A brief review on Brennan's conjecture

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1. Basic notation

$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, The extened complex plane.

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, The unit disc.

$\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, The unit circle.

$\mathbb{D}_e = \{z \in \mathbb{C}_\infty : |z| > 1\}$, The exterior of the unit disc.

$\mathcal{H}(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C},\ f \text{ analytic}\}$.

We consider $f : \mathbb{D} \to \mathbb{C}$ analytic and univalent.

Theorem (Koebe)

Let $f : \mathbb{D} \to \mathbb{C}$ univalent, then for $z \in \mathbb{D}$

$$(a) \quad |f'(0)|\frac{|z|}{(1+|z|)^2} \le |f(z) - f(0)| \le |f'(0)|\frac{|z|}{(1-|z|)^2},$$

$$(b) \quad |f'(0)|\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le |f'(0)|\frac{1+|z|}{(1-|z|)^3}.$$

Classes of analytic functions

- $S = \{f \in \mathcal{H}(\mathbb{D}) : f \text{ univalent and } f(0) = f'(0) - 1 = 0\}$

Example 1. The Koebe function

$$k(z) = \frac{z}{(1-z)^2}, \quad z \in \mathbb{D},$$

with range $k(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$.

- $S_b = \{f \in \mathcal{H}(\mathbb{D}) : f \text{ univalent and bounded with } f(0) = 0\}$

- $\Sigma = \{f : \mathbb{D}_e \to \mathbb{C}_\infty : f \text{ univalent with } f(z) = z + b_0 + \Sigma_{n=1}^{\infty} b_n z^{-n}\}$

Example 2.

$$f(z) = z + \frac{1}{z}, \quad z \in \mathbb{D}_e,$$

with range $f(\mathbb{D}_e) = \mathbb{C}_\infty \setminus [-2, 2]$.

Brennan's Conjecture

Let G be a simply connected planar domain and $g : G \to \mathbb{D}$ conformal. The conjecture is that, for all such G and g,

$$\int_G |g'|^p dA < \infty \tag{1}$$

holds for $4/3 < p < 4$. It is an easy consequence of the Koebe distortion theorem that (1) holds when $4/3 < p < 3$. Brennan (1978) extended this to $4/3 < p < 3 + \delta$ where $\delta > 0$, and conjectured that (1) holds for $4/3 < p < 4$.

The roots of the conjecture can be found in the work of T. A. Metzger (1973) in connection with a question in approximation theory.

Brennan's conjecture can also be formulated for analytic and univalent maps of $\mathbb{D}$ by setting $f = g^{-1}$. Thus the conjecture becomes:

$$\int_{\mathbb{D}} |f'|^p dA < \infty \tag{2}$$

for $-2 < p < 2/3$ and for all univalent $f : \mathbb{D} \to \mathbb{C}$.

This is known for the range $-1,78 \lesssim p < 2/3$ (S. Shimorin (2005)).

Example 3.

(i) The Koebe function $k(z) : \mathbb{D} \to \mathbb{C}$, where

$$k(z) = \frac{z}{(1-z)^2}, \quad z \in \mathbb{D},$$

shows that the range of p cannot be extended outside $(-2, 2/3)$.

(ii) The conjecture holds for close to convex domains. This is due to B. Dahlberg and J. Lewis (1978).

Let $f : \mathbb{D} \to \mathbb{C}$ univalent and $t \in \mathbb{R}$. The integral means of f' is

$$M_t(r, f') = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^t d\theta, \quad (0 \le r < 1). \tag{3}$$

Define

$$\beta_f(t) = \limsup_{r \to 1} \frac{\log M_t(r, f')}{\log \frac{1}{1-r}}, \quad (t \in \mathbb{R}). \tag{4}$$

Thus $\beta_f(t)$ is the smallest number such that

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^t d\theta = O(\frac{1}{(1-r)^{\beta_f(t)+\varepsilon}}), \quad (r \to 1) \tag{5}$$

for every $\varepsilon > 0$. The function β_f is called the integral means spectrum of f.

