Intersections of Fréchet spaces and (LB)–spaces

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Abstract

This article presents results about the class of locally convex spaces which are defined as the intersection $E \cap F$ of a Fréchet space F and a countable inductive limit of Banach spaces E. This class appears naturally in analytic applications to linear partial differential operators. The intersection has two natural topologies, the intersection topology and an inductive limit topology. The first one is easier to describe and the second one has good locally convex properties. The coincidence of these topologies and its consequences for the spaces $E \cap F$ and $E + F$ are investigated.

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The aim of this paper is to investigate spaces $E \cap F$ which are the intersection of a Fréchet space F and an (LB)-space E. They appear in several parts of analysis whenever the space F is determined by countably many necessary (e.g. differentiability of integrability) conditions and E is the dual of such a space, in particular E is defined by a countable sequence of bounded sets which may also be determined by concrete estimates. Two natural topologies can be defined on $E \cap F$: the intersection topology, which has seminorms easy to describe and which permits direct estimates, and a finer inductive limit topology which is defined in a natural way and which has good locally convex properties, e.g. $E \cap F$ with this topology is a barrelled space. It is important to know when these two topologies coincide. It turns out that the locally convex properties of $E \cap F$ with the intersection topology are related to the completeness of the (LF)-space $E + F$. The present setting provides us with new interesting examples of (LF)-spaces. Recent progress on the study of (LF)-spaces, see [4, 14, 25, 27], is very important in our work. Our main results are Theorems 4 and 7. The Examples 5 and 10 show the main difficulties.

Our article continues the research in [5], where the case when F is a Fréchet Schwartz space (an (FS)-space) and E is the dual of an (FS)-space (a (DFS)-space) was analyzed. One of the original motivations for the research in [5] came up in the investigations of Langenbruch [18, 19, 20] about the surjectivity of linear partial differential operators $P(D)$ with constant coefficients on spaces of ultradifferentiable functions and ultradistributions. The extension of $\{\omega\}$-ultradifferentiable regularity for $\{\omega\}$- ultradistributions u such that $P(D)u = 0$ across a hypersurface has consequences for the surjectivity of $P(D)$, in the same way that Holmgren's uniqueness theorem is relevant in the discussions of surjectivity problems for $P(D)$ on $C^\infty(\Omega)$, see [15, Chapters 8 and 10]. The extension of regularity for ultradistributions was studied by Langenbruch in [18]. In the classical case, treated by Hörmander in [15, Section 11.3], one seeks conditions on a linear subspace V of R^N and on

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the polynomial $P(z)$ to assure the existence of $u \in C^n(R^N), n \in N$, with $P(D)u = 0$ such that the singular support of u coincides with V. In the proof of [15, 11.3.1], Hörmander considers the intersection $G := \{u \in C^n(R^N) | \ P(D)u = 0\} \cap C^\infty(R^N \setminus V)$ which is a Fréchet space, and uses the closed graph theorem to get concrete estimates. More than a converse of [15, 11.3.1] is obtained in [15, 11.3.6], which yields the extension of C^∞-regularity across hypersurfaces. The extension of this last result was obtained by Langenbruch in [18, Theorem 2.2] both for Beurling (ω)-regularity and Roumieu $\{\omega\}$-regularity. However, only a partial converse of [18, Theorem 2.2] is obtained in [18, Theorem 2.6] in which a Fréchet intersection, similar to the Fréchet space G above, plays an important role. As Langenbruch points out in the Remark after the proof of his Theorem 2.6, a full converse of [18, Theorem 2.2] would have required that every sequentially continuous linear form on the intersection of a Fréchet Schwartz space and a (DFS)-space was continuous. This property does not hold in general for intersections of (FS) and (DFS) spaces as was shown in [5].

