Sums and products of bad functions

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Abstract. We examine the question of whether there are algebras of functions having special properties. The types of functions studied will be everywhere surjective functions, hypercyclic entire functions, and continuous functions that attain their maximum at a unique point.

2000 MSC: Primary 46E25, 32A15; Secondary 15A03.
Keywords and phrases: Entire functions, everywhere surjective functions, lineability, algebrability.

1. Introduction

This expository note deals with the following type of general problem. Suppose that we have a set of functions, all having a special property. When does it occur that the set contains a "large" vector space? When does it contain a large algebra?

A typical situation occurs with $\mathcal{ND} = \{f \in \mathcal{C}[0,1] \mid f \text{ is nowhere differentiable}\}$. It was first observed by V. Gurariy [**10**] that $\mathcal{ND}$ contains an infinite dimensional Banach space. Later, L. Rodríguez-Piazza [**15**] proved that, in fact, there is an isometric copy of every separable Banach space contained in $\mathcal{ND}$. (See [**13**] for an even stronger result concerning nowhere Hölder functions.) Recently, F. Bayart and L. Quarta [**6**] have proved that $\mathcal{ND}$ contains an infinite dimensional algebra. On the other hand, it is known that the set of everywhere differentiable functions on $[0,1]$ does not contain a complete, infinite dimensional vector space (see, e.g., [**11**]). In short, we see that the set $\mathcal{ND}$ of continuous functions that are nowhere differentiable contains large Banach spaces (in fact, every separable Banach space) as well as large algebras. On the other hand, no infinite dimensional vector subspace of everywhere differentiable functions is complete.

In this article, we describe several other such instances. In particular, we discuss the so-called *algebrability* of the following sets:

(i) the everywhere surjective functions on $\mathbb{C}$,

and

The second author was supported in part by GVA, grant CTESPP/2005.

The second and third authors were supported in part by MEC and FEDER, Project MTM2004-02262 and Research Net MTM2004-21420-E.

(ii) the entire functions that are hypercyclic with respect to either the Birkhoff translation operator or the MacLane differentiation operator.

We will conclude this brief article with a discussion of one result that seems to be in the same spirit and a somewhat surprising one that is not!

2. Everywhere surjective functions

Let $\mathcal{S}$ denote the set of *everywhere surjective functions on* $\mathbb{C}$, that is functions $f : \mathbb{C} \to \mathbb{C}$ with the property that for every open set $U \subset \mathbb{C}, f|_U$ is surjective. It was shown in [**2**] that $\mathcal{S} \cup \{0\}$ contains a large vector space. In fact, the cardinality of this vector space is the same as that of the set of *all* functions $\mathbb{C} \to \mathbb{C}$, and so this vector space clearly has the largest possible dimension. In fact, in [**4**], the following is proved.

THEOREM 2.1. ([**4**]) *The set $\mathcal{S}$ contains an infinitely generated algebra $\mathcal{A}$. That is, there is an algebra $\mathcal{A} \subset \mathcal{S}$ having a (countably) infinite number of generators.*

PROOF. We provide a brief sketch of the proof. First, it is not hard to see that $\mathcal{S} \neq \emptyset$. Indeed, enumerate the squares in $\mathbb{C}^2$ having only complex rational coordinates, $(S_j)_{j=1}^{\infty}$, and for each j let $C_j \subset S_j$ be a copy of the Cantor set that is disjoint from $C_1 \cup \cdots \cup C_{j-1}$. Define $f_j : C_j \to \mathbb{C}$ to be any one-to-one correspondence. Since any open subset of C contains some S_j, the function f defined by

$$f(z) = \begin{cases} f_j(z) & \text{if } z \in C_j \\ 0 & \text{if } z \notin \cup_j C_j \end{cases}$$

is easily seen to be in $\mathcal{S}$.

We remark in passing that the algebra generated by an element $f \in \mathcal{S}$ has the property that any non-zero element in it is everywhere surjective. To see this, consider an arbitrary such element $P(f) = \sum_{j=1}^{k} a_j f^j$ and the associated polynomial $P(z) = \sum_{j=1}^{k} a_j z^j$. Given any $w \in \mathbb{C}$, let z_0 be a solution to $\sum_{j=1}^{k} a_j z^j = w$. If $U \subset \mathbb{C}$ is an arbitrary open set, we can find $u_0 \in U$ so that $f(u_0) = z_0$, and it is clear that $P(f)(u_0) = w$.

In order to find an infinitely generated algebra, we must do more. First, it suffices to find functions $f, g : \mathbb{C} \to \mathbb{C}$ with the following properties:

• f and g are algebraically independent,

and

• Any non-zero function in the algebra $\mathcal{A}$ generated by f, g is onto.

To see this, let $F : \mathbb{C} \to \mathbb{C}$ be a fixed function in $\mathcal{S}$. Then it is easy to verify that every non-zero function in the algebra $\{h \circ F \mid h \in \mathcal{A}\}$ is everywhere surjective.

