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CESARO SUMMABILITY
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JOINT WORK WITH:
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DEFINITIONS

Let \( S_n(f,\xi)(z) = \sum_{k=0}^{n} c_k(z - \xi)^k, n = 0, 1, 2, \ldots, \)
be the sequence of partial sums of the Taylor development of
\( f(z) = \sum_{k=0}^{\infty} c_k(z - \xi)^k \) in the open disk \( D(\xi, R), 0 < R < \infty. \)

If \( K \subset \mathbb{C} \) with \( K \cap D(\xi, R) = \emptyset, \) then we say that the function \( f \) (or the Taylor series \( \sum_{k=0}^{\infty} c_k(z - \xi)^k \)) is universal with respect to \( K \) (or \( K \)-universal), if for every function \( h: K \rightarrow \mathbb{C} \) continuous on \( K \) and holomorphic in the interior of \( K \) (if \( K^o \neq \emptyset \)), there exists a sequence of natural numbers \( \{k_n\}, n \in \mathbb{N} \), such that the subsequence \( S_{k_n}(f,\xi)(z) \) converges to \( h(z) \) uniformly on \( K. \)

Let \( f \) be a \( K \)-universal function, \( z_0 \in K \) and \( a > -1. \) We say that \( f \) is \((C,a)\)-summable at \( z_0 \) to a finite sum \( s, \) if we have that:
\[ \sigma_n^a(f,\xi)(z_0) \xrightarrow{n \to \infty} s, \]
for the sequence of \( a \)-Cesaro means.
\[ \sigma_n^a(f, \xi)(z_0) = \sum_{k=0}^{a} \frac{A_{n-k}^a}{A_n} c_k (z_0 - \xi)^k, \]

where \( A_m^a = \frac{(a+1)(a+2)\cdots(a+m)}{m!} \).

**EXAMPLES**

1. Let \( \Omega \neq \emptyset \) be a simply connected domain in \( \mathbb{C} \) and \( \xi \in \Omega \). We denote by \( H(\Omega) \) the set of holomorphic functions on \( \Omega \), endowed with the topology of uniform convergence on compact sets. We also denote by \( K = \text{c.c.c.} \) any subset \( K \) of \( \mathbb{C} \), which is Compact with Connected Complement. Then:

   (a) The following class of universal functions

   \[ U_f(\Omega, \xi) = \left\{ f \in H(\Omega) : f(z) \text{ is } K\text{-universal}, \forall K = \text{c.c.c.}, K \cap \overline{\Omega} = \emptyset \right\} \]

   is a \( G_\delta \)-dense subset of \( H(\Omega) \). These classes studied, initially in the case where \( \Omega = D(0,1) \), by Luh and independently by Chui and Parnes – in the early of 70’s.

   Universal functions in \( U_f(D,0) \) have as natural boundary the unit circle \( T = \partial D(0,1) = \{ z \in \mathbb{C} : |z| = 1 \} \), but they may be smooth on \( T \).

   There are functions in this class, which have the highest order of regularity and summability at the boundary of \( D(0,1) \); their Taylor developments are \((C,a)\)-summable for every \( a > -1 \), at any point of \( T \).

   (b) In 1996, V. Nestoridis strengthened the results of Luh and Chui – Parnes by allowing to the compact set \( K \) to meet \( T \) and the universal approximation was obtained on the boundary of \( D(0,1) \) as well. Then, the following new class of universal functions, defined by
\[ U(\Omega,\xi) = \{ f \in H(\Omega) : f(z) = K - universal, \forall K = c.c.c., K \cap \Omega = \emptyset \} , \]

is again a \( G_\delta \)–dense subset of \( H(\Omega) \). It is a proper subset of the previous class and the universal functions in \( U(\Omega,\xi) \) have several wild properties. For instance, when \( \Omega = D(0,1) \), their Taylor coefficients cannot have polynomial growth and so their Taylor development cannot be \((C,a)\)−summable, for every \( a > -1 \), at any point of \( T \). This argument gives the \((C,a)\)−non-summability simultaneously for all points of \( T \) and it is not possible to have the result for some such points. There is a different proof of this fact, based on an extension of Rogosinski’s formula, which enables one to distinguish some points on \( T \) and so we shall follow this method here.

