+1}]$ and we have either $w_j = 0$ a.e. or $w_j > 0$ a.e. in $[a_i, a_{i+1}]$.

By Lemma 4.2, we just need to show that there exists a positive constant c such that

$$\|f^{(j-1)}\|_{L^p([a_i,a_{i+1}],w_j)} \le c\,\|f\|_{W^{k,p}(\mu)}\,,$$

for every $1 \le i < n$, $1 \le j \le k$ and $f \in V_{ko}^{k,p}(\mu)$.

Let us fix $1 \le i < n$.

If $w_j = 0$ a.e. in $[a_i, a_{i+1}]$ for some $1 \le j \le k$, then we have

$$\|f^{(j-1)}\|_{L^p([a_i,a_{i+1}],w_j)} = 0 \le c\,\|f\|_{W^{k,p}(\mu)}\,,$$

for every positive constant c.

Fix $1 \le j \le k$ with w_j comparable to a non-decreasing function in $[a_i, a_{i+1}]$ and $w_j > 0$ a.e. in $[a_i, a_{i+1}]$.

Without loss of generality we can assume that w_j is a non-decreasing function in $[a_i, a_{i+1}]$. Then, we have $w_j > 0$ in $(a_i, a_{i+1}]$.

Let us notice that, applying Minkowski's integral inequality, if $1 < p < \infty$,

$$\begin{aligned}
\Big\| \int_x^{a_{i+1}} f^{(j)} \Big\|_{L^p([a_i,a_{i+1}],w_j)} &= \Big(\int_{a_i}^{a_{i+1}} \Big| \int_{a_i}^{a_{i+1}} \chi_{[x,a_{i+1}]}(t)\, f^{(j)}(t)\,dt \Big|^p w_j(x)\,dx \Big)^{1/p} \\
&\le \int_{a_i}^{a_{i+1}} \Big(\int_{a_i}^{a_{i+1}} \chi_{[a_i,t]}(x)\, \big|f^{(j)}(t)\big|^p w_j(x)\,dx \Big)^{1/p} dt \\
&= \int_{a_i}^{a_{i+1}} \big|f^{(j)}(t)\big| \Big(\int_{a_i}^{t} w_j(x)\,dx \Big)^{1/p} dt \\
&\le \int_{a_i}^{a_{i+1}} \big|f^{(j)}(t)\big|\, w_j(t)^{1/p}\,(t-a_i)^{1/p}\,dt \\
&\le \Big(\int_{a_i}^{a_{i+1}} \big|f^{(j)}(t)\big|^p w_j(t)\,dt \Big)^{1/p} \Big(\int_{a_i}^{a_{i+1}} (t-a_i)^{1/(p-1)}\,dt \Big)^{(p-1)/p} \\
&= \Big(\frac{p-1}{p} \Big)^{(p-1)/p} (a_{i+1}-a_i)\, \big\|f^{(j)}\big\|_{L^p([a_i,a_{i+1}],w_j)}\,.
\end{aligned}$$

Since $F(x) = x^x \le 1$ for every $x \in (0,1]$, we obtain

$$\Big\| \int_x^{a_{i+1}} f^{(j)} \Big\|_{L^p([a_i,a_{i+1}],w_j)} \le (a_{i+1}-a_i)\, \big\|f^{(j)}\big\|_{L^p([a_i,a_{i+1}],w_j)}\,.$$

If $p = 1$, with a similar argument, we also obtain

$$\begin{aligned}
\Big\| \int_x^{a_{i+1}} f^{(j)} \Big\|_{L^1([a_i,a_{i+1}],w_j)} &\le \int_{a_i}^{a_{i+1}} \big|f^{(j)}(t)\big|\, w_j(t)\,(t-a_i)\,dt \\
&\le (a_{i+1}-a_i)\, \big\|f^{(j)}\big\|_{L^1([a_i,a_{i+1}],w_j)}\,.
\end{aligned}$$

Then, for any $1 \le p < \infty$,

$$\begin{aligned}
\big\|f^{(j-1)}(a_{i+1}^-) - f^{(j-1)}\big\|_{L^p([a_i,a_{i+1}],w_j)} &\le (a_{i+1}-a_i)\, \big\|f^{(j)}\big\|_{L^p([a_i,a_{i+1}],w_j)}\,, \\
c_1\, \big\|f^{(j-1)}\big\|_{L^p([a_i,a_{i+1}],w_j)} &\le \big\|f^{(j)}\big\|_{L^p([a_i,a_{i+1}],w_j)} + \big|f^{(j-1)}(a_{i+1}^-)\big|\,.
\end{aligned}$$

Since w_j is a non-decreasing function in $[a_i, a_{i+1}]$ and $w_j > 0$ in $(a_i, a_{i+1}]$, we have that $w_j \ge c > 0$ in $[a'_{i+1}, a_{i+1}]$ for any fixed $a'_{i+1} \in (a_i, a_{i+1})$, and consequently $w_j \in B_p([a'_{i+1}, a_{i+1}])$ for any $a'_{i+1} \in (a_i, a_{i+1})$.