On the other hand, the investigations in [1] about the relation of the hypoellipticity of a linear partial differential operator with variable coefficients with the local solvability of its transpose operator in the setting of Gevrey classes made the authors consider when the intersection of a Banach space and a (DFS)–space is barrelled. This is one of the reasons why we treat here a setting which is more general than the one of [5]. On the other hand, the intersections of function spaces appear also in other areas of functional analysis. It turned out that the locally convex structure of the intersection of two Fréchet spaces was rather intricate. Indeed, Taskinen [23] showed that the strong dual of the space $C(\mathbb{R}) \cap L^1(\mathbb{R})$ is not an (LB)–space. The structure of the intersection spaces $C^m(\Omega) \cap H^{k,p}(\Omega)$ was studied in detail (see, for example, [2, 7]).

Our notation for locally convex spaces is standard and we refer the reader, for example, to [16]. Given a locally convex space G, we denote by $\mathcal{U}(G)$ the system of all closed absolutely convex 0–neighbourhoods in G and by $\mathcal{B}(G)$ the system of all closed absolutely convex bounded sets of G. The general frame in which we work is the following. Let F be a Fréchet space with a basis $(U_K)_K$ of closed absolutely convex 0–neighbourhoods. The canonical Banach space generated by U_K is denoted by F_K. Let $E = \mathrm{ind}_n E_n$ be a regular (LB)–space, i.e. every bounded set in E is contained and bounded in a step E_n. The unit ball of the Banach space E_n is denoted by B_n. We suppose that both E and F are continuously included in a Hausdorff locally convex space H. The intersection $E \cap F$ is endowed with the intersection topology, which is denoted by s. A basis of closed absolutely convex 0–neighbourhoods of $(E \cap F, s)$ is given by the sets of the form $U \cap V$ as U and V run over bases of closed absolutely convex 0–neighbourhoods of E and F respectively. The sum $E + F$ (as a subspace of H) can be considered as a quotient of the locally convex direct sum $E \oplus F$ via the linear map $q \colon E \oplus F \mapsto H$, $q(x, y) := x - y$. On the other hand the space $E \cap F$ can be identified with a closed subspace of $E \oplus F$ by means of the linear injection $i \colon E \cap F \mapsto E \oplus F$, $i(x) := (x, x)$. Clearly $i(E \cap F) = \ker q$ and we have the short exact sequence

$$0 \to E \cap F \xrightarrow{i} E \oplus F \xrightarrow{q} E + F \to 0.$$

There is also an (LF)–topology t on $E \cap F$ finer than s. Indeed, if for each $n \in \mathbb{N}$ we endow $E_n \cap F$ with the intersection topology s_n for which it is a Fréchet space, then the inclusion map $(E_n \cap F, s_n) \hookrightarrow (E \cap F, s)$ is continuous. Define $(E \cap F, t) := \mathrm{ind}_n(E_n \cap F, s_n)$, and observe that, for each $n \in \mathbb{N}$, a basis of closed absolutely convex 0–neighbourhoods of $(E_n \cap F, s_n)$ is given by $(K^{-1}B_n \cap U_K)_K$.

An (LF)–space $G = \mathrm{ind}_n G_n$ is said to satisfy condition (M) (resp. condition (M_0)) of Retakh if there is an increasing sequence $(V_n)_n$, with V_n absolutely convex 0–neighbourhood in G_n, such that for each $n \in \mathbb{N}$ there is $m > n$ such that G and G_m induce the same topologies (resp. the same weak topologies) on V_n. By [21, Corollaries 2.1 and 7.2] (see also [25, Theorem 3.2 and Corollary 3.3]) if G satisfies condition (M), then G is a complete, hence regular, (LF)–space. On the other hand, G is said to be *boundedly stable* (resp. *weakly boundedly stable*) if on every set which is bounded in some step almost all step topologies (resp. almost all step weak topologies) coincide. Finally, G is said to satisfy condition (wQ) introduced by Vogt [25] (resp. condition (wQ*) introduced by Bierstedt and the second author [4]) if

$$\begin{aligned} &\forall n \in \mathbb{N}\ \exists m \geq n,\ V_n \in \mathcal{U}(G_n)\ \forall k \geq m,\ U \in \mathcal{U}(G_m) \\ &\exists S > 0,\ W \in \mathcal{U}(G_k)\ \text{ such that }\ V_n \cap W \subseteq SU \end{aligned}$$