2. There are unbounded non-simply connected domains \( \Omega \) in \( \mathbb{C} \), where universal Taylor series exist. On the other hand, if \( \Omega \) is a bounded annulus the class of universal functions is empty. This leads us to look for weaker approximation results on arbitrary planar domains.

   From their main results we need the following:
   Let \( \Omega \neq \mathbb{C} \) be an arbitrary planar domain, \( \xi \in \Omega \),
   \[ R = \text{dist}(\xi, \partial \Omega) > 0 , \]
   \[ J(\Omega,\xi) = C(\xi, R) \cap \partial \Omega , \]
   and \( C(\xi, R) = \{ z \in \mathbb{C} : |z - \xi| = R \} \).

   Then, we define the following class of universal functions:

   \[ B(\Omega,\xi) = \{ f \in H(\Omega) : f(z) = K - universal, \forall K = c.c.c., K \subset J(\Omega,\xi) \} \]

   This class is again a \( G_\delta \)–dense subset of \( H(\Omega) \).

   If \( J(\Omega,\xi) \) contains an arc of \( C(\xi, R) \) with strictly positive opening, then the Taylor coefficients of any \( f \in B(\Omega,\xi) \) can not have polynomial growth and so the
Taylor development of \( f \), with center \( \xi \), can not be \((C,a)\)–summable for any \( z \in J(\Omega, \xi) \) and every \( a > -1 \).

**CESARO SUMMABILITY IN THE GENERAL CASE**

If \( f(z) = \sum_{k=0}^{\infty} c_k (z - \xi)^k \) \((z \in D(\xi, R), \ 0 < R < \infty)\) is a universal function with respect some \( K \subset \mathbb{C}, \ K \cap D(\xi, R) = \emptyset \) and \( z_0 \in K \), then the question is under what conditions on \( K \) and \( z_0 \) the above Taylor series is not \((C,a)\)–summable at \( z_0 \) to a finite sum \((a > -1)\)?

It is well known that for \( z_0 \) with \( |z_0 - \xi| > R \) the above series is not \((C,a)\)–summable at \( z_0 \), for every \( a > -1 \). So, from now on, we assume that \( z_0 \in K \cap C(\xi, R) \). Moreover, without loss of generality we may assume that:

\[ D(\xi, R) = D(0, 1), \ K \subset T \quad \text{and} \quad z_0 = 1 \in K \cap T. \]

We recall now the classical formula of Rogosinski:

Rogosinski’s formula: \( z_n \in \mathbb{C}, \ n \in \mathbb{N}: 1 - z_n = O(\frac{1}{n}) \) and \( \sum_{k=0}^{\infty} c_k \) is \((C,1)\)–summable to \( s \in \mathbb{C} \). If \( S_n(z) = \sum_{k=0}^{n} c_k z^k \), then we have:

\[ S_n(z_n) - s - (S_n(1) - s)z_n^{n} \xrightarrow[n \to \infty]{} 0. \]

From this we obtain easily the following:

**Proposition 1**: Let \( K \subset T, \ 1 \in K \) and \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) be a \( K \)-universal function. If there exists a sequence \( \{z_n\} \subset K \) such that:
(i) \( 1 - z_n = O\left(\frac{1}{n}\right) \) and

(ii) \( \{z^n\} \) has not 1 as a limit point,

then the series \( \sum_{k=0}^{\infty} c_k \) is not (C,1) – summable.

