HYPERCYCLIC SUBSPACES IN OMEGA
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Abstract. We show that any countable family of operators of the form $P(B)$, where $P$ is a non-constant polynomial and $B$ is the backward shift operator on the countably infinite product of lines $\omega$, has a common hypercyclic subspace.

The space $\omega = \mathbb{K}^\mathbb{N}$ -i.e., the countably infinite product of the (real or complex) scalar field $\mathbb{K}$, endowed with the product topology- is perhaps the most elementary infinite dimensional Fréchet space. Even so, because it does not support a dense subspace with a continuous norm, it sometimes requires to be considered separately when showing hypercyclic properties for all (separable, infinite dimensional) Fréchet spaces, see for example [12, p. 587-8].

A continuous linear operator $T$ acting on a Fréchet space $X$ is said to be hypercyclic provided there is some vector $z$ in $X$ whose orbit $\{z, Tz, T^2z, \ldots\}$ is dense in $X$. Such vector $z$ is called a hypercyclic vector for $T$. A hypercyclic manifold for $T$ is a dense, invariant subspace of $X$ consisting entirely -except for the origin- of hypercyclic vectors for $T$. A hypercyclic subspace for $T$ is a closed, infinite dimensional subspace of $X$ consisting entirely -except for the origin- of hypercyclic vectors for $T$.

Every separable, infinite dimensional Fréchet space supports a hypercyclic operator; see the works of Ansari [2], Bernal [5], and of Bonet and Peris [12]. It is also well known that once an operator on a Fréchet space has a hypercyclic vector, the smallest manifold invariant for $T$ containing that vector is a hypercyclic manifold; see the works of Bourdon [13], Herrero [19], and Wengenroth [29]. The situation...
for hypercyclic subspaces is different. Consider the backward shift
\[(x_1, x_2, x_3, \ldots, x_k) \mapsto (x_2, x_3, x_4, \ldots, x_k).\]

While Rolewicz [27] showed that each scalar multiple \(\lambda B\) is hypercyclic on \(\ell_2\) whenever the scalar \(\lambda\) has modulus strictly larger than 1, Montes [24] showed that no such operators have a hypercyclic subspace.

Read [26] and Bernal and Montes [7] constructed the first examples of hypercyclic subspaces. In fact, Read’s examples include an operator on \(\ell_1\) for which every non-zero vector in \(\ell_1\) is hypercyclic. González, León, and Montes [17] showed that if an operator \(T\) acting on a Banach space \(X\) satisfies that \(T \oplus T\) is hypercyclic on \(X \times X\), then \(T\) has a hypercyclic subspace if and only if there exists a closed, infinite dimensional subspace \(X_0\) of \(X\) and integers \(1 < n_1 < n_2 < \ldots\) so that

\[(1) \quad T^{n_k} x \to 0 \quad \text{as } k \to \infty \quad \text{for each } x \in X_0\]

and, moreover, if and only if the essential spectrum of \(T\) meets the closed unit disk. Let us stress here that the condition of \(T \oplus T\) being hypercyclic on \(X \times X\) is very mild, as all hypercyclic operators that we know seem to have this property; see [9]. In fact, the spectral characterization was used by León and Montes to test the existence of hypercyclic subspaces among a wide variety of classes of hypercyclic operators [22]. They also used this characterization to show that every separable, infinite dimensional Banach space supports an operator with a hypercyclic subspace [21].

Moreover, Condition (1) is sufficient to ensure the existence of a hypercyclic subspace well beyond the Banach space setting, as long as the Fréchet space \(X\) supports a continuous norm, see [11, Theorem 3.5] and [17, p. 177]. Indeed, L. Bernal [6, Theorem 2.5] and independently, Petersson [25, Theorem 7], used this fact to show that every separable infinite dimensional Fréchet space with a continuous norm supports a hypercyclic subspace.