(resp. there is an increasing sequence $(V_n)_n$ with the above property). In [8] it was pointed out that condition (wQ) is equivalent to the following one: there is a sequence $(V_n)_n$ of absolutely convex 0–neighbourhoods V_n in G_n, such that for each $n \in \mathbb{N}$ there is $m > n$ such that for every $k > m$ the topologies of G_m and G_k have the same bounded sets in V_n. Every (LB)-space satisfies condition (wQ*). Regular (LF)–spaces always satisfy condition (wQ) as it was proved in [25, Theorem 4.7]. Moreover, by [28, Theorem 6.4] (or see [26, Theorem 3.8]) an (LF)–space has condition (M) if and only if it is boundedly stable and satisfies condition (wQ). By [26, Theorem 5.6] an (LF)–space has condition (M_0) if and only if it is weakly boundedly stable and satisfies condition (wQ*). It was conjectured in [26] (see also [14]) that an (LF)–space has condition (M_0) if and only if it is weakly boundedly stable and satisfies condition (wQ).

We have the following preliminary observations.

Lemma 1 *Let $E = \mathrm{ind}_n E_n$ be a regular (resp. complete) (LB)–space and F be a Fréchet space. Then the following holds.*

(i) The space $E \oplus F = \mathrm{ind}_n(E_n \oplus F)$ is a regular (resp. complete) (LF)–space. The space $(E \cap F, s)$ is a (resp. complete) locally convex space and the space $E + F = \mathrm{ind}_n(E_n + F)$ is an (LF)–space.

(ii) The topologies t and s have the same bounded sets, and $(E \cap F, t)$ is a regular (LF)–space.

A (DF)-space space G is said to satisfy the *dual density condition* (DDC) [3, Definition 1.1] if each bounded subset of G is metrizable. The following characterization was proved in [13, § 5].

Lemma 2 *Let G be a (DF)–space with an increasing fundamental sequence $(B_n)_n$ of absolutely convex bounded sets. The space G satifies the (DDC) if and only if there is $U \in \mathcal{U}(G)$ such that for every $n \in \mathbb{N}$ and $V \in \mathcal{U}(G)$ there is $\epsilon > 0$ with $B_n \cap \epsilon U \subset V$.*

If we denote by p_U the Minkowski functional of the neighbourhood U in G given by Lemma 2, then p_U is clearly a norm on G and the topologies of G and of the normed space (G, p_U) coincide on B_n for all $n \in \mathbb{N}$.

Proposition 3 *Let $E = \mathrm{ind}_n E_n$ be an (LB)–space with the (DDC). Let F be a Fréchet space. Then:*

(i) There is $U \in \mathcal{U}(E)$ such that the topologies of $(E \cap F, s)$ and of $(E \cap F, s_U)$ coincide on $B_n \cap F$ for all $n \in \mathbb{N}$, where s_U denotes the intersection topology defined by F and the normed space (E, p_U).

(ii) If $(E \cap F, s)$ is barrelled, then every sequentially linear map from $(E \cap F, s)$ into a Banach space is continuous.

