Málaga Economic Theory Research Center Working Papers

How to Add Apples and Pears: Non-Symmetric Nash Bargaining and the Generalized Joint Surplus

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June 2010

Abstract

We generalize the equivalence of the non-symmetric Nash bargaining solution and the linear division of the joint surplus when bargainers use different utility scales. This equivalence in the general case requires the surplus each agent receives to be expressed in compatible, or comparable, units. This result is valid in the case of bargaining over multiple-issues. Our conclusions have important implications for comparative static exercises and calibrated work. For example, when comparing the joint surplus of economies with different preferences, it is crucial to lay out the surplus in terms of one utility unit or the other. On the other hand, while it is necessary to transform the units when expressing the non-symmetric Nash bargaining solution as a share of the joint surplus, it is not necessary to perform a unit transformation when maximizing directly the generalized Nash product. Finally, we discuss the requirements on the curvatures of the agents' utility functions, or, in other words, on the bargainers' attitudes towards risk.

Keywords: Bargaining problems, Non-Symmetric Nash Bargaining Solution, Linear Sharing

JEL Class.: C7, J5

1 Introduction

In order to compute the non-symmetric Nash bargaining solution there is an equivalence that is well-known to, and widely-used by, economists.[1] This equivalence allows to choose between maximizing the generalized Nash product or finding the unique solution that linearly shares the joint surplus. This equivalence holds, however, only where preferences of all involved agents are identical.

The purpose of this work is to show that the surplus from bargaining obtained by each individual at the non-symmetric Nash bargaining solution does in fact coincide with a linear sharing of the total surplus from bargaining. In the general case, special attention has to be taken to express the total surplus correctly. More precisely, we show that the linear sharing equivalence holds when one applies a transformation to express all utility levels in the same units. Our conclusions have important implications for comparative static exercises and calibrated work. For example, when comparing the joint surplus of economies with different preferences, it is crucial to lay out the surplus in terms of one utility unit or the other. An obvious illustration of this situation is when one wishes to look at the effect on an economy of a change in risk aversion. In this case, it is primordial to evaluate the joint surplus in similar utility units. In addition, it is useful to stress that, while it is necessary to transform the units when expressing the non-symmetric Nash bargaining solution as share of the joint surplus, it is not necessary to perform a unit transformation when maximizing the generalized Nash product.

To see what we mean by "linear sharing equivalence" and why this problematic is important, consider the following example. Think of two individuals bargaining about the division of 10 units of money. If they are not able to reach an agreement, both agents obtain zero. Assume that individuals, named A and B, only care about their share of the units, y_A and y_B, with $y_A + y_B = 10$, and that they are risk neutral.

Example 1. For simplicity, let us take the utility of individuals A and B to be equal to the amount of money they receive, y_A and y_B, respectively. The non-symmetric Nash bargaining solution suggests that bargainers will agree on a division $y^* = (y_A^*, y_B^*)$ that solves

$$\max_{y_A, y_B: \ y_A + y_B = 10} y_A^\alpha y_B^{1-\alpha},$$

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[1]See Pissarides (1985) , Blanchard and Diamond (1990), and Pissarides (2003). See also Rogerson *et al.* (2005) for bargaining in matching models, and Rupert *et. al.* (2001) for bargaining in monetary economics.

with α a number between 0 and 1. This solution gives agent A 10α units of money and agent B $10(1 - \alpha)$ units of money. It is therefore clear that each agent gets their corresponding share of the 10 units according to α and $(1 - \alpha)$.

As in Example 1 above, the linear sharing equivalence is often used in situations when all agents have linear preferences, and it is only valid when agents have **identical preferences at the margin**. With more general preferences, the division of the joint surplus might not be so neatly done. In fact, a conversion rate (at the margin) has to be applied in order to express each bargainer's preferences the units of the other bargainer's preferences. The next example illustrates this. As before, two individuals bargain about the division of 10 units of money. In case of disagreement, both agents obtain no payoff and individuals, A and B, only care about their share of the units, y_A and y_B, with $y_A + y_B = 10$.

Example 2. Assume now that the utility of agent B stays as in Example 1, but the utility of A is such that $u_A(y_A) = \sqrt{y_A}$. The non-symmetric Nash bargaining solution (y_A^*, y_B^*) will now solve

$$\max_{y_A, y_B:\ y_A + y_B = 10} (y_A)^{\frac{\alpha}{2}}\, y_B^{1 - \alpha},$$

with α a number between 0 and 1. At the new specification agent A obtains $\frac{10\alpha}{2 - \alpha}$ units of money, while agent B receives $\frac{20}{2 - \alpha}$ units of money. It now seems that agents are not sharing the surplus according to the corresponding weights α and $(1 - \alpha)$ of the generalized Nash product. If we look at the surplus in terms in utilities, we observe that $u_A(y_A^*) - u_A(0) = \sqrt{\frac{10\alpha}{2 - \alpha}}$ and $u_B(y_B^*) - u_B(0) = \frac{20}{2 - \alpha}$, which does not seem an α, $(1 - \alpha)$ split either.