On the other hand, Bonet, Martínez-Giménez and Peris [11, Remark 3.6] showed that, in general, Condition (1) is no longer sufficient in the case of Fréchet spaces without a continuous norm: the operator \((x_i)_{i \in \mathbb{Z}} \mapsto (2x_{i+1})_{i \in \mathbb{Z}}\) acting on \(X = \{(x_i)_{i \in \mathbb{Z}} \in \ell^\infty : (x_i)_{i=1}^\infty \in \ell_2\}\) satisfies Condition (1) and yet does not have a hypercyclic subspace.
We show in this note that \( \omega \) supports operators with a hypercyclic subspace too, even that \( \omega \) is known not to have dense subspaces with a continuous norm [23, Corollary 1]. Indeed, we show the following.

**Theorem 1.** Let \( (P_k)_{k=1}^{\infty} \) be any sequence of non-constant polynomials, and let \( B \) be the backward shift acting on \( \omega \). Then the operators \( P_k(B) \) \( (k \in \mathbb{N}) \) have a common hypercyclic subspace. That is, there exists a closed infinite dimensional subspace \( S \) of \( \omega \) satisfying that

\[
\{x, P_k(B)x, P_k^2(B)x, \ldots \}
\]

is dense in \( \omega \) for each \( 0 \neq x \in S \) and each \( k \in \mathbb{N} \).

Theorem 1 also improves a result by Herzog and Lemmert [20, Bemerkungen 1], who showed that each operator on \( \omega \) of the form \( P(B) \), where \( P \) is a non-constant polynomial and \( B \) the backward shift, has a hypercyclic vector.

For more on hypercyclicity results we refer to the surveys by Grosse-Erdmann [15, 16] and by Bonet, Martínez-Giménez and Peris [10]. For work on common hypercyclic vectors and common hypercyclic subspaces, we refer to the articles of Abakumov and Gordon [1], Bayart [4], Costakis and Sambarino [14], and by Aron et al. [3].

Before proving Theorem 1 we first show two lemmas. For each \( m \in \mathbb{N} \), we let \( \Pi_m \) denote the standard projection of \( \omega \) onto \( \mathbb{K}^m \); that is, \( \Pi_m x = (x_1, \ldots, x_m) \) for each \( x = (x_i)_{i=1}^{\infty} \) in \( \omega \).

**Lemma 2.** Let \( T = P(B) \), where \( B \) is the backward shift on \( \omega \) and \( P(t) = a_1 + a_2 t + \cdots + a_{d+1} t^d \) is any polynomial of degree \( d \geq 1 \). Then for each \( l, m \in \mathbb{N} \), \( (y_1, y_2, \ldots, y_l) \in \mathbb{K}^l \) and \( (x_1, x_2, \ldots, x_{md}) \in \mathbb{K}^{md} \), there exists a unique \((z_1, z_2, \ldots, z_l) \in \mathbb{K}^l \) so that

\[
\Pi_l T^m (x_1, x_2, \ldots, x_{md}, z_1, z_2, \ldots, z_l, h_1, h_2, \ldots) = (y_1, y_2, \ldots, y_l)
\]

for each \( h_1, h_2, \ldots \) in \( \mathbb{K} \).

**Proof.** Notice that each \( x = (x_i)_{i=1}^{\infty} \) in \( \omega \) we have

\[
Tx = ((a_1 x_j + a_2 x_{j+1} + \cdots + a_d x_{j+d-1}) + a_{d+1} x_{j+d})_{j=1}^{\infty}.
\]
Thus the Lemma follows, since $a_{d+1} \neq 0$. □

**Lemma 3.** Let $[f_{i,j}] \in \mathbb{K}^{N \times N}$ be an infinite matrix with coefficients in $\mathbb{K}$ and no row of zeroes. For each row $f_n = (f_{n,1}, f_{n,2}, \ldots)$, let $a_n := \min \{ j \in \mathbb{N} : f_{n,j} \neq 0 \}$. Then if $(a_m)_{m=1}^{\infty}$ is strictly increasing

i) $\{f_1, f_2, \ldots\}$ is linearly independent, and

ii) $\text{span}\{f_1, f_2, \ldots\} = \{ \sum_{n=1}^{\infty} \alpha_n f_n : (\alpha_n)_{n=1}^{\infty} \in \mathbb{K}^N \}$.