Proof. (i) By Lemma 2 there is $U \in \mathcal{U}(E)$ such that, for each $n \in \mathbb{N}$, the topology induced by $(E \cap F, s)$ on B_n has a basis of closed absolutely convex 0–neighbourhood given by $\{r^{-1}U;\ r \in \mathbb{N}\}$. Let $V \in \mathcal{U}(E)$ and $W \in \mathcal{U}(F)$. Then there is $r \in \mathbb{N}$ such that $B_n \cap r^{-1}U \subset V$, hence

$$(B_n \cap F) \cap (r^{-1}U \cap W) = (B_n \cap r^{-1}U) \cap W \subset V \cap W.$$

(ii) Let Y be a Banach space and T be a sequentially continuous linear map from $(E \cap F, s)$ into Y. If B_Y is the closed unit ball of Y, $T^{-1}(B_Y)$ is an absolutely convex absorbing subset of $E \cap F$. Moreover, by (i) for each $n \in \mathbb{N}$ $(2^n B_n \cap F, s)$ is metrizable and hence $T_{|2^n B_n \cap F}\colon (2^n B_n \cap F, s) \to Y$ is continuous; thereby implying that $T^{-1}(B_Y) \cap (2^n B_n \cap F)$ is a 0–neighbourhood in $(2^n B_n \cap F, s)$ for every $n \in \mathbb{N}$. The so–called localization property for barrelled spaces (see [22, Corollary 8.2.5] or [16, Theorem 12.3.5]) implies that $T^{-1}(B_Y)$ is also a 0–neighbourhood in $(E \cap F, s)$. □

Theorem 4 *Let $E = \mathrm{ind}_n E_n$ be a regular and weakly boundedly stable (LB)–space satisfying condition (DDC) and let F be a Fréchet space. The following conditions are equivalent:*

(1) $(E \cap F, s) = \mathrm{ind}_n (E_n \cap F, s_n)$ holds topologically,

(2) $(E \cap F, s)$ is an (LF)–space,

(3) $(E \cap F, s)$ is bornological,

(4) $(E \cap F, s)$ is barrelled.

Moreover, if E is boundedly stable, these equivalent conditions imply

(5) $E + F$ is a complete (LF)–space.

Proof. Clearly (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4). Let us assume (4). We first prove that $(E \cap F, s)' = (E \cap F, t)'$. Since E is regular and weakly boundedly stable, the (LF)–space $E \oplus F$ is regular and satisfies condition (M_0). This easily implies that every $u \in (E \cap F, t)'$ is sequentially s–continuous, hence s–continuous by Proposition 3 (ii), i.e. $u \in (E \cap F, s)'$. Next, since $(E \cap F, s)$ is barrelled and $(E \cap F, s)' = (E \cap F, t)'$, the topologies t and s coincide, and (1) holds.

Suppose now that condition (1) is satisfied and that E is also boundedly stable. In this case, the (LF)-space $E \oplus F$ is acyclic (or equivalently satisfies condition (M) of Retakh, see [25]). We consider the short exact sequence

$$0 \to \mathrm{ind}_n (E_n \cap F) \xrightarrow{i} \mathrm{ind}_n (E_n \oplus F) \xrightarrow{q} \mathrm{ind}_n (E_n + F) \to 0.$$

Condition (1) holds if and only if i is a monomorphism. By a theorem of Palamodov [21] in the version of [25, Theorems 1.4, 1.5 and 2.10], this implies that $E+F=\mathrm{ind}_n(E_n+F)$ satisfies the condition (M), hence it is complete. $\square$

Example 5 *We show that $(E\cap F,t)'$ may be different from $(E\cap F,s)'$ although $E+F$ is a complete (LF)–space, even for a (DFS)-space E. This yields that, in general, condition (5) does not imply condition (1) in Theorem 4.*