**Proof:**

Suppose that the series \( \sum_{k=0}^{\infty} c_k \) is (C,1) – summable to \( s \in \mathbb{R} \). Put \( h(z) = s + 1 \), \( z \in K \). Since \( f \) is \( K \)-universal there exists a sequence \( \{k_n\} \subset \mathbb{N} \), such that \( S_{k_n}(z) \xrightarrow{n \to \infty} h(z) = s + 1 \) uniformly on \( K \). Then, Rogosinski’s formula implies that \( z_{k_n}^{k_n} \xrightarrow{n \to \infty} 1 \), a contradiction by (ii).

Since we used Rogosinski’s formula, condition (i) in Prop. 1 is necessary.

The question is if condition (ii) is necessary, or we can replace it by a weaker one.

Of course, there are stronger conditions which imply conditions (i) and especially (ii) and we give two such examples.

(C.1) If \( \hat{K} = \{t \in [-\pi, \pi] : z = e^{it} \in K\} \), then 0 is a right (resp. left) density point of \( \hat{K} \).

(C.1) implies also the following condition (C.2), which in turn implies (i) and (ii).

(C.2) \( \exists \) \( z_n = e^{it_n} \in K \) such that (a) \( n|1 - z_n| \xrightarrow{n \to \infty} c > 0 \) and (b) \( \exists n_0 \in \mathbb{N} : n|t_n| < 2\pi \), \( \forall n \geq n_0 \).

**Remark:** As we shall see, (a) of (C.2) guarantees the (C,1) – non – summability of the series \( \sum_{k=0}^{\infty} c_k \). Thus, for example, if \( z_n = \exp\left(\frac{2\pi i}{n}\right) \in K \), then \( n|1 - z_n| \xrightarrow{n \to \infty} 2\pi > 0 \) and so \( \sum_{k=0}^{\infty} c_k \) is
not (C,1) - summable. Moreover, \( z^n = \exp(2\pi i) = 1 \), which implies that the answer to the previous question is negative and condition (ii) is not necessary.

Now, if we examine more carefully the above conditions and especially (a) of (C.2), we see that the following condition (C*) is a (weaker) conclusion of (a) of (C.2) and it is sufficient for our purposes.

Let \( \hat{K} = \{ t \in [-\pi, \pi] : z = e^t \in K \} \). We assume that:

\[
\forall N \in \mathbb{N}, \ N \geq 1 \text{ and } \forall S \subseteq \mathbb{N}, \ S \text{ infinite, } \]

(C*) \[
\exists a_N = a_N(S), \ b_N = b_N(S) \quad \text{with } 0 < a_N < b_N < \frac{1}{N}, \]

\[
\exists \{ k_n \} \subseteq S \quad \text{and} \quad \exists \{ t_n \} \subseteq \hat{K} \quad \text{such that} \]

\[
k_n | t_n | \in (a_N, b_N), \ \text{for every } n=1,2,3,\ldots.
\]

Then, we have the following:

**Theorem:** Let \( K \subset T, \ 1 \in K \) and \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) be a K-universal function. If K satisfies condition (C*), then the series \( \sum_{k=0}^{\infty} c_k z^k \) is not \((C,a)\)–summable for every \( a > -1 \).

It is sufficient to prove this Theorem for \( a = m, \) an arbitrary natural number, since, as it is well known, if a series is \((C,m)\)-summable, then it is \((C,a)\)–summable, for every \( a \geq m \).

(So, the \((C,m)\)-non-summability, for every \( m \in \mathbb{N} \), implies the \((C,a)\)-non-summability for every \( a > -1 \)).

We shall not give the proof here, which is much more complicated than that of Prop. 1, because of the complexity
of condition \((C^*)\) and the use of the following extension of Rogosinski’s formula:

\[
\sum_{k=0}^{\infty} c_k = (C,m)\text{-summable for some natural number } m, \text{ to a finite sum } s \text{ and let }
\]
\[
S \subset \mathbb{N}, S \text{ infinite and } \forall n \in S, \forall j=1,2,\ldots,m,
\exists z_{j,n} \in \mathbb{N} \text{ such that }
\]
\[
\lim_{n \to \infty} n (1 - z_{j,n}) = \zeta_j, \text{ with } 0 < |\zeta_1| < |\zeta_2| < \cdots < |\zeta_m|.
\]