In the next section, we lay out the one-issue two-person bargaining problem, present the non-symmetric Nash bargaining solution, explain the way to convert the scale of one individual into the scale of the other one in order to measure the joint surplus, and show how the corresponding linear shares of these measures coincide with the non-symmetric Nash bargaining solution. In Section 3, we show that there is not always concordance between convexity of the feasible set of the bargaining problem and the risk aversion of individual preferences. Furthermore, we discuss in Section 4 the possibility that the maximization of the generalized Nash product on one-issue settings can still be well defined in terms of the linear sharing equivalence even if the feasible set of the bargaining problem is not convex. Finally, in Section 5 we comment briefly on how the linear sharing equivalence still holds, and the transformation of units required for that, when agents bargain over multiple issues.

2   Two-Person Bargaining Problems: The One-Issue Setting

A two-person bargaining problem is generically defined as a pair (S, d), where S is a closed, upper-bounded, comprehensive subset of $\Re^2$ and d is an interior point in S. The set S represents the set of feasible utilities, while vector d is the disagreement point, also called disagreement payoff allocation, outside option or status quo.

We are interested in situations where bargaining takes place in terms of a variable $x \in \Re$ and where individuals A and B, engaged in bargaining, have utilities $a(x)$ and $b(x)$, respectively, for each possible choice of x. For simplicity, assume that both $a(x)$ and $b(x)$ are monotonous, continuous functions in $\Re$. Depending on the application, an interval of $\Re$, denoted X, is available to bargainers if agreement is reached. The utility possibility frontier is therefore the set $F \in \Re^2$ defined as

$$F = \left\{ (a(x), b(x)) \in \Re^2, \text{ where } x \in X \right\}. \tag{1}$$

Assumptions on the interval X and on functions $a(x)$ and $b(x)$

1. Both $a(x)$ and $b(x)$ are differentiable, with $a'(x) > 0$ and $b'(x) < 0$ for all $x \in X$. This implies that both functions a and b have an inverse, namely a^{-1} and b^{-1}, respectively.[2]

2. The disagreement point can be written as $d = (a(x_A), b(x_B))$, where x_A and x_B are shadow values of x at disagreement. Typically, if no agreement is reached no decision about x will be implemented. The values x_A and x_B are computed by means of the inverse functions a^{-1} and b^{-1}, namely $x_A = a^{-1}(d_A)$ and $x_B = b^{-1}(d_B)$, where $d = (d_A, d_B)$. Furthermore, $x_A < x_B$.

Assumption 1 restricts the problem to differentiable functions. This allows us to compute a closed-form solution and infer its properties. Assumption 2 means that there is a subinterval of X, namely $[x_A, x_B]$, such that feasible utility levels for both agents are over the disagreement point. Note that given that $a(x)$ is an increasing function of x, anything to the right of x_A yields a higher utility than d for agent A, while, given that $b(x)$ is a decreasing function of x, anything to the left of x_B yields a higher utility than d for agent B. We check now that S fulfills the usual assumptions found in the bargaining theory literature.

Assumptions on S

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[2]See Sydsaeter and Hammond (1995) Section 7.6, p. 240.

1. *Essentiality.* We say that the bargaining problem is essential if there is at least one $v \in F$ such that $d_i < v_i$ for $i = A, B$. By assumption 2 above we know that any point $x \in [x_A, x_B]$ generates a pair of utilities $v = (a(x), b(x))$ satisfying the condition.

2. *Closeness.* The utilities on the utility possibilities frontier F are feasible, i.e., $F \subset S$. F is continuous, which is true for $a(x)$ and $b(x)$ continuous on X, where X is an interval.

3. *Upper Boundedness.* F is the upper frontier of S: If a vector of utilities u satisfies that $u \geq v$ with $u \neq v$ for a vector $v \in F$, then $u \notin S$, or, in words, u cannot be feasible. We have nevertheless to check that F draws a decreasing line in the orthant of utilities $(a(x), b(x))$. Let $f(z) = b\left(a^{-1}(z)\right)$, with $z = a(x)$, for any $x \in X$. As we mentioned before, the inverse function a^{-1} exists. Furthermore, it is differentiable since a is. By applying the chain rule and computing the derivative of an inverse function,

$$f'(z) = b'\left(a^{-1}(z)\right) \, \left(a^{-1}\right)'(z) = \frac{b'\left(a^{-1}(z)\right)}{a'\left(a^{-1}(z)\right)},$$

or, to make it easier to read,

$$f'(z) = \frac{b'(x)}{a'(x)},$$

where $a(x) = z$.[3] Recall that, by assumption, $b'(x) < 0$ and $a'(x) > 0$, for any x in the interval X. Hence, $f'(z) < 0$.