Example 4 (Pommerenke). Let $f(\mathbb{D})$ be a heart-shaped domain with an inward-pointing cusp and an outward-pointing cusp. Then

$$\beta_f(t) = \begin{cases} |t| - 1, & \text{for } |t| \geq 1 \\ 0, & \text{for } |t| < 1. \end{cases} \tag{6}$$

Proposition (3.1)

The function β_f is continuous and convex in $\mathbb{R}$. Moreover if the domain $G = f(\mathbb{D})$ is bounded then

(i) $\beta_f(t \pm s) \leq \beta_f(t) + s$, *for* $t \neq 0, s \geq 0$,

(ii) $\beta_f(t) \leq t - 1$, *for* $t \geq 2$.

For $-\infty < t < +\infty$, the universal integral means spectrum of bounded univalent functions is defined as follows:

$$B_{S_b}(t) = \sup\{\beta_f(t) : f \text{ is bounded and univalent in } \mathbb{D}\}.$$

Theorem (Properties of B_{S_b})

The following hold:

(i) $B_{S_b}(t)$ *is a convex function,*

(ii) $B_{S_b}(t) = t - 1$ *for* $t \geq 2$, [*Pommerenke* (*1999*)]

(iii) $B_{S_b}(t) = t - 1 + O((t-2))^2$ *as* $t \to 2$, [*Jones and Makarov* (*1995*)],

(iv) $B_{S_b}(t) \geq 0.117t^2$, *for small* $|t|$, [*Makarov* (*1986*) *and Rohde* (*1989*)],

(v) $B_{S_b}(t) \geq t^2/5$, *for* $0 < t \leq \frac{2}{5}$, [*Kayumov* (*2006*)].

Remark. The following are equivalent:

(i) Brennan's conjecture is true,

(ii) $B_{S_b}(-2) = 1$,

(iii) $B_{S_b}(t) = |t| - 1$, for $t \leq -2$.

$(i) \Leftrightarrow (ii)$. Assume that the conjecture is true. Then because the integral means $M_t(r, f')$ are non decreasing in r,

$$\int_0^{2\pi} |f'(re^{i\theta})|^t d\theta \leq \frac{1}{(1-r)} \int_{\{r<|z|<1\}} |f'(z)|^t dA(z)$$

and $B_{S_b}(t) \leq 1$. Therefore by continuity $B_{S_b}(-2) \leq 1$ and since $B_{S_b}(t) \geq |t| - 1$ for $|t| \geq 1$ by the heart-shaped example, we have

$$B_{S_b}(-2) = 1.$$

Conversely assume $B_{S_b}(-2) = 1$ and thus $B_S(-2) = 1$. Let $-2 < t < 0$. Then $B_S(t) < \alpha < 1$ by convexity, so that for all $f \in S$

$$\int_0^{2\pi} |f'(re^{i\theta})|^t d\theta \leq C \frac{1}{(1-r)^{\alpha}}.$$

Then an application of Fubini's theorem implies Brennan's conjecture.

$(ii) \Leftrightarrow (iii)$. Since

$$|t| - 1 \leq B_{S_b}(t) \leq B_{S_b}(-2) + |t| - 2, \quad \text{for } t \leq -2$$

the equivalence of (ii) and (iii) is obvious.

Towards this direction L. Carleson and N. G. Makarov (1994) proved (iii) for large $|t|$, namely:

Theorem (Carleson and Makarov)

There exists a $t_0 \leq -2$ *such that*

$$B_{S_b}(t) = |t| - 1, \quad \textit{for} \quad t \leq t_0.$$

In (1999) N. G. Makarov gave a complete description of the functions that may occur as $\beta_f(t)$.

Theorem (Makarov)

Let $\beta : \mathbb{R} \to [0, +\infty)$ be convex. Then there exists a bounded univalent function f such that $\beta = \beta_f$, if and only if:

(i) $\beta(t) \leq B_{S_b}(t)$ *for* $-\infty < t < +\infty$,

(ii) $|\beta'(t \pm 0)| \leq (\beta(t) + 1)/|t|$ *for* $t \neq 0$.

In (1996) P. Kraetzer using computational methods proposed the following conjecture.

Kraetzer conjecture: $B_{S_b}(t) = t^2/4$, for $|t| \leq 2$.