Then,
\[
\forall n \in S, \forall j=1,2,\ldots,m,
\exists \lambda_{j,n} \in \mathbb{N} \text{ such that }
\]
\[
\sum_{j=1}^{m} \left[ \frac{1}{z_{j,n}^n} \left( S_n(z_{j,n}) - s \right) - (S_n(1) - s) \right] \lambda_{j,n} \xrightarrow{n \in S, n \to \infty} 0 \quad \text{and}
\]
\[
\lim_{n \in S, n \to \infty} \lambda_{j,n} = \frac{1}{\zeta_j \prod_{k=j}^{m} (\zeta_j - \zeta_k)}.
\]

On the other hand, it seems that the situation is quit different in the case where the set \(K\) is only a finite subset of \(\mathbb{N}\), say \(K = \{z_1, z_2, \ldots, z_n\}\), with \(|z_j| \geq 1, j=1,2,\ldots,n\).

Then, there are \(K\)-universal functions \(f(z) = \sum_{k=0}^{\infty} c_k z^k\) in \(D(0,1)\), which are \((C,a)\)-summable at every point \(z_j \in K \cap T\) and every \(a \geq 1\).

For example, let \(K = \{1\}\) and let \(\{q_n\}, n=0,1,2,\ldots,\) be an enumeration of the rational points of \(\mathbb{N}\), i.e. \(q_n \in \mathbb{N} \setminus \mathbb{N} + \mathbb{N} i\). We choose a sequence \(\{\lambda_n\}\) with \(\lambda_0 = 0, \lambda_{n+1} > \lambda_n + m\), where \(m\) is some suitable natural number and moreover
\[
\frac{|q_0 + q_1 + \cdots + q_n|}{\lambda_n + 1} < \frac{1}{n+1}, \text{ for every natural number } n.
\]
Then we define
\[ f(z) = \sum_{n=0}^{\infty} (q_n - q_n z^2 + q_n z^3) z^{\lambda_n}, \]

which, for \( m \geq 2 \), has the following properties:

(i) \( S_{\lambda_n}(1) = q_n, \ n=0,1,2,\ldots \)

(ii) \[ \frac{|S_o + S_1 + \cdots + S_k|}{k+1} < \frac{1}{n+1}, \]

for every \( k \in \mathbb{N} \) with \( \lambda_n \leq k < \lambda_{n+1} \) and every \( n=0,1,2,\ldots \).

Since \( \{q_n\} \) is dense in \( \mathbb{D} \cup \{\infty\} \), it follows from (i) that \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) is a K–universal function with radius of convergence \( R \leq 1 \). By (ii) we have that the Taylor series of \( f \) is (C,1)–summable, at \( z_0=1 \), to the finite sum 0 and so \( R=1 \).

Moreover, for every \( a \geq 1 \) the series \( \sum_{k=0}^{\infty} c_k \) is (C, a)–summable.

Further, if \( m \geq 4 \), we can consider the function

\[ f(z) = \sum_{n=0}^{\infty} (q_n - q_n z^2 + q_n z^3) z^{\lambda_n}, \]

which is K–universal and has a K–universal and (C,1)–summable, at \( z_0=1 \), derivative and so on.

In the general case where \( K=\{z_1, z_2, \ldots, z_n\} \), with \( |z_j| \geq 1 \), \( j=1,2,\ldots,n \), the construction of K–universal functions \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) in \( D(0,1) \), which are (C, a)–summable at every point \( z_j \in K \cap T \) and every \( a \geq 1 \), is similar to the above example. In fact, there are two ways to construct such examples. One, by using the following result of Dirichlet:

If \( \{z_1, z_2, \ldots, z_m\} \subset T \), then \( (1,1,\ldots,1) \) is a limit point of \( \{(z_1^n, z_2^n, \ldots, z_m^n), n = 1,2,3,\ldots\} \).
and second without this, but with some more complicated technique.