But if $f(z)$ is a decreasing line in the orthant $(z, f(z))$ then F draws a decreasing line in the orthant of utilities since, by definition of F and f, $F = \{(z, f(z)), \text{ where } z = a(x), \ x \in X\}$.

4. *Free disposal.* Agents can choose to dispose of resources, i.e., the set S is comprehensive. This implies that for any vector of utilities v in the frontier F it has to be that all vectors $u \leq v$ are in S.

It is the case that S satisfies Assumptions 1-4, we can therefore define the one-issue bargaining problem as a pair (S, d), where

$$S = \left\{u \in \Re^2 \text{ such that } u \leq v \text{ for some } v \in F\right\}, \tag{2}$$

where the set F is defined as in (16), and $d = (a(x_A), b(x_B))$.

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[3]See Sydsaeter and Hammond (1995) Section 7.6, Theorem 7.9 on page 243 for the derivative of an inverse function, and Section 5.2 for the chain rule.

The assumption of convexity of S

Note that in order to define a one-issue bargaining problem we have not imposed the feasible set S to be a convex set. In the seminal works on bargaining theory, the set S of feasible utilities is usually assumed to be convex (see among others Nash (1950), Kalai-Smorodinsky (1975), Kalai (1977a), but also reference text books as Mas-Colell et al (1995) or Myerson (1991).) Convexity is justified by assuming that players can agree to jointly randomized strategies, so that, if the utility allocations x and y are feasible then the expected utility allocation $\theta x + (1 - \theta) y$ can also be achieved by bargainers by means of a randomizing device that implements x with probability θ and y with probability $1 - \theta$. In addition, the usual justifications for the use of the non-symmetric Nash bargaining solution, Kalai (1977b) –using a replica argument– and Rubinstein (1982) –from a strategic point of view–, assume that the set of feasible utilities is convex. Up to our knowledge, only the work by Zhou (1997) studies the non-symmetric Nash bargaining solution for non convex problems, and he does so from an axiomatic point of view.[4]

In our one-issue setting, convexity of S requires that the function $f(z) = b(a^{-1}(z))$, for $z = a(x)$ and $x \in [x_A, x_B]$ to be concave. Assume that a and b are twice differentiable, $f(z)$ has to be such that $f''(z) \leq 0$.

Recall that $f'(z) = \frac{b'(x)}{a'(x)}$, where $x = a^{-1}(z)$. Therefore,

$$df'(z) = \frac{b''(x)a'(x) - b'(x)a''(x)}{[a'(x)]^2} dx$$

and that

$$dz = a'(x)dx.$$

This means that

$$f''(z) = \frac{df'(z)}{dz} = \frac{b''(x)a'(x) - b'(x)a''(x)}{[a'(x)]^3}.$$

Hence, $f''(z) \leq 0$ if and only if

$$b''(x) \leq b'(x)\frac{a''(x)}{a'(x)}. \tag{3}$$

Given that $a'(x) > 0$ and $b'(x) < 0$ for $x \in [x_A, x_B]$, condition (3) above is true if both $a''(x) \leq 0$ and $b''(x) \leq 0$, but the converse is not true in general.

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[4]The seminal works by Nash (1950), (1953) deal with symmetric solutions, which in our setting means $\alpha = \frac{1}{2}$. Some papers have dealt with the symmetric Nash bargaining solution for non convex sets. See for example Roth (1977), Kaneko (1980) and Conley and Wilkie (1996).

2.1 The Non-Symmetric Nash Bargaining Solution

For generic two-person bargaining problems (S, d), the non-symmetric Nash bargaining solution (Nash (1950) and (1953) and Kalai (1977)) solves

$$\max_{v \in S} (v_A - d_A)^\alpha (v_B - d_B)^{1-\alpha}, \tag{4}$$

where $v = (v_A, v_B)$, $d = (d_A, d_B)$ and $\alpha \in (0, 1)$. In our one-issue setting, maximizing this generalized Nash product is equivalent to agreeing on the issue x^*, where $x^* = a^{-1}(z^*)$ and z^* solves

$$\max_{z \in [a(x_A), a(x_B)]} (z - a(x_A))^\alpha (f(z) - b(x_B))^{1-\alpha}. \tag{5}$$

Recall that $f(z) = b(a^{-1}(z))$, or, in other words, $f(z) = b(x)$ for $a(x) = z$.