**Proof.** Notice that since $(a_m)_{m=1}^{\infty}$ is strictly increasing, for each $s \in \mathbb{N}$ we have $f_{s,a_s} \neq 0$ and $f_{n,j} = 0$ for each $(n,j) \in (s, \infty) \times [1, a_s]$. Hence (i) follows, and $\sum_{n=1}^{\infty} \alpha_n f_n$ converges in $\omega$ for any $(\alpha_n)_{n=1}^{\infty} \in \mathbb{K}^N$. Now, let $g \in \text{span}\{f_1, f_2, \ldots\}$. There exist integers $1 < r_1 < r_2 < \ldots$ and sequences $(\alpha_{n,1})_{n=1}^{\infty}, (\alpha_{n,2})_{n=1}^{\infty}, \ldots$ in $\mathbb{K}$ so that

\[
(2) \quad P_n := (\alpha_{n,1} f_1 + \alpha_{n,2} f_2 + \cdots + \alpha_{n,r_n} f_{r_n}) \to_{n \to \infty} g.
\]

It remains to show that there exists a strictly increasing sequence $(\alpha_s)_{s=1}^{\infty}$ in $\mathbb{N}$ so that

\[
(3) \quad \Pi_{a_s} (\alpha_1 f_1 + \cdots + \alpha_s f_s) = \Pi_{a_s} (g) \quad (s \in \mathbb{N}).
\]

Now, for $n > a_1 \alpha_{n,1} (f_{1,1}, f_{1,2}, \ldots, f_{1, a_1}) = \Pi_{a_1} (P_n)$, and so by (2) $\alpha_n \to_{n \to \infty} \alpha_1$ and $\Pi_{a_1} (g) = \Pi_{a_1} (\alpha_1 f_1)$, where $\alpha_1 = \frac{g_{a_1}}{f_{1,a_1}}$. Inductively, suppose that we found $\alpha_j \in \mathbb{K} (1 \leq j \leq s-1)$ so that

\[
(4) \quad \alpha_{n,j} \to_{n \to \infty} \alpha_j \quad \text{and} \quad \Pi_{a_j} (g) = \Pi_{a_j} (\alpha_1 f_1 + \cdots + \alpha_j f_j)
\]

for each $(1 \leq j \leq s-1)$. Again, since $(a_m)_{m=1}^{\infty}$ is strictly increasing, $\Pi_{a_s} (\alpha_{n,1} f_1 + \cdots + \alpha_{n,s} f_s) = \Pi_{a_s} (P_n)$ for each $n > s$ and so by (4) and (2) we have $\alpha_{n,s} \to_{n \to \infty} \alpha_s$ and $\Pi_{a_s} (g) = \Pi_{a_s} (\alpha_1 f_1 + \cdots + \alpha_s f_s)$, where $\alpha_s = \frac{g_{a_s}}{f_{s,a_s}} - (\alpha_{s-1} f_{s,a_{s-1}} + \cdots + \alpha_{s-1} f_{1,a_{s-1}})$. So (3) follows. □
Proof of Theorem 1: Let \( \{ r_l : l \in \mathbb{N} \} \) be a countable dense set in \( \omega \) so that each \( r_l = (r_{l,j})_{j=1}^{\infty} \) satisfies \( r_{l,j} \neq 0 \) if and only if \( 1 \leq j \leq l \). For each \( k \in \mathbb{N} \), let \( T_k := P_k(B) \) and \( d_k := \text{deg}(P_k) \). We make use of the following claim.

Claim 4. There exists an infinite, upper triangular matrix \( F = [f_{i,j}] \in \mathbb{K}^{\mathbb{N} \times \mathbb{N}} \) satisfying

a) No row \( f_n = (f_{n,1}, f_{n,2}, \ldots) \) is zero.

b) The sequence \( (a_n)_{n=1}^{\infty} \) given by \( a_n := \min \{ j \in \mathbb{N} : f_{n,j} \neq 0 \} \) is strictly increasing.

c) For each \( (k,i,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) with \( k < i + l \), there exists a positive integer \( m_{k,i,l} \) so that

\[
\Pi_l T_k^{m_{k,i,l}} f_n = \begin{cases}
(r_{l,1,1}, r_{l,2,1}, \ldots, r_{l,l,1}) & \text{if } n = i \\
(0, 0, \ldots, 0) & \text{if } n \neq i.
\end{cases}
\]

Suppose the Claim holds. We show now that \( S := \text{span}\{f_1, f_2, \ldots\} \) is a hypercyclic subspace for each \( T_k \) \((k \in \mathbb{N})\).