This would imply that

$$B_{S_b}(t) = \begin{cases} t^2/4, & \text{for } |t| \leq 2 \\ |t| - 1, & \text{for } |t| > 2. \end{cases}$$

Kraetzer's conjecture reduces to Brennan's conjecture for $t \leq -2$ and moreover states that

$$B_{S_b}(t) = t - 1 + (t-2)^2/4$$

which is in agreement with the known properties of B_{S_b}. Furthermore $t^2/4$ is the only polynomial of degree at most 3 that satisfies the known values:

$$B_{S_b}(0) = B'_{S_b}(0) = 0, \;\; B_{S_b}(2) = B'_{S_b}(2) = 1.$$

For the other classes of functions i.e S and Σ, we can define $\beta_f(t)$ in an analogous way. Thus

$$B_S(t) = \sup\{\beta_f(t) : f \in S\}$$

and

$$B_\Sigma(t) = \sup\{\beta_f(t) : f \in \Sigma\}.$$

Theorem (3.1)

We have

(i) $B_\Sigma(t) = B_{S_b}(t)$, *for* $t \in \mathbb{R}$, [*Kraetzer (1995)*],

(ii) $B_S(t) = \max\{B_{S_b}(t), 3t-1\}$, [*Makarov (1999)*],

(iii) $B_S(t) = 3t-1$, *for* $2/5 \leq t < \infty$, [*Feng and MacGregor (1976)*].

Now since for univalent f the derivative f' is never zero in $\mathbb{D}$, we can define $(f'(z))^\tau$ for arbitrary $\tau \in \mathbb{C}$ and thus the notions of $\beta_f(\tau)$ and $B(\tau)$ are well defined.

Theorem (3.2)

We have:

- (i) $B_{S_b}(\tau) = B_\Sigma(\tau)$, *for* $\tau \in \mathbb{C}$, [*Hedenmalm and Sola* (*2008*)],
- (ii) $B_S(\tau) = B_\Sigma(\tau)$, *for* $Re\,\tau \leq 0$, [*Binder* (*1998*)],
- (iii) $B_S(\tau) = \max\{B_\Sigma(\tau), |\tau| + 2Re\,\tau - 1\}$, *for* $Re\,\tau > 0$ [*Binder* (*1998*)],
- (iv) $B_{S_b}(2-\tau) \leq 1 - Re\,\tau + (9e^2/2 + o(1))|\tau|^2 \log|\tau|$, *as* $|\tau| \to 0$, [*Baranov and Hedenmalm* (*2006*)].

The Brennan conjecture in the complex case as suggested by J. Becker and Ch. Pommerenke (1987) is whether

$$B_{S_b}(\tau) = 1, \quad \text{for all} \quad |\tau| = 2.$$

For the universal integral means spectrum I. A. Binder conjectured:

$$B_{S_b}(\tau) = \begin{cases} |\tau|^2/4, & \text{for } |\tau| \le 2 \\ |\tau| - 1, & \text{for } |\tau| > 2. \end{cases}$$

To this direction I. A. Binder in (2009) proved the following theorem:

Theorem (Binder)

For each θ, $0 \le \theta < 2\pi$ there exists $T_\theta > 0$ such that

$$B_{S_b}(te^{i\theta}) = t - 1, \quad \textit{for} \quad t \ge T_\theta.$$

The Bergman space A^p_α

For $0 < p < +\infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}(\mathbb{D})$ with the property

$$\| f\|^p_{p,\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$.

For $1 \leq p < \infty$ and $\alpha > -1$ fixed, A^p_α is a Banach space and for $p = 2$, A^2_α is a Hilbert space with inner product given by

$$\langle f, g\rangle = (\alpha+1)\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha dA(z).$$

The reproducing kernel of A^2_α at the point $a \in \mathbb{D}$ is

$$K_a(z) = \frac{1}{(1-\overline{a}z)^{2+\alpha}}, \quad z \in \mathbb{D}.$$

Thus for every $f \in A^2_\alpha$ and every $a \in \mathbb{D}$ we have

$$\langle f, K_a \rangle = f(a).$$

For $\alpha = 0$, we get the standard unweighted Bergman space A^p.