Convexity of S ensures that the Nash bargaining solution, in general bargaining settings, is unique and interior. Namely, we prove the following statement in the mathematical appendix:

Proposition 1 *Let $f(z)$ be defined as above, with $z = a(x)$, for any $x \in [x_A, x_B]$. If $f'(z) < 0$ and $f''(z) \le 0$, then the generalized Nash product as a function of z is strictly concave in $[a(x_A), a(x_B)]$.*

Proposition 1 states that, if the set of feasible utilities S is convex, maximizing the generalized Nash product, as a function of $z = a(x)$, yields a **unique, interior** solution $z^* \in [a(x_a), a(x_B)]$. Given that a is a monotone, increasing function of x, this means that $x^* = a^{-1}(z^*)$ is the non-symmetric Nash bargaining solution and is unique and interior to $[x_a, x_B]$. In particular and very importantly for our results, the first order condition of the maximization problem in our one-issue setting is necessary and sufficient, and therefore the linear sharing equivalence will hold.

2.2 Measuring the Joint Surplus

Fix a bargaining agreement or solution x^*. We can define the bargaining surplus for agent A as the difference $a(x^*) - a(x_A)$. Similarly, we can also define the bargaining surplus for agent B as the difference $b(x^*) - b(x_B)$. In situations where the surplus of both individuals is measured in the same units, as for example in terms of money (see Example 1 of the introduction), the joint surplus from bargaining can be defined as the sum of both surpluses, i.e.,

$$JS(x^*) = a(x^*) - a(x_A) + b(x^*) - b(x_B).$$

Unfortunately, when utility units are not the same across individuals, the joint surplus cannot be the sum of utilities. A transformation of utils of A into utils of B, and vice versa, has to be applied. A natural transformation makes use of the marginal utilities evaluated at the bargaining agreement.

The change of utils of A per marginal change of utils of B at a given agreement x, or the marginal rate of substitution of utils of B into utils of A at the frontier F, formally $\frac{d\ a(x)}{d\ b(x)}$, is equal to $\frac{a'(x)}{b'(x)}$. Note that this number is negative in general, since $a'(x) > 0$ and $b'(x) < 0$. Therefore, starting at a given agreement x, if we want to increase the utility of B by one unit (in utils of B), the utility of A will decrease in $\left|\frac{a'(x)}{b'(x)}\right|$ units (in utils of A). At the margin and in terms of exchange (or in terms of opportunity costs), one util of B is equivalent to $\left|\frac{a'(x)}{b'(x)}\right|$ utils of A. Applying this transformation rule to the bargaining surplus, the joint surplus at the Nash solution x^* measured either in utils of A, JS_A, or in utils of B, JS_B, looks as follows

$$JS_A(x^*) = a(x^*) - a(x_A) + \frac{a'(x^*)}{|b'(x^*)|}\left[b(x^*) - b(x_B)\right],$$

and

$$JS_B(x^*) = b(x^*) - b(x_B) + \frac{|b'(x^*)|}{a'(x^*)}\left[a(x^*) - a(x_a)\right].$$

In Example 1 of the Introduction, $|b'(x^*)| = a'(x^*)$ and therefore $JS_A(x^*) = JS_B(x^*)$. Intuitively, x measures units of money and for both agents A and B in Example 1 one unit of money transforms into one util. Hence, the exchange of utils of A into utils of B is a one-to-one transformation.

2.3 From the Non-Symmetric Nash Bargaining Solution to the Joint Surplus

We show that, at the non-symmetric Nash bargaining solution x^*, each of the agents obtains a share of the generalized joint surplus. This share is equal to the corresponding weights in the Nash product, α and $1 - \alpha$, exactly like in the well-known case of agents using the same units presented in Example 1. The difference is that the generalized joint surplus has to be measured in utils of A if we are computing the share for agent A. Equivalently, the generalized joint surplus has to be measured in units of B if we are computing the share for agent B.

Recall that any number z^* solving (5) has to lie interior of the interval $[a(x_A), a(x_B)]$. This means that the first order condition of the maximization problem has to be satisfied with equality. Therefore, z^* solves:

$$(z^* - a(x_A))^\alpha \left(f(z^*) - b(x_B)\right)^{1-\alpha}\left(\frac{\alpha}{z^* - a(x_A)} + (1-\alpha)\frac{f'(z^*)}{f(z^*) - b(x_B)}\right) = 0. \tag{6}$$

Given that $z^* = a(x^*)$ and $f'(z^*) = \frac{b'(x)}{a'(x)}$ we can rewrite the first order condition in (6) as:

$$(a(x^*) - a(x_A))^\alpha (b(x^*) - b(x_B))^{1-\alpha} \left(\alpha \frac{a'(x^*)}{a(x^*) - a(x_A)} + (1-\alpha) \frac{b'(x^*)}{b(x^*) - b(x_B)} \right) = 0. \quad (7)$$