Let s be the Fréchet space of rapidly decreasing sequences and s' its strong dual. Let $E=(\mathsf{s}')^{(\mathbb{N})}$ and $F=\ell^1(\ell^1)$. Using Hahn–Banach theorem and the exact sequence

$$0\to(\mathsf{s}')^{(\mathbb{N})}\cap\ell^1(\ell^1)\xrightarrow{i}(\mathsf{s}')^{(\mathbb{N})}\oplus\ell^1(\ell^1)\xrightarrow{q}(\mathsf{s}')^{(\mathbb{N})}+\ell^1(\ell^1)\to 0,$$

it is easy to show that the topological dual of $\big((\mathsf{s}')^{(\mathbb{N})}\cap\ell^1(\ell^1),s\big)$ coincides with $\mathsf{s}^{\mathbb{N}}+\ell^\infty(\ell^\infty)$. Moreover, $(\mathsf{s}')^{(\mathbb{N})}\cap\ell^1(\ell^1)=(\ell^1)^{(\mathbb{N})}$ algebraically and the (LF)–topology t on it clearly coincides with the one of $(\ell^1)^{(\mathbb{N})}$.

We prove that $\big((\mathsf{s}')^{(\mathbb{N})}\cap\ell^1(\ell^1),s\big)$ and $(\ell^1)^{(\mathbb{N})}$ have a different dual, thus obtaining that $\big((\mathsf{s}')^{(\mathbb{N})}\cap\ell^1(\ell^1),s\big)$ is not barrelled. For each $k\in\mathbb{N}$ let z_k be the constant sequence $(k,k,\ldots)$. Then $z:=(z_k)_k$ belongs to the dual of $(\ell^1)^{(\mathbb{N})}$. Suppose that it belongs to the dual of $\big((\mathsf{s}')^{(\mathbb{N})}\cap\ell^1(\ell^1),s\big)$. Consequently, $z=x+y$ for some $x=((x_k^j)_j)_k\in\ell^\infty(\ell^\infty)$ and $y=((y_k^j)_j)_k\in(\mathsf{s})^{\mathbb{N}}$. For each $k\in\mathbb{N}$, there is $j(k)$ such that $|y_k^j|<1$ for all $j\geq j(k)$. Thus, for each $k>2$, the norm of x_k in ℓ^∞ is greater or equal than $k-1$. This implies that $x=(x_k)_k=(z_k-y_k)_k$ does not belong to $\ell^\infty(\ell^\infty)$, a contradiction.

On the other hand, using again Hahn–Banach theorem and the exact sequence

$$0\to(\mathsf{s})^{\mathbb{N}}\cap c_0(c_0)\xrightarrow{i_1}(\mathsf{s})^{\mathbb{N}}\oplus c_0(c_0)\xrightarrow{q_1}(\mathsf{s})^{\mathbb{N}}+c_0(c_0)\to 0,$$

where $i_1(x):=(x,-x)$ and $q_1(x,y):=x+y$, it is easy to show that the topological dual of the Fréchet space $\big((\mathsf{s})^{\mathbb{N}}\cap c_0(c_0),s\big)$ coincides with $(\mathsf{s}')^{(\mathbb{N})}+\ell^1(\ell^1)$. Moreover, by [11, Proposition 2] (see also [12]) its strong dual is barrelled so that $\big((\mathsf{s})^{\mathbb{N}}\cap c_0(c_0),s\big)'_\beta=(\mathsf{s}')^{(\mathbb{N})}+\ell^1(\ell^1)$ holds topologically, thereby implying that $(\mathsf{s}')^{(\mathbb{N})}+\ell^1(\ell^1)$ is a complete, regular (LB)–space.

An (LB)–space $E=\mathrm{ind}_nE_n$ is called an (LS_w)–space if for each n there is $m>n$ such that the inclusion map $E_n\hookrightarrow E_m$ is weakly compact (see [22, Definition 8.5.3]). By the well-known factorization theorem for weakly compact maps of [10], the weakly compact operator $E_n\hookrightarrow E_m$ factors over a reflexive Banach space which is algebraically a subspace of E_m. Accordingly, we may and will suppose that the Banach spaces E_n are reflexive for all $n\in\mathbb{N}$ if $E=\mathrm{ind}_nE_n$ is an (LS_w)–space. It is well-known that, *if $E=\mathrm{ind}_nE_n$ is an (LS_w)–space and F is a reflexive Banach space, then every sequentially continuous form on $(E\cap F,s)$ is continuous, i.e. $E\cap F$ is a well–located subspace of $E\oplus F$* (cf. [22, Proposition 8.6.7]). This result and Example 5 suggest that some additional assumptions on F must be required to connect the completeness of $E+F$ and the fact that $E\cap F$ is barrelled.