Hence, $a_{i+1}^- \in \Omega(j-1)$. Since $\mathcal{K}_{ko}(\mu)=0$, Theorem 2.1 gives
$$\big|f^{(j-1)}(a_{i+1}^-)\big| \le c_2 \,\|f\|_{W^{k,p}(\mu)}\,,$$
and we conclude
$$\big\|f^{(j-1)}\big\|_{L^p([a_i,a_{i+1}],w_j)} \le c_3 \,\|f\|_{W^{k,p}(\mu)}\,.$$
If we fix $0<j\le k$ with w_j comparable to a non-increasing function in $[a_i,a_{i+1}]$ and $w_j>0$ a.e. in $[a_i,a_{i+1}]$, we obtain a similar inequality. Consequently,
$$\big\|f^{(j-1)}\big\|_{L^p([a_i,a_{i+1}],w_j)} \le c_4 \,\|f\|_{W^{k,p}(\mu)}\,,$$
for every $1\le i<n$, $1\le j\le k$ and $f\in V_{ko}^{k,p}(\mu)$.

Hence, Lemma 4.2 finishes this implication. □

Remarks.

1. The conclusion of Theorem 4.3 also holds without the hypothesis $\operatorname{supp}\mu_0 \subseteq [\alpha,\beta]$; we just need $\operatorname{supp}\mu_j \subseteq [\alpha,\beta]$ for $1\le j\le k$ (as shows the proof of Theorem 4.3).

2. Let us notice that the equivalence of (B), (C) and (D), and (A) implies (B) holds even if we remove the hypotheses on μ (as shows the proof of Theorem 4.3).

3. If we remove the hypotheses on μ, (B) does not imply (A), as the example after Theorem 4.2 shows.

5. Examples.

We present in this section some examples which show the scope of application of Theorem 4.3.

1. Let us choose any finite measure μ_0 with compact support. For each $1\le j\le k$, let us consider
$$w_j(x) := u_j(x)\big|x-a_{j1}\big|^{\alpha_{j1}}\big|x-a_{j2}\big|^{\alpha_{j2}}\cdots\big|x-a_{jn_j}\big|^{\alpha_{jn_j}}\chi_{J_j}(x)\,,$$
with $a_{j1}<a_{j2}<\cdots<a_{jn_j}$, $\alpha_{j1},\alpha_{j2},\dots\alpha_{jn_j}>-1$, J_j a finite union of compact intervals (we allows $J_j=\varnothing$, and then $w_j:=0$), and $u_j,u_j^{-1}\in L^\infty(J_j)$. Then, $\mu=(\mu_0,\dots,\mu_k)$ verifies the hypothesis of Theorem 4.3.

In order to apply Theorem 4.3, let us notice that $a_{ji}\in\Omega_j$ if and only if a_{ji} belongs to the interior of J_j and $|x-a_{ji}|^{-\alpha_{ji}}\in L^{1/(p-1)}([a_{ji}-\varepsilon,a_{ji}+\varepsilon])$ for some $\varepsilon>0$. The latter condition is equivalent either to $-1<\alpha_{ji}<p-1$ (if $p>1$) or to $-1<\alpha_{ji}\le 0$ (if $p=1$).

2. The last example can be widely generalized. Let us consider the functions defined inductively by $l_1(x):=\log(1/x)$, $l_n(x):=\log(l_{n-1}(x))$, and $L_n(x):=\max\{1,\,l_n(x)\}$.

We can substitute each $|x-a_{ji}|^{-\alpha_{ji}}$ by $|x-a_{ji}|^{-\alpha_{ji}}$ multiplied or divided by any finite number of factors $L_{n_{jim}}(|x-a_{ji}|)^{\beta_{jim}}$.

3. We can consider the first example with $k=1$, i.e., μ_0 has compact support,
$$w_1(x) := u(x)\big|x-a_1\big|^{\alpha_1}\big|x-a_2\big|^{\alpha_2}\cdots\big|x-a_n\big|^{\alpha_n}\chi_J(x)\,,$$
with $a_1<a_2<\cdots<a_n$, $\alpha_1,\alpha_2,\dots\alpha_n>-1$, J a finite union of compact intervals, and $u,u^{-1}\in L^\infty(J)$. We know that $a_i\in\Omega_1$ if and only if a_i belongs to the interior of J and we have either to $-1<\alpha_i<p-1$ (if $p>1$) or to $-1<\alpha_i\le 0$ (if $p=1$).

Then the multiplication operator $\mathcal{M}$ is bounded in $W_{ko}^{1,p}(\mu)$ if and only if $\mu_0(A)>0$ for every connected component A of Ω_1.

4. Finally, let us consider the case of Jacobi weights for the derivatives: Let us choose any finite measure μ_0 with compact support. For each $1\le j\le k$, let us consider
$$w_j(x) := c_j(1+x)^{\alpha_j}(1-x)^{\beta_j}\chi_{(-1,1)}(x)\,,$$
with $\alpha_j,\beta_j>-1$, and c_j verifies either $c_j=1$ or $c_j=0$.

Then the multiplication operator $\mathcal{M}$ is bounded in $W^{k,p}_{ko}(\mu)$ if and only if

$$\sharp \operatorname{supp}\big(\mu_0|_{(-1,1)}\big) \geq \min\{1 \leq j \leq k : \, c_j = 1\} .$$

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