Since $x^* \in (x_A, x_B)$ we know that $a(x^*) > a(x_A)$ and $b(x^*) > b(x_B)$. This implies that equation (7) is true if and only if:

$$\alpha a'(x) (b(x^*) - b(x_B)) + (1-\alpha) b'(x) (a(x^*) - a(x_A)) = 0.$$

Rearranging terms,

$$a(x^*) - a(x_A) = \alpha \left[a(x^*) - a(x_A) + \frac{a'(x^*)}{|b'(x^*)|} [b(x^*) - b(x_B)] \right], \quad (8)$$

given that $b'(x^*) \neq 0$ and $|b'(x^*)| = -b'(x^*)$. But equation (8) means that the surplus of agent A is the α share of the joint surplus measured in utils of A, i.e.,

$$a(x^*) - a(x_A) = \alpha JS_A(x^*). \quad (9)$$

Equivalently, and given that $a'(x^*) \neq 0$,

$$b(x^*) - b(x_B) = (1-\alpha) JS_B(x^*). \quad (10)$$

Controlling for utility units, therefore, the Nash Bargaining solution does in fact prescribe the $\alpha, 1-\alpha$ split!

3 Convexity of the Feasible Set and Bargainers Risk Aversion

As the reader may well be aware by now, the shapes of a and b cannot be directly interpreted as the attitudes of the bargainers towards risk. In Examples 1 and 2, where the function $a(x)$ is a direct utility function over x it is the case. A risk averse A-type bargainer will have $a''(x) < 0$, a risk neutral A-type bargainer will have $a''(x) = 0$ and, finally, a risk lover A-type bargainer will have $a''(x) > 0$. What about B-type bargainers?

In the example of money division, for instance, a B-type individual might have the usual utility function u_B defined on another variable y negatively correlated with x (for example, $y = 10 - x$). Assume $y'(x) < 0$. Then our one-issue bargaining problem takes $b(x) = u_B(y(x))$. Applying the chain rule:

1. $b'(x) = u'_B(y)y'(x) < 0$, and

2. $b''(x) = u_B''(y) [y'(x)]^2 + u_B'(y)y''(x)$.

It is easily seen how the convexity of b is not exactly equal to the convexity of u_B and viceversa. Although $u_B''(y) [y'(x)]^2 \leq 0$ for a risk averse individual, $b''(x)$ might be negative or not depending on the sign of $y''(x)$. In the money division problems as in Examples 1 and 2, $y''(x) = 0$, and therefore a B-type bargainer is (i) risk averse if $b''(x) < 0$, (ii) risk neutral if $b''(x) = 0$, and (iii) risk lover if $b''(x) > 0$. In more complex situations the relationship is not as straightforward, as we show in the next example.

Example 3. A Firm-Worker Bargaining over Wages

Assume the A-type bargainer is a worker that values money with some risk averse utility function on money. The firm's owner has utility over profit with a risk loving attitude, i.e., $u_B(y) = y^2$, where y is profit. Profit is obviously related to the salary paid to the worker, but not in a linear way. Assume $y = \sqrt{x} - x$, where x is a salary between 0 and 1. If we assume that the worker has the right to a subsidy equal to $\frac{1}{16}$ and that the firm could earn a profit of at most $\frac{9}{16}$ if he does not hire the current worker with whom he is about to bargain the salary, we have that (i) $b'(x) < 0$ and (ii) $b''(x) \leq 0$. Therefore, the one-issue bargaining problem looks like bargaining between two risk averse individuals in the simple money division problem, where the Nash product is concave on z (and on x, see next section for that) and the set of feasible utilities S is convex.

4 Convexity of the Feasible Set and concavity of the generalized Nash product as a function of x

We have shown that the convexity of S implies that the generalized Nash product is concave as a function of $z = a(x)$. Nevertheless, we are interested in the Nash bargaining solution written in terms of the issue x. This is why in many applications the generalized Nash product is maximized as a function of x. It happens that for one-issue settings, the first order condition for an interior solution is equivalent whether we maximize with respect to x or with respect to z. Unfortunately, one can be tempted to check the concavity of the generalized Nash product as a function of x, and not as a function of z. We now show that the fact that the generalized Nash product, expressed as a function of z, is concave in z does not automatically imply that the generalized Nash product, expressed as a function of x, is concave in x. The shape of $a(x)$ is very relevant.

If $G(z)$ denotes the generalized Nash product as a function of z, and $N(x)$ stands as the generalized Nash product as a function of x, it is clear that $G(a(x)) = N(x)$. Note that the order of composition of the functions G and a should be inverted in order to preserve concavity, but unfortunately, it is not the case here.[5] Applying the chain rule,

$$N'(x) = G'(a(x)) a'(x),$$

and that

$$N''(x) = G''(a(x)) \left[a'(x)\right]^2 + G'(a(x)) a''(x).$$

The assumption about convexity of S implies that $G(z)$ is concave and therefore $G''(a(x)) < 0$. This implies that $G''(a(x)) [a'(x)]^2 < 0$. Unfortunately, depending on the value of $G'(a(x)) a''(x)$ the function $N(x)$ could be concave or not.