We thank the referee for the following lemma and its proof.

Lemma 6 *Let $E=\mathrm{ind}_nE_n$ be an (LB)–space and let F be a Fréchet space. Then the (LF)–space $E+F$ satisfies condition (wQ) if and only if it satisfies condition (wQ*).*

Proof. Since condition (wQ*) always implies condition (wQ), we only have to prove that the converse holds. By replacing the constant S appropriately we obtain that condition (wQ) is equivalent to the following slightly simpler one

$$\forall n \in \mathbb{N}\ \exists m \geq n,\ N \in \mathbb{N}\ \forall k \geq m,\ M \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0$$
$$(B_n + U_N) \cap (B_k + U_K) \subseteq S(B_m + U_M).$$

Since we clearly can assume $N(1) = 1$ this yields

$$\exists m(1) \geq n\ \forall k \geq m(1),\ M \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0 \qquad (0.1)$$
$$U_1 \cap (B_k + U_K) \subseteq S(B_{m(1)} + U_M).$$

For $n \in \mathbb{N}$ we set $m = \max\{n, m(1)\}$ and obtain for $k \geq m$ and $M \in \mathbb{N}$ with K and S as in (0.1)

$$(B_n + U_1) \cap (B_k + U_K) \subseteq S(B_m + U_M).$$

Indeed, given $x = b + u = d + v$ with $b \in B_n$, $u \in U_1$, $d \in B_k$, and $v \in U_k$ we have

$$u = d - b + v \in U_1 \cap (2B_k + U_K) \subseteq 2S(B_m + U_M),$$

hence $x = b + u \in (2S + 1)(B_m + U_M)$. □

Lemma 6 settles positively in the context considered in this article the conjecture of Wengenroth that an (LF)-space has condition (M_0) if and only if it is weakly boundedly stable and satisfies condition (wQ).

Theorem 7 *Let $E = \mathrm{ind}_n E_n$ be a regular and weakly boundedly stable (LB)–space and let F be a Fréchet space. The following conditions are equivalent:*

(1) every sequentially continuous linear form on $(E \cap F, s)$ is continuous,

(2) the (LF)–space $E + F$ satisfies condition (M_0) of Retakh,

(3) the (LF)–space $E + F$ is weakly boundedly stable and is of type (wQ),*

(4) the (LF)–space $E + F$ is weakly boundedly stable and is of type (wQ), i.e.

$$\forall n \in \mathbb{N}\ \exists m \geq n,\ N \in \mathbb{N}\ \forall k \geq m,\ M \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0$$
$$(N^{-1}B_n + U_N) \cap (K^{-1}B_k + U_K) \subset S(M^{-1}B_m + U_M).$$

Proof. Since E is regular and weakly boundedly stable, the (LF)–space $E \oplus F$ is regular and clearly satisfies condition (M_0) of Retakh. Consider the short exact sequence

$$0 \to \mathrm{ind}_n(E_n \cap F) \xrightarrow{i} \mathrm{ind}_n(E_n \oplus F) \xrightarrow{q} \mathrm{ind}_n(E_n + F) \to 0.$$

Condition (1) holds if and only if i is a weak monomorphism. Therefore a theorem of Palamodov [21] in the version of [25, Theorems 1.4, 1.5 and 2.10] implies that (1) is equivalent to (2). By [26, Theorem 5.6], (2) is equivalent to (3). Finally, by Lemma 6, (3) is equivalent to (4). □

Remark 8 It is desiderable to have easy sufficient conditions for E and F which allow us to conclude that the (LF)–space $E+F$ is of type (wQ). For instance, for an (LS$_w$)–space E it is enough that F has a 0–neighbourhood U which is closed in H, as it happens in Example 5. Then B_n+U is closed in H as the sum of a weakly compact set and a closed set and one can apply [14, Proposition 2.6] to conclude . In this case, if the (LF)–space is also weakly boundedly stable, it is regular by [17, Theorem 2].