A^p_α-Carleson measures

A finite measure μ in $\mathbb{D}$ is called $A^p_\alpha-$Carleson measure if the inclusion map $i(f) = f : A^p_\alpha \to L^p(\mathbb{D}, d\mu)$ is continuous.
Equivalently if the inequality

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha dA(z)$$

holds for every $f \in A^p_\alpha$.

Theorem (A^p_α-Carleson measures)

The following are equivalent:

(i) *The measure μ is an $A^p_\alpha-$Carleson measure.*

(ii) $\int_{\mathbb{D}} \frac{d\mu(z)}{|1-\bar{\lambda}z|^\gamma} \leq \frac{C(\mu,\gamma)}{(1-|\lambda|^2)^{\gamma-\alpha-2}}$, *for some $\gamma > \alpha+2$ and any $\lambda \in \mathbb{D}$.*

Integral means spectrum revised

The integral means spectrum admits an easy description in terms of weighted Bergman spaces A^2_α. Namely if $(f')^{t/2} \in A^2_\alpha$ then

$$\beta_f(t) \le \alpha + 1.$$

On the other hand if $\beta_f(t) \le \beta_0$ then

$$(f')^{t/2} \in A^2_\alpha$$

for any $\alpha > \beta_0 - 1$. Therefore

$$\beta_f(t) = \inf\{\alpha > 0 : (f')^{t/2} \in A^2_{\alpha-1}\}. \tag{7}$$

Shimorin's approach (2005)

Let $\varphi : \mathbb{D} \to \mathbb{D}$, with $\varphi \in S_b$. We define the weighted composition operator W_φ^t on $\mathcal{H}(\mathbb{D})$ as follows:

$$W_\varphi^t(f)(z) = (\varphi'(z))^t f(\varphi(z)), \quad z \in \mathbb{D}.$$

Theorem (Boundedness)

Let φ be a univalent self-map of $\mathbb{D}$. Then the following are equivalent:

(i) *W_φ^t is bounded on A_α^p.*

(ii) *The measure μ defined as*

$$\mu(E) := \int_{\varphi^{-1}(E)} |\varphi'(z)|^{pt}(1-|z|^2)^\alpha dA(z)$$

is an A_α^p−Carleson measure.

Observe that

$$W_\varphi^t \circ W_\psi^t = W_{\psi\circ\varphi}^t, \quad \text{(semigroup property)}.$$

S. Shimorin was interested in the restriction of $W_\varphi^{t/2}$ on the weighted Bergman spaces A_α^2. He introduced the following functions:

$$\alpha_\varphi(t) =: \inf\{\alpha > 0 : W_\varphi^{t/2} \text{ is bounded on } A_{\alpha-1}^2\} \tag{8}$$

and

$$A(t) := \sup_\varphi \alpha_\varphi(t). \tag{9}$$

By applying the operator $W_\varphi^{t/2}$ to the constant function 1 it is easy to see that

$$\beta_\varphi(t) \le \alpha_\varphi(t) \quad \text{and} \quad B_{S_b}(t) \le A(t).$$

The functions $\alpha_\varphi(t)$ and $A(t)$ have the following properties:

Proposition (Properties of $\alpha_\varphi(t)$ and $A(t)$)

The following holds:

(i) *The functions $\alpha_\varphi(t)$ and $A(t)$ are both convex.*

(ii) $\alpha_\varphi(t \pm r) \leq \alpha_\varphi(t) + r$, *for* $r \geq 0$ *and* $t \in \mathbb{R}$.

(iii) $|A(t_1) - A(t_2)| \leq |t_1 - t_2|$.

Since W^1_φ is bounded on A^2, we have

$$A(2) \leq 1,$$

and thus

$$A(t) = A(t-2+2) \leq t-2+A(2) \leq t-1, \quad \text{for} \quad t \geq 2.$$

Since $A(t) \geq B_{S_b}(t) = t-1$, for $t \geq 2$, we have

$$A(t) = t-1, \quad \text{for} \quad t \geq 2.$$

Theorem (Shimorin (2005))

Let t_0 be the critical value in Carleson-Makarov theorem then

$$A(t) = |t|-1, \quad \textit{for} \quad t \leq t_0.$$

Remark. According to Shimorin's theorem, the validity of Brennan's conjecture is equivalent to the fact that

$$A(-2) = 1,$$

or to the property

$$W_\varphi^t \text{ is bounded on } A^2, \text{ for every } t \in (-1, 0).$$

The proof of the theorem is based on the following result of Bertilsson:

Theorem (Bertilsson) For $t \leq t_0$, there is a constant $C = C(t)$ such that for any $f \in S$

$$\int_0^{2\pi} \left| r^2 \frac{f'(re^{i\theta})}{f^2(re^{i\theta})} \right|^t d\theta \leq \frac{C}{(1-r)^{|t|-1}}. \tag{10}$$

Proof. Let $\varphi : \mathbb{D} \to \mathbb{D}$ univalent with $\varphi(0) = 0$. Fix $\lambda \in \mathbb{D}$. Thus

$$f(z) = \frac{\varphi'(0)^{-1}\varphi(z)}{(1-\overline{\lambda}\varphi(z))^2} \in S$$

Then for $t \leq t_0$, we have by Bertilsson's theorem

$$\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|^t}{|1-\overline{\lambda}\varphi(re^{i\theta})|^{|t|}} d\theta \leq C(\varphi, t) \int_0^{2\pi} \left| r^2 \frac{f'(re^{i\theta})}{f^2(re^{i\theta})} \right|^t d\theta$$
$$\leq \frac{C_1(\varphi, t)}{(1-r)^{|t|-1}},$$

where

$$\frac{z^2 f'(z)}{f(z)^2} = \varphi'(0) \left(\frac{z}{\varphi(z)} \right)^2 \varphi'(z)(1+\overline{\lambda}\varphi(z))(1-\overline{\lambda}\varphi(z)).$$

For $0 < \varepsilon < 1$ and $\gamma > |t| + \varepsilon + 1$, we have

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|\varphi'(z)|^t}{|1-\overline{\lambda}\varphi(z)|^\gamma}(1-|z|^2)^{|t|-2+\varepsilon}dA(z)\\
&= \int_0^1\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|^t}{|1-\overline{\lambda}\varphi(re^{i\theta})|^{|t|}}\frac{(1-r^2)^{|t|-2+\varepsilon}}{|1-\overline{\lambda}\varphi(re^{i\theta})|^{\gamma-|t|}}2rd\theta dr\\
&\leq C_2\int_0^1\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|^t}{|1-\overline{\lambda}\varphi(re^{i\theta})|^{|t|}}\frac{(1-r^2)^{|t|-2+\varepsilon}}{(1-r|\lambda|)^{\gamma-|t|}}d\theta dr\\
&\leq C_3\int_0^1 \frac{(1-r)^{\varepsilon-1}}{(1-r|\lambda|)^{\gamma-|t|}}dr\\
&= C_3\left(\int_0^{|\lambda|}+\int_{|\lambda|}^1\right)\frac{(1-r)^{\varepsilon-1}}{(1-r|\lambda|)^{\gamma-|t|}}dr = C_3(I_1+I_2).
\end{aligned}$$

Moreover

$$I_1 \leq (1-|\lambda|)^{\varepsilon-1}\int_0^{|\lambda|}\frac{dr}{(1-r|\lambda|)^{\gamma-|t|}} \leq \frac{C_4}{(1-|\lambda|)^{\gamma-|t|-\varepsilon}}$$

and

$$I_2 \leq (1-|\lambda|)^{|t|-\gamma}\int_{|\lambda|}^1 (1-r)^{\varepsilon-1}dr = \frac{C_5}{(1-|\lambda|)^{\gamma-|t|-\varepsilon}}.$$

Thus we have

$$\int_{\mathbb{D}}\frac{|\varphi'(z)|^t}{|1-\overline{\lambda}\varphi(z)|^{\gamma}}(1-|z|^2)^{|t|-2+\varepsilon}dA(z) \leq \frac{C_6}{(1-|\lambda|)^{\gamma-|t|-\varepsilon}},$$

where the constant C_6 is independent of λ. Thus by the boundedness theorem and the $A^p_\alpha-$Carleson measures theorem the operator $W_\varphi^{t/2}$ is bounded on $A^2_{|t|-2+\varepsilon}$, which shows $\alpha_\varphi(t) \leq |t|-1+\varepsilon$. Hence by letting $\varepsilon \to 0$ the proof is complete.