Furthermore, it could be that the function $N(x)$ is concave in x but the set of feasible alternatives S is not convex. This would still guarantee a unique, interior solution of the maximization of $N(x)$ as a function of x, without the requirement of convexity. The intuition behind this fact is that there are some bargaining problems where the set of alternatives is not convex but still the non-symmetric Nash bargaining solution is interior and unique. To see how this could happen, recall that the condition for S to be convex, in terms of the primitives of the model, is that $f'(z) = \frac{b(x)}{a(x)} < 0$ and $f''(z) \leq 0$. The latter implies, in particular, as we saw in equation (3), that

$$b''(x) \leq b'(x) \frac{a''(x)}{a'(x)}. \tag{11}$$

It is easy to see that

$$N'(x) = N(x) \left[\frac{\alpha a'(x)}{a(x) - a(x_A)} + \frac{(1 - \alpha) b'(x)}{b(x) - b(x_B)} \right], \tag{12}$$

and that

$$N''(x) = N(x) \left[\alpha \frac{a''(x)}{a(x) - a(x_A)} + (1 - \alpha) \frac{b''(x)}{b(x) - b(x_B)} - \alpha (1 - \alpha) \left\{ \frac{a'(x)}{a(x) - a(x_A)} - \frac{b'(x)}{b(x) - b(x_B)} \right\}^2 \right]. \tag{13}$$

Hence, $N''(x) < 0$ if and only if

$$\alpha \frac{a''(x)}{a(x) - a(x_A)} + (1 - \alpha) \frac{b''(x)}{b(x) - b(x_B)} < \alpha (1 - \alpha) \left\{ \frac{a'(x)}{a(x) - a(x_A)} - \frac{b'(x)}{b(x) - b(x_B)} \right\}^2. \tag{14}$$

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[5]By preservation of concavity we mean the following. Take three functions f, g and h with compatible domains such that $f = g \circ h$. If h is concave and g is nondecreasing, then f is concave. Note that $N = G \circ a$, therefore what we call the inverted order, since G is the one concave, and a is nondecreasing.

If both $a''(x)$ and $b''(x)$ are positive the assumption of concavity with respect to x holds. But the assumption on concavity with respect to x does not imply that the set S of utilities is convex, although when the assumption of concavity holds the non-symmetric Nash bargaining solution x^* is unique. Consider the following example of money division.

Example 4. A money division problem where the generalized Nash product is concave in x but the set of feasible alternatives is not convex in $\Re_+^2$.

Consider, as in Examples 1 and 2 of the introduction, two individuals bargaining about the division of 10 units of money. If they are not able to reach an agreement, both agents obtain zero. Assume now that $u_A(y_A) = \sqrt{y_A}$ and $u_B(y_B) = y_B^2$ with $y_A + y_B = 10$. Fix $x = y_A$. Then, $a(x) = u_A(x) = \sqrt{x}$ and $b(x) = u_B(10 - x) = (10 - x)^2$. Furthermore, $x_A = 0$ and $x_B = 10$.

It is easy to check that condition (14), for concavity with respect to x to be true, becomes in this case

$$\alpha \frac{-1}{4x^2} + (1 - \alpha) \frac{2}{(10 - x)^2} < \alpha (1 - \alpha) \left[\frac{1}{2x} + \frac{2}{10 - x} \right]^2. \tag{15}$$

But

$$\alpha (1 - \alpha) \left[\frac{1}{2x} + \frac{2}{10 - x} \right]^2 \geq \alpha (1 - \alpha) \left[\frac{1}{4x^2} + \frac{4}{(10 - x)^2} \right],$$

and

$$(1 - \alpha) \left[\frac{\alpha}{4x^2} + \frac{4\alpha}{(10 - x)^2} \right] > (1 - \alpha) \frac{2}{(10 - x)^2},$$

for $\alpha \geq \frac{1}{2}$ and given that $(1 - \alpha) \frac{\alpha}{4x^2} > 0$. Therefore, if $\alpha \geq \frac{1}{2}$ condition (15) above is satisfied (strictly) and the function $N(x)$ is strictly concave.

Nevertheless, condition (3) guaranteeing convexity of S is not satisfied for all $x \in [0, 10]$. To guarantee convexity of S, the following condition is needed

$$2 \leq \frac{10 - x}{x},$$

which is true only for $x \leq \frac{10}{3}$.

5   Extensions to Multiple-issue Bargaining Settings

What if bargaining takes place over multiple issues? Clearly, the joint surplus has to be measured in terms of units of individual A or in terms of units of individual B as before. But since there are multiple issues, any of these issues could be taken as a reference (at the margin). We show now that at the Nash bargaining solution, it does not matter which issue is taken as a reference, since

at the margin they all give the same rate of exchange of units of A for units of B and viceversa. Let us now be more formal.