By (a), (b), and Lemma 3(i), the closed subspace \( S \) is infinite dimensional. Let \( 0 \neq f \in S \). We show that \( f \) is hypercyclic for \( T_k \), \( k \in \mathbb{N} \). By Lemma 3, \( f = \sum_{n=1}^{\infty} \alpha_n f_n \) for some sequence of scalars \( (\alpha_n)_{n=1}^{\infty} \). Multiplying \( f \) by a nonzero scalar if necessary, we may assume without loss of generality that \( \alpha_i = 1 \) for some \( i \in \mathbb{N} \). But by (c), for each \( l > \max\{k - i, 1\} \)

\[
\Pi_l T_k^{m_{k,i,l}} f = \sum_{n=1}^{\infty} \alpha_n \Pi_l T_k^{m_{k,i,l}} f_n = \Pi_l T_k^{m_{k,i,l}} f_i = (r_{l,1,1}, r_{l,2,1}, \ldots, r_{l,l,1}).
\]

It follows that \( f \) is hypercyclic for \( T_k \). We finish the proof of Theorem 1 by showing the Claim.

Proof of Claim. Let \( M_{0,0} := 1 \). Inductively, for each \( N \in \mathbb{N} \) define

\[
\begin{cases}
M_N := d_N M_{(N-1),(N-1)^2} \\
M_{N,i} := 2^{N+i} M_N \quad (1 \leq i \leq N^2) \\
M_{(N-1),(N-1)^2+1} := M_{N,1}.
\end{cases}
\]
Also, for each \((k, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\) with \(1 \leq k \leq (i + l) - 1\), let
\[
m_{k, i, l} := \frac{M_{(i+1)((k-1)(i+l)-1)}}{d_k}.
\]
Finally, let \(f_{n, j} = 0\) for each \((n, j) \in \mathbb{N} \times [1, M_{1, 1}]\). We complete the definition of the matrix \(F = [f_{n, j}]\) inductively. At each step \(N\) we define \(f_{n, j}\) for all \((n, j) \in \mathbb{N} \times (M_{N, 1}, M_{N+1, 1}]\).

**Step \(N = 1\)**. We define \(f_{n, j}\) for all \((n, j) \in \mathbb{N} \times (M_{1, 1}, M_{2, 1}]\) so that
\[
(5) \quad \Pi T_{1}^{m_{1,1}} f_{n, 1, 2, \ldots, f_{n, M_{2,1}}, *, *, \ldots} = \begin{cases} r_{1,1} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}
\]
By Lemma 2 (letting \(l = 1, m = m_{1,1,1}, T = T_{1}, d = d_{1}, y_{1} = r_{1,1}\) and \(x_{j} = f_{1,j}\) \((1 \leq j \leq M_{1,1})\), there exists a unique \(z \in \mathbb{K}\) so that
\[
\Pi T_{1}^{m_{1,1}} (f_{1,1,1}, f_{1,2}, \ldots, f_{1,M_{1,1}}, z, *, *, \ldots) = r_{1,1}.
\]
So (5) is satisfied if we define \(f_{1,M_{1,1}+1} := z\), and \(f_{n, j} = 0\) for each \((1, M_{1,1}+1) \neq (n, j) \in \mathbb{N} \times (M_{1,1}, M_{2,1}]\).

**Step \(N \geq 2\)**.

We divide this step into \(N^2\) substeps; one for each \((k, i) \in [1, N] \times [1, N]\). We start with substep \(N, 1, 1\), and follow with the “lexicographic” order given by the relation \((k', i') < (k, i)\) if and only if either \(k' < k\) or both \(k' = k\) and \(i' < i\).