So, all that remains to show is that we can find functions f, g with the two properties indicated above. For each $p, q \in \mathbb{N}$, let $\Phi_{p,q}$ be a homeomorphism from $U_{p,q} = \{z = x + iy \mid p - 1 < x < p; q - 1 < y < q\}$ to $\mathbb{C}$. Define $f, g : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} [\Phi_{p,q}(z)]^p & \text{if } z \in U_{p,q} \\ 0 & \text{otherwise} \end{cases};$$

$$g(z) = \begin{cases} [\Phi_{p,q}(z)]^q & \text{if } z \in U_{p,q} \\ 0 & \text{otherwise} \end{cases}.$$

We now verify that $\{f, g\}$ is an algebraically independent set and that every non-trivial function $h \in \mathcal{A}$ is surjective. In fact, one argument will suffice to show that both requirements hold. Let

$P:\mathbb{C}^2\to\mathbb{C}$ be a non-constant polynomial, and recall the observation [**1**] that for some $m_1,m_2\in\mathbb{N}$, the polynomial $z\in\mathbb{C}\rightsquigarrow P(z^{m_1},z^{m_2})\in\mathbb{C}$ is onto. Then consider $h=P\circ(f,g)|_{U_{m_1,m_2}}$. For any $z\in\mathbb{C}$, since $\Phi^{-1}_{m_1,m_2}(z)\in U_{m_1,m_2}$, the definitions of f and g imply that $h(z)=P(z^{m_1},z^{m_2})$. In other words, not only is $P\circ(f,g)$ non-trivial but it is in fact a surjective mapping.

■

Problem 2.1. *We remark that the same techniques will show that the cone of functions* $\{f:\mathbb{R}^+\to\mathbb{R}^+ \mid$ *for all* $(a,b)\subset\mathbb{R}^+,\ f(a,b)=\mathbb{R}^+\}$ *is closed under multiplication. In addition, we mention that we believe that there is an algebra* $\mathcal{A}\subset\mathcal{S}$ *having uncountably many generators.*

3. Entire functions

Recall that an operator $T:X\to X$ between Fréchet spaces is said to be *hypercyclic* if there is a vector $x\in X$ whose orbit under T, $orb(T,x)=\{x,T(x),T^2(x),...\}$ is dense in X. Although they may seem rare and counterintuitive at first, there are many natural instances when hypercyclic operators arise, and their relations to the theory of chaos and dynamical systems are well established.

In general, we will restrict our attention here to the two most basic hypercyclic operators T acting on the Fréchet space $\mathcal{H}(\mathbb{C})$ of entire functions in one complex variable. Namely, we will consider only the Birkhoff translation operator $T=\tau:\mathcal{H}(\mathbb{C})\to\mathcal{H}(\mathbb{C}),\ \tau(f)(z)=f(z+1)$, and the MacLane differentiation operator $T=D:\mathcal{H}(\mathbb{C})\to\mathcal{H}(\mathbb{C}),\ D(f)=f'$. It is known that for each of these (and every convolution operator that is not a multiple of the identity), the set $HC(T)$ of entire functions that are hypercyclic contains an infinite dimensional dense vector space (see, e.g., [**9**]). Here, we ask the question of whether $HC(T)$ contains an algebra. Namely, our focus here is on the size and behavior of the set of functions $f,g\in HC(T)$ such that fg is again in $HC(T)$. We will present some partial results from [**1**]. However, very many problems remain.

Proposition 3.1. ([**1**]) . *Suppose that* $f\in\mathcal{H}(\mathbb{C})$ *is such that for some* $k\in\mathbb{N}, f^k\in HC(\tau)$. *Then* $k=1$.

In particular, if f is hypercyclic for the Birkhoff translation operator τ, then f^k is not for every $k\geq 2$. In [**1**], the authors characterize $Orb(\tau,f^k)$ for an arbitrary $k\in\mathbb{N}$ and $f\in\mathcal{H}(\mathbb{C})$.

Proof. Suppose that $f^k\in HC(\tau)$ for some $k\geq 2$. Thus, the set $\{f^k(z),f^k(z+1),...,f^k(z+n),...\}$ is dense in $\mathcal{H}(\mathbb{C})$, with the usual compact-open topology. Consequently, for some sequence (n_j), it must be that

$$\sup_{|z|\leq 1}|f^k(z+n_j)-z|\to 0 \text{ as } j\to\infty.$$

Now, by Hurwitz's theorem (see, e.g, [**7**], p. 152), for sufficiently large j, $f^k(z+n_j)$ and z have the same number of zeros, namely one, in the unit disk. However, this is impossible if $k\geq 2$.

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We now turn to the same question for $T=D$. For each $k\geq 1$, set

$$M_k=\{f\in\mathcal{H}(\mathbb{C})\mid f^k\in HC(D)\}.$$

In other words, M_k is the set of those entire functions whose k^{th} power is hypercyclic for the differentiation operator.

Proposition 3.2. ([**1**]). *For each* $k\geq 1,\ M_k$ *is a dense* $G_\delta-$*set in* $\mathcal{H}(\mathbb{C})$.