Assume for this section that bargaining takes place in terms of a vector of variables $x \in \Re^m$, where $m \geq 2$ is the number of issues that are relevant in the bargaining process. In the firm-worker setting, bargaining can take place in terms of both wage per hour and number of hours of labor, and therefore m would be equal to 2. Similar to before, individuals A and B, engaged in bargaining, have utilities $a(x)$ and $b(x)$, respectively, for each possible choice of x. For simplicity, assume that both $a(x)$ and $b(x)$ are continuous functions in $\Re$. Depending on the application, a compact set of $\Re^m$, denoted X, is available to bargainers if agreement is reached. The utility possibility frontier is therefore the set $F \in \Re^2$ defined as

$$F = \left\{ (a(x), b(x)) \in \Re^2, \text{ where } x \in X \right\}. \tag{16}$$

Assumptions for the m case

1. Both $a(x)$ and $b(x)$ are twice differentiable for all $x \in X$. Furthermore, for any agreement x and any of its issues x_i we have that if $\frac{\partial a(x)}{\partial x_i} > 0$ then $\frac{\partial b(x)}{\partial x_i} < 0$ and viceversa.

2. The disagreement point can be written as $d = (a(x_A), b(x_B))$, where x_A and x_B are shadow values of x at disagreement.

3. The generalized Nash product $(z - a(x_A))^\alpha (f(z) - b(x_B))^{1-\alpha}$ is a concave function on $z = a(x) \in Z$, where $Z = z \in \Re$ such that $z = a(x)$ and $x \in X^m$ and $f(z) = b(x)$ for x such that $a(x) = z$.

The assumption on the opposite signs of the partial derivatives with respect to the same issue is related to the concavity assumption. If both partial derivatives are of the same sign, say positive, it means that both individuals would like to agree on a higher value of x_i. Any agreement maximizing the generalized Nash product would set the value of x_i as high possible, and therefore there is no interior maximizer of the generalized Nash product. The same applies if one of the derivatives is null, the other being positive. Similarly, if at least one of the partial derivatives is negative and the other non positive, the maximizer of the generalized Nash product will fix x_i^* equal to the lowest value possible, and again there is no interior solution of the maximization problem.

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In this multiple-issue setting, we propose the following modifications of the joint surplus:

$$JS_A^i(x) = a(x) - a(x_A) - \frac{\frac{\partial a(x)}{\partial x_i}}{\frac{\partial b(x)}{\partial x_i}} \left[b(x) - b(x_B) \right],$$

and

$$JS_B^i(x) = b(x) - b(x_B) - \frac{\frac{\partial b(x)}{\partial x_i}}{\frac{\partial a(x)}{\partial x_i}} \left[a(x) - a(x_a) \right].$$

In words, $JS_A^i(x)$ is the joint surplus of A and B at the agreement x when measured in utils of A where the ith issue is taken as a reference. The joint surplus measured in utils of A may differ if we fix another issue as a reference, say j, as far as

$$\frac{\frac{\partial a(x)}{\partial x_i}}{\frac{\partial b(x)}{\partial x_i}} \neq \frac{\frac{\partial a(x)}{\partial x_j}}{\frac{\partial b(x)}{\partial x_j}}. \tag{17}$$

Proposition 2 *Take the multi-issue two-person bargaining procedure defined just above. If the assumptions for the m case hold, the nonsymmetric Nash bargaining solution $x^* = (x_i^*)_{i \in 1,...,m} \in X^m$ is interior to X^m and satisfies*

$$a(x^*) - a(x_A) = \alpha JS_A^i(x^*), \tag{18}$$

and

$$b(x^*) - b(x_B) = (1 - \alpha) JS_B^i(x^*), \tag{19}$$

for any issue i.

Proposition 2 implies in particular that the joint surplus measured in utils of A is the same independently of the issue that we take as a reference at the nonsymmetric Nash bargaining solution. The same can be said about the joint surplus measured in utils of B. This is not true in general for any agreement x of the domain X^m, but it has to be true for the nonsymmetric Nash bargaining solution x^* when the latter is interior to X^m. This implies very importantly that the rate of exchange of utils of A into utils of B at the non-symmetric Nash bargaining solution is the same no matter which of its issues is taken as a reference.

Proof of Proposition 2. Given the assumptions, there is an interior solution z^* to the generalized Nash product in the utility space and the first order as in equation 6 is necessary and sufficient. Unfortunately, we are also interested in the optimal value of the issues x^*, and

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not only on the optimal value of the utility z^*, where $z^* = f(x^*)$. Given that x is in this case a vector, we have that $f'(z)$ is equal to

$$\frac{df(z)}{dz} = \frac{\sum_{i=1}^{m} b_i(x)dx_i}{\sum_{i=1}^{m} a_i(x)dx_i}, \tag{20}$$

where $a_i(x)$ (resp. $b_i(x)$) is the partial derivative of $a(x)$ (resp. $b(x)$) with respect to the i-th element, x_i.