At each substep \(N, k, i\) we define the coordinates \(f_{n, j}\) for all indexes \((n, j)\) in \(\mathbb{N} \times (M_{N,(k-1)N+i}, M_{N,(k-1)N+i+1}]\), so that
\[
(6) \quad \Pi T_{k}^{m_{k-1,i}} g_{n} = \begin{cases} (r_{l,1}, \ldots, r_{l,t}) & \text{if } n = i \\ (0, \ldots, 0) & \text{if } n \neq i, \end{cases}
\]
for any \(g_{n}\) of the form \(g_{n} = (f_{n, 1, \ldots, f_{n,M_{N,(k-1)N+i+1}}, *, *, \ldots})\) and \(t = N + 1 - i\).

**Substep \(N, 1, 1\)**.

Applying \(N\) times Lemma 2 (Taking, for each \(1 \leq n \leq N\): \(l = N, m = m_{1,1,N}, T = T_{1}, d = d_{1}, x_{j}^{(n)} = f_{n,j}\) \((1 \leq j \leq M_{N,1})\), and \((y_{1}^{(n)}, \ldots, y_{N}^{(n)}) = (r_{N,1}, \ldots, r_{N,N})\) if \(n = 1\) and \((y_{1}^{(n)}, \ldots, y_{N}^{(n)}) = (0, \ldots, 0) \in \mathbb{K}^{N}\) if \(n \neq 1\), we get \((z_{1}^{(n)}, z_{2}^{(n)}, \ldots, z_{N}^{(n)}) \in\).
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(7) \( \Pi_N T_{n}^{m_{1},N} g_n = \begin{cases} (r_{N,1}, \ldots, r_{N,N}) & \text{if } n = 1 \\ (0, \ldots, 0) & \text{if } n \neq 1. \end{cases} \)

for any \( g_n \) of the form \( g_n = (f_{n,1}, \ldots, f_{n,M_N}, z_{1}^{(n)}, \ldots, z_{N}^{(n)}, *, *, \ldots) \). Hence (6) is satisfied for \((k, i) = (1, 1)\) if we define

\[
(f_{n,M_N+1}, \ldots, f_{n,M_N+1+N}) = (z_{1}^{(n)}, \ldots, z_{N}^{(n)}) \quad (1 \leq n \leq N)
\]

and \( f_{n,j} = 0 \) for each \((n, j)\) in either \((\mathbb{N} \setminus \{1, \ldots, N\}) \times (M_{N,1}, M_{N,2})\) or in \(\mathbb{N} \times (M_{N,1} + N + 1, M_{N,2})\).

Substep \(N.k.i\).

We have already defined \( f_{n,j} \) for each \((n, j) \in \mathbb{N} \times [1, M_{N,(k-1)N+i+1}]\), so that equation (6) holds for each \((1, 1) \leq (k', i') < (k, i)\). That is, so that

\[
(8) \quad \Pi_{\ell} T_{k'}^{m_{k',\ell}} g_n = \begin{cases} (r_{\ell,1}, \ldots, r_{\ell,l}) & \text{if } n = i' \\ (0, \ldots, 0) & \text{if } n \neq i'. \end{cases}
\]

for any \( g_n \in \omega \) of the form \( g_n = (f_{n,1}, \ldots, f_{n,M_{N,(k'-1)N'+i+1}}, *, *, \ldots) \) and \( l = N + 1 - i' \).