W. Smith 2005

Let G be a simply connected planar domain and $\tau : \mathbb{D} \to G$ be a conformal map. For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, W. Smith defined the weighted composition operator given by

$$A_{\varphi,p}(f)(z) = (Q_\varphi(z))^p f(\varphi(z)), \quad \text{where} \quad Q_\varphi(z) = \frac{\tau'(\varphi(z))}{\tau'(z)}.$$

He showed that the Brennan conjecture is equivalent to the problem of determining for which values of the parameter p there is a choice of φ that make $A_{\varphi,p}$ compact on A^2.

Theorem (Smith (2005))

Let τ be holomorphic and univalent on $\mathbb{D}$ and let $p \in \mathbb{R}$. $(1/\tau')^p \in A^2$ if and only if there exists an analytic self map φ of $\mathbb{D}$ such that $A_{\varphi,p}$ is compact on A^2.

Sketch of the proof.

$(\Longrightarrow)$ If $(1/\tau')^p \in A^2$ then for $b \in \mathbb{D}$, let $\varphi_b(z) \equiv b$. Thus

$$A_{\varphi,p}(z) = (\tau'(b))^p f(b)/(\tau')^p,$$

and therefore $A_{\varphi,p}$ is compact on A^2 as a bounded rank one operator.

$(\Longleftarrow)$ Suppose that $A_{\varphi,p}$ is compact on A^2. Then it follows that:

(i) φ is not an automorphism,

(ii) φ must have a fixed point in $\mathbb{D}$.

The first assertion follows from the fact that if φ was an automorphism then $C_{\varphi^{-1}}$ is well defined and

$$(A_{\varphi,p}C_{\varphi^{-1}})(f)(z) = (Q_\varphi(z))^p f(z).$$

But the only compact multiplication operator is the one with symbol identically 0. So $A_{\varphi,p}C_{\varphi^{-1}}$ is not compact and since $C_{\varphi^{-1}}$ is bounded on A^2, $A_{\varphi,p}$ is not compact.

The second assertion is proved again by contradiction. The main steps are:

(i) If φ does not have a fixed point in $\mathbb{D}$, then by the Denjoy Wolff theorem there exist a unique boundary fixed point ζ with the angular derivative $\varphi'(\zeta)$ finite.

(ii) $A_{\varphi,p}$ is compact $\Longrightarrow$ $A^*_{\varphi,p}$ is compact.

(iii) Since $A^*_{\varphi,p}$ is compact we have $\|A^*_{\varphi,p}k_a\| \to 0$, as $|a| \to 1$, where $\{k_a : a \in \mathbb{D}\}$ is the set of the normalized reproducing kernels of A^2.

(iv) The existence of a sequence $\{a_n\} \in \mathbb{D}$ with $a_n \to \zeta$ nontangentially and $\liminf_{n\to\infty} \|A^*_{\varphi,p} k_{a_n}\| \geq C(\zeta, \varphi'(\zeta), p) > 0$.

Let $b \in \mathbb{D}$ the fixed point of φ, then

$$A^*_{\varphi,p} K_b = K_b.$$

Hence the number 1 is an eigenvalue of $A^*_{\varphi,p}$ and so it is in the spectrum of $A_{\varphi,p}$. But $A_{\varphi,p}$ is compact, thus 1 is an eigenvalue of $A_{\varphi,p}$. Therefore there exists a nonzero $f \in A^2$ such that

$$A_{\varphi,p} f = f,$$

or equivalently the function $g = (\tau')^p f$ satisfies

$$g \circ \varphi = g.$$

We already know that φ is not an automorphism, so its iterates $\varphi_n \to b$ $(n \to \infty)$ uniformly on compact subsets of $\mathbb{D}$. Hence for $z \in \mathbb{D}$ we have

$$g(z) = g(\varphi(z)) = g(\varphi_n(z)) \to g(b), \quad \text{as} \quad n \to \infty.$$

Thus g is the constant function $g(z) \equiv g(b) \neq 0$, since $f \not\equiv 0$. Hence

$$(1/\tau')^p = g(b)^{-1} f \in A^2.$$

Gracias!