Consider the maximization of the generalized product as a function of x, and not as a function of $z = a(x)$. Formally, take an x^* that solves

$$\max_{x \in X} \left(a(x) - a(x_A) \right)^{\alpha} \left(b(x) - b(x_B) \right)^{1-\alpha}. \tag{21}$$

If x^* satisfies the first order solution of this problem it has to be true, for each issue x_i^*

$$\alpha a_i(x^*) \left(b(x^*) - b(x_B) \right) + (1 - \alpha) b_i(x^*) \left(a(x^*) - a(x_A) \right) = 0, \tag{22}$$

which is the same as

$$a(x^*) - a(x_A) = \alpha \left[a(x) - a(x_A) - \frac{a_i(x^*)}{b_i(x^*)} \left[b(x) - b(x_B) \right] \right],$$

for any issue i. It remains to check that condition 22, for each issue i, implies the first order condition of the maximization problem in the utility space. It is easy to see that equation 22, for each issue i, implies that

$$\alpha a_i(x^*)dx_i \left(b(x^*) - b(x_B) \right) + (1 - \alpha) b_i(x^*)dx_i \left(a(x^*) - a(x_A) \right) = 0,$$

for each i and therefore

$$\alpha \sum_{i=1}^{m} a_i(x^*)dx_i \left(b(x^*) - b(x_B) \right) + (1 - \alpha) \sum_{i=1}^{m} b_i(x^*)dx_i \left(a(x^*) - a(x_A) \right) = 0,$$

which is the same as

$$\alpha \left(b(x^*) - b(x_B) \right) + (1 - \alpha) \frac{\sum_{i=1}^{m} b_i(x^*)dx_i}{\sum_{i=1}^{m} a_i(x^*)dx_i} \left(a(x^*) - a(x_A) \right) = 0. \tag{23}$$

Fix $z^* = a(x^*)$ and $f(z^*) = b(x^*)$ and recall that $f'(z) = \frac{\sum_{i=1}^{m} b_i(x)dx_i}{\sum_{i=1}^{m} a_i(x)dx_i}$. Rewriting (23) we obtain

$$\alpha \left(f(z^*) - b(x_B) \right) + (1 - \alpha) f'(z^*) \left(z^* - a(x_A) \right) = 0, \tag{24}$$

i.e., $z^* = a(x^*)$ satisfies the first order condition of 5 and it is therefore the non-symmetric Nash bargaining solution under Assumptions 1 to 3 of the multi-issue setting. ∎

6   Final Comments

We have shown that the solution to the Nash Bargaining problem takes the form of a linear split of the joint surplus even when agents differ in preferences, provided the joint surplus is expressed in terms of similar utility units. The necessity of spelling out the joint surplus in compatible utility units is also crucial when comparing the surplus, or the total welfare, of economies with agents differing in utility. An important example is when comparing the joint surplus of economies calibrated to different risk aversion parameters.

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Mathematical Appendix

Proposition 1. Let $f(z)$ be as defined above, with $z = a(x)$, for any $x \in [x_A, x_B]$. If $f'(z) < 0$ and $f''(z) \leq 0$, then the generalized Nash product, say $G(z)$, as a function of z is concave in $[a(x_A), a(x_B)]$.

Proof. By definition, $G(z) = (z - z_A)^\alpha (f(z) - f(z_B))^{1-\alpha}$, where $z_A = a(x_A)$ and $z_B = a(x_B)$. Taking derivatives,

$$G'(z) = G(z) \left[\frac{\alpha}{z - z_A} + \frac{(1 - \alpha) f'(z)}{f(z) - f(z_B)} \right],$$

and

$$G''(z) = G(z) \left\{ \left[\frac{\alpha}{z - z_A} + \frac{(1 - \alpha) f'(z)}{f(z) - f(z_B)} \right]^2 - \frac{\alpha}{(z - z_A)^2} - \frac{(1 - \alpha) [f'(z)]^2}{[f(z) - f(z_B)]^2} + \frac{(1 - \alpha) f''(z)}{f(z) - f(z_B)} \right\}. \tag{25}$$

Since $f'(z) < 0$ we know that

$$\left[\frac{\alpha}{z - z_A} + \frac{(1 - \alpha) f'(z)}{f(z) - f(z_B)} \right]^2 < \frac{\alpha}{(z - z_A)^2} + \frac{(1 - \alpha) [f'(z)]^2}{[f(z) - f(z_B)]^2}.$$

Since $f''(z) \leq 0$

$$\frac{(1 - \alpha) f''(z)}{f(z) - f(z_B)} \leq 0.$$

These last two things together imply, by (25), that $G''(z) < 0$. Hence, the generalized Nash product is a (strict) concave function on $z = a(x)$ if the set of utilities S is convex. ∎