Although the details of the proof are too technical to repeat here, the basic idea is to fix an arbitrary countable collection of basic open sets $(U_n)_{n=1}^{\infty}$ for $\mathcal{H}(\mathbb{C})$. For each n, one shows that the set $S(n)$ of entire functions f, such that for some $j \in \mathbb{N}$, $D^j(f^k) \in U_n$, is a dense open subset of $\mathcal{H}(\mathbb{C})$. Once this is shown, the rest of the argument is easy, since $M_k = \bigcap_{n=1}^{\infty} S(n)$. The proof of the denseness of each $S(n)$ is the key, difficult point. In very rough terms, the idea is to let $\varepsilon > 0$, $K \subset \mathbb{C}$ compact, $p(z) = \sum_{i=0}^{m'} a_i z^i$ and $q(z) = \sum_{i=0}^{m} b_i z^i \in U_n$ all be arbitrary. After some calculations, one shows that there exist $f(z) \in \mathcal{H}(\mathbb{C})$ and $j \in \mathbb{N}$ such that for $||f(z) - p(z)||_K < \varepsilon$ and $D^j f^k(z) = q(z)$. The reader is referred to [**1**] for complete details.

By taking intersections of the sets M_k, we obtain the following result.

Proposition 3.3. *The set of entire functions f such that $f^k \in HC(D)$ for every $k \in \mathbb{N}$ is a dense $G_\delta-$set in $\mathcal{H}(\mathbb{C})$.*

We conclude this section with several comments and questions.

Problem 3.1. *Although the preceding proposition can be regarded as positive "experimental evidence," it is unknown if $HC(D)$ contains an* algebra. *In particular, it is not even known if there is $f \in \mathcal{H}(\mathbb{C})$ such that every non-zero function in the algebra generated by f is hypercyclic for the differentiation operator.*

Problem 3.2. *The question of whether the set $HC(T)$ contains algebras for other natural hypercyclic operators T on $\mathcal{H}(\mathbb{C})$ remains open. In particular, the behavior of $\tau \circ D$, which takes $f(z) \to f'(z+1)$, relative to this question is unknown. On the other hand, the argument that is given here to show that no power of an entire function can be hypercyclic for τ has been used by J. H. Shapiro* [**16**] *to show that for any $f \in HC(\tau)$, $f : \mathbb{C} \to \mathbb{C}$, is surjective.*

Problem 3.3. *In addition, it is unknown whether $HC(D)$ contains a* **closed** *subspace. In connection with this, we mention recent work of H. Petersson* [**14**]*, in which the following is proved:*

Theorem 3.4. ([**14**]) *Let $T : \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^n)$ be a linear continuous mapping that satisfies the following condition: $T(\tau_b)(f) = \tau_b(T)(f)$, for every $f \in \mathcal{H}(\mathbb{C}^n)$ and every $b \in \mathbb{C}^n$. Then, if $n \geq 2$, $HC(T)$ contains an infinite dimensional closed subspace.*

4. Related results

An obvious requirement for large subspaces of functions having an unusual property is for there to exist a large set with the property. For this, a very frequent tool is the Baire category theorem. We describe here two situations in which the Baire theorem applies, but with different conclusions.

First, following Gurariy and Quarta [**12**], define the set $UM[0,1]$ to consist of those real-valued $f \in \mathcal{C}[0,1]$ such that f attains its maximum at a unique point. It is not difficult to show that $UM[0,1]$ is a dense $G_\delta-$set in $\mathcal{C}[0,1]$. Despite this, one has the following:

Theorem 4.1. ([**12**]). *If $V \subset UM[0,1]$ is a non-trivial vector space, then* $\dim V = 1$.

To conclude this note, we turn to the question of convergence of Fourier series on $\mathbb{T}$. In [**5**], F. Bayart shows that the set of functions in $L_1(\mathbb{T})$ whose Fourier series diverge everywhere on $\mathbb{T}$ is *spaceable*; i.e. there is a closed, infinite dimensional subspace $M \subset L_1(\mathbb{T})$ such that for every $f \in M$, $f \neq 0$, the Fourier series of f is everywhere divergent. In addition, Bayart shows that given a subset $E \subset \mathbb{T}$ of measure 0, there is a dense vector space of functions in $\mathcal{C}(\mathbb{T})$, such that the Fourier series of every non-zero function in this vector space diverges at every point of E. In fact, in a recent paper, the authors show the following:

THEOREM 4.2. ([**3**]). *Given any subset $E \subset \mathbb{T}$ of measure 0, there is an infinitely generated* dense algebra $\mathcal{A} \subset \mathcal{C}(\mathbb{T})$ *such that for all $f \in \mathcal{A}$, $f \neq 0$, the Fourier series of f diverges at every point of E.*

We conclude this paper with the following:

PROBLEM 4.1. *Characterize when there exists a* **closed** *infinite dimensional algebra of functions with a particular "strange" property? Note that in all the examples given above, none of the algebras constructed is closed. On the other hand, there do exist situations in which one can find Banach algebras of functions with a very special property. For instance, in* [**8**]*, it is shown that the collection of so-called Brodén type functions on $\mathcal{C}[a,b]$ contains a Banach algebra.*

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