We apply \(N\) times Lemma 2 ( taking, for each \(1 \leq n \leq N\): \( l = N + 1 - i\), \( m = m_{k,l,i}, T = T_k, d = d_k, x_j^{(n)} = f_{n,j} (1 \leq j \leq M_{N,(k-1)N+i})\), \((y_1^{(n)}, \ldots, y_l^{(n)}) = (r_{1,1}, \ldots, r_{1,l})\) if \( n = i \) and \((y_1^{(n)}, \ldots, y_l^{(n)}) = (0, \ldots, 0) \in \mathbb{K}^l\) if \( n \neq i \), to obtain

\[
(9) \quad \Pi_{\ell} T_{k}^{m_{k,\ell}} g_n = \begin{cases} (r_{\ell,1}, \ldots, r_{\ell,l}) & \text{if } n = i \\ (0, \ldots, 0) & \text{if } n \neq i, \end{cases}
\]

for any \( g_n \in \omega \) of the form \( g_n = (f_{n,1}, \ldots, f_{n,M_{N,(k-1)N+i+1}}, z_{1}^{(n)}, \ldots, z_{l}^{(n)}, *, *, \ldots) \) and \( l = N + 1 - i \). So equation (6) is satisfied if we define \( f_{n,M_{N,(k-1)N+i+s}} = z_s^{(n)} \) when \((n, s) \in [1, N] \times [1, l]\), and \( f_{n,j} = 0 \) for all indexes \((n, j)\) in either \((\mathbb{N} \setminus \{1, \ldots, N\}) \times (M_{N,(k-1)N+i}, M_{N,(k-1)N+i+1})\) or in \([1, \ldots, N] \times (M_{N,(k-1)N+i+1}, M_{N,(k-1)N+i+1})\).

We have now completely defined the matrix \([f_{n,j}] \in \mathbb{K}^{N \times N}\). Notice that for each \(N \in \mathbb{N}\), \( f_{N,j} = 0 \) for \(1 \leq j \leq M_{N,N}\), and (as defined on substep \(N.1.N\) of step \(N\)) \( f_{N,M_{N,N+1}} \neq 0\). So \( a_N = \min\{j \in \mathbb{N} : f_{N,j} \neq 0\} = M_{N,N} + 1\), and (a) and
(b) of the Claim hold. Finally, given any \((k, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\) with \(k < i + l\), our definitions on substep \(N.k.i\) of step \(N = i + l - 1\) given by (6) ensure that

\[
\Pi_{k} T_{k}^{m_{k},i,l} f_n = \begin{cases} 
(r_{1,1}, \ldots, r_{l,l}) & \text{if } n = i \\
(0, \ldots, 0) & \text{if } n \neq i.
\end{cases}
\]

So Part (c) of the Claim holds, and the proof of Theorem 1 is now complete.

\[\square\]

**Corollary 5.** The set of operators on \(\omega\) that have a hypercyclic subspace is dense, with respect to the Strong Operator Topology (S. O. T.), in the algebra \(L(\omega)\) of all continuous linear operators on \(\omega\).

**Proof.** By a result of Hadwin, Nordgreen, Radjavi and Rosenthal [18] (cf. [8, Corollary 6]), the set of operators on \(\omega\) having a hypercyclic subspace, which is invariant under conjugations, must be either empty or S.O.T.-dense in \(L(\omega)\). Theorem 1 then gives the desired conclusion. \[\square\]

**Remark 6.** A simple modification to Lemma 2 allows to generalize Theorem 1 to backward shifts \(B_b\) with non-zero weights. Namely, if \((b_n)_{n=2}^{\infty}\) is a sequence of nonzero weights and \((x_1, x_2, x_3, \ldots) \mapsto (b_2x_2, b_3x_3, b_4x_4, \ldots)\) is its associated weighted shift on \(\omega\), then any countable collection of operators of the form \(P(B_b)\), where \(P\) is a non-constant polynomial, has a common hypercyclic subspace in \(\omega\).

Solving a problem by Salas [28], Abakumov and Gordon [1] showed that the family \(\{\lambda B : |\lambda| > 1\}\) of all scalar multiples of the backward shift \(B\) on \(\ell_2\) (with the scalars of modulus strictly larger than 1) have a common hypercyclic vector. Hence (cf. also [14, Remark 8.3]) it is natural to ask

**Problem 7.** Let \(\mathcal{F}\) be the collection of all operators on \(\omega\) of the form \(P(B)\), where \(P\) is a non-constant polynomial and \(B\) is the backward shift. Do the operators in \(\mathcal{F}\) have a common hypercyclic vector in \(\omega\)? Do they share a common hypercyclic subspace?

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