On the other hand, there are examples of sums of (LB) and Fréchet spaces which are regular (LF)–spaces of type (wQ), but not weakly boundedly stable, e.g. $(\mathsf{s}')^{(\mathbb{N})}+\ell^1(\ell^1)$ by Theorem 7 and Example 5.

In the next result, we suppose that $E=\mathrm{ind}_n E_n$ is an (LS$_w$)–space and $F=\mathrm{proj}_K F_K$ is a (FS$_w$)–space (i.e. for all K there is $M>K$ such that the linking map $p_{K,M}\colon F_M\to F_K$ is weakly compact, see [22, Definition 8.5.2]).

Proposition 9 *Let $E=\mathrm{ind}_n E_n$ be an (LS$_w$)–space and let $F=\mathrm{proj}_K F_K$ be a (FS$_w$)–space. The following conditions are equivalent:*

(1) every sequentially continuous linear form on $(E\cap F,s)$ is continuous,

(2) the (LF)–space $E+F$ satisfies condition (M_0) of Retakh,

(3) the (LF)–space $E+F$ is of type (wQ),*

(4) the (LF)–space $E+F$ is of type (wQ), i.e.

$$\forall n\in\mathbb{N}\ \exists m\geq n,\ N\in\mathbb{N}\ \forall k\geq m,\ M\in\mathbb{N}\ \exists K\in\mathbb{N},\ S>0$$
$$(N^{-1}B_n+U_N)\cap(K^{-1}B_k+U_K)\subset S(M^{-1}B_m+U_M).$$

(5) the (LF)–space is regular,

(6) the (LF)–space is complete.

Proof. Since E is an (LS$_w$)–space and F is (FS$_w$)–space, by the factorization theorem of weakly compact maps of [10], we may assume that all the Banach spaces E_n and F_K are reflexive. For each $n\in\mathbb{N}$, E_n+F is then a reflexive Fréchet space. Indeed, it is a quotient of $\mathrm{proj}_K(E_n\oplus F_K)$ which is totally reflexive as a subspace of a product of reflexive Banach spaces; see e.g. [24]. Consequently, $E+F$ is an (LF)–space with reflexive steps, hence it is weakly boundedly stable.

By Theorem 7 conditions (1), (2), (3) and (4) are equivalent. Moreover, (6) implies (5) as it is well-known, and by [25, Theorem 4.7] (5) implies (4). Finally, since each step E_n+F is a reflexive Fréchet space, (2) implies (6) by [25, Proposition 4.4] . □

Example 10 *We give an example of a reflexive Banach space F and a (DFS)-space E such that $E+F$ is a complete reflexive (LB)–space and $(E\cap F,s)$ is not barrelled. This shows that the equivalent conditions of Proposition 9 need not imply condition (1) of Theorem 4.*

Let $E=(\mathsf{s}')^{(\mathbb{N})}$ and $F=\ell^p(\ell^p)$, $1<p<+\infty$. Since $(\mathsf{s}')^{(\mathbb{N})}$ is an (LS$_w$)–space and $\ell^p(\ell^p)$ is a reflexive Banach space, every sequentially continuous form on $\big((\mathsf{s}')^{(\mathbb{N})}\cap\ell^p(\ell^p),s\big)$ is continuous (cf. [22, Proposition 8.6.7]), but it is not a topological subspace of $(\mathsf{s}')^{(\mathbb{N})}\oplus\ell^p(\ell^p)$

or equivalently, it is not barrelled. Indeed, $(\mathsf{s}')^{(\mathbb{N})} \cap \ell^p(\ell^p) = (\ell^p)^{(\mathbb{N})}$ algebraically and the (LF)–topology t on it coincides with the one of $(\ell^p)^{(\mathbb{N})}$. Moreover, the exact sequence

$$0 \to (\mathsf{s}')^{(\mathbb{N})} \cap \ell^p(\ell^p) \xrightarrow{i} (\mathsf{s}')^{(\mathbb{N})} \oplus \ell^p(\ell^p) \xrightarrow{q} (\mathsf{s}')^{(\mathbb{N})} + \ell^p(\ell^p) \to 0$$

is the dual of the exact sequence

$$0 \to \mathsf{s}^{\mathbb{N}} \cap \ell^q(\ell^q) \xrightarrow{i_1} \mathsf{s}^{\mathbb{N}} \oplus \ell^q(\ell^q) \xrightarrow{q_1} \mathsf{s}^{\mathbb{N}} + \ell^q(\ell^q) \to 0,$$

where $q = \frac{p}{p-1}$ and, as it is easy to show, $\mathsf{s}^{\mathbb{N}} + \ell^q(\ell^q) = (\ell^q)^{\mathbb{N}}$ algebraically and topologically. By [6, Corollary 3] $\big((\mathsf{s}')^{(\mathbb{N})} \cap \ell^p(\ell^p), s\big)$ is barrelled if and only if q_1 lifts bounded sets (i.e. for every bounded set B of $(\ell^q)^{\mathbb{N}}$ there is a bounded subset A of $\mathsf{s}^{\mathbb{N}} \oplus \ell^q(\ell^q)$ such that $q_1(A) \supset B$). If this were the case, since $\mathsf{s}^{\mathbb{N}} \oplus \ell^q(\ell^q)$ is quasinormable, $\mathsf{s}^{\mathbb{N}} \cap \ell^q(\ell^q)$ endowed with the intersection topology would be a quasinormable Fréchet space by the following result of [9]: *a subspace L of a quasinormable Fréchet space X is itself quasinormable if and only if the quotient map $q\colon X \to X/L$ lifts bounded sets.* Now, by [12, Proposition 3] $\mathsf{s}^{\mathbb{N}} \cap \ell^q(\ell^q)$ is not quasinormable with respect to the intersection topology and then $\big((\mathsf{s}')^{(\mathbb{N})} \cap \ell^p(\ell^p), s\big)$ is not barrelled.

Finally, we give a precise characterization of the completeness of $E+F$ in the setting of Köthe sequence spaces. Its proof is obtained adapting the methods of [5] or with a direct argument to show that conditions (1) and (3) are equivalent.

Proposition 11 *Let $1 < p < +\infty$. Let $\mathcal{W} = (w_n)_n$ be a decreasing sequence of strictly positive weights on $\mathbb{N}$. Let $E_n = \ell^p(w_n)$ for all $n \in \mathbb{N}$, and let $E = \mathrm{ind}_n \ell^p(w_n)$ be an (LS_w) echelon space of order p. Let $A = (a_K)_K$ be an arbitrary increasing sequence of strictly positive weights on $\mathbb{N}$. Let $F = \mathrm{proj}_K \ell^p(a_K)$ be an (FS_w) echelon space of order p. The the following conditions are equivalent:*

(1) the (LF)–space $E+F$ is of type (wQ),

(2) the (LF)–space $E+F$ is complete,

(3) the following condition is satisfied by the weights:

$$\begin{aligned} &\forall n \in \mathbb{N}\ \exists m \geq n,\ N \in \mathbb{N}\ \forall k \geq m,\ M \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0 \\ &\forall i \in \mathbb{N} \quad \min(w_m(i), a_M(i)) \leq S\left(\min(w_n(i), a_N(i)) + \min(w_k(i), a_K(i))\right). \end{aligned}$$

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