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Lineability of sets of nowhere analytic functions

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Abstract

Although the set of nowhere analytic functions on $[0,1]$ is clearly not a linear space, we show that the family of such functions in the space of C^∞-smooth functions contains, except for zero, a dense linear submanifold. The result is even obtained for the smaller class of functions having Pringsheim singularities everywhere. Moreover, in spite of the fact that the space of differentiable functions on $[0,1]$ contains no closed infinite-dimensional manifold in $C([0,1])$, we prove that the space of real C^∞-smooth functions on $(0,1)$ does contain such a manifold in $C((0,1))$.

Keywords: Nowhere analytic function; C^∞-smooth function; Dense linear submanifold; Closed linear submanifold; Müntz spaces

1. Introduction, terminology and known results

In this paper, we are mainly concerned with the class of real or complex functions being infinitely many differentiable on an interval of the real line $\mathbb{R}$ but nowhere analytic.

Although at first glance the existence of such functions is somewhat surprising, the truth is that such existence is known at least from an example given in 1876 by du Bois Reymond [13]. An excellent survey about the first results on the subject (up to 1955) is the paper [32] by Salzmann and Zeller. An early, nice example is the following one due to M. Lerch (see [24] and [33]):

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(a^n x)}{n!},$$

where a is an odd positive integer.

Before going on, let us fix some notation and terminology. The symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Q}, \mathbb{C}$ will stand, respectively, for the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the set of rational numbers, and the complex plane, while $\mathbb{K}$ is any of the fields $\mathbb{R}, \mathbb{C}$. If X is a topological space and $A \subset X$, then A^0, $\bar{A}$ denote, respectively, its interior and its closure

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in X. A subset $A \subset X$ is said to be *meager*, or *of first category*, provided that there are subsets $A_n \subset X$ ($n \in \mathbb{N}$) such that $A = \bigcup_{n=1}^{\infty} A_n$ and $\bar{A}^0 = \emptyset$ for every $n \in \mathbb{N}$. The space X is a *Baire space* is no nonempty open subset is of first category. If X is a Baire space, then a subset $A \subset X$ is called *residual*, or *comeager*, if $X \setminus A$ is of first category, or equivalently, if A contains some dense G_δ subset. Every completely metrizable space is a Baire space (see [27]).

In the present paper, functions are primarily defined on the unit interval $I := [0, 1]$. Nevertheless, many results to be proved here (or already proved in the literature) can easily be extended to real or complex functions defined on a (closed or not) interval of $\mathbb{R}$, including the whole real line. By $C^\infty(I)$ we denote, as usual, the Fréchet space of all smooth (:= infinitely many times differentiable) $\mathbb{K}$-valued functions on I. It becomes a Fréchet space (:= complete metrizable locally convex space) if it is endowed with the topology generated by the seminorms

$$p_k(f) = \left\| f^{(k)} \right\|_I \quad (k \in \mathbb{N}_0),$$

where $\|h\|_A := \sup\{|h(x)|\colon x \in A\}$ for any set A and any function $h : A \to \mathbb{K}$. Let us denote by $\mathcal{K}(A)$ the family of all compact subsets of A. If J is an open interval of $\mathbb{R}$, then the space $C^\infty(J)$ of smooth $\mathbb{K}$-valued functions on J is a Fréchet space whenever it is endowed with the seminorms $f \mapsto \|f^{(k)}\|_L$ ($k \in \mathbb{N}_0, L \in \mathcal{K}(J)$) (see [22, p. 136]). Hence $C^\infty(I)$ and $C^\infty(J)$ are Baire spaces. The same is true for the Banach space $C(I)$ of continuous functions on I (endowed with the supremum norm $\|\cdot\|_I$) and the Fréchet space $C(I^0) = C((0, 1))$ (endowed with the seminorms $\|\cdot\|_L$, $L \in \mathcal{K}(I^0)$).

If $f \in C^\infty(I)$ and $x_0 \in I$, then f is said to be *analytic* at x_0 provided that the Taylor series

$$T(f, x_0) := \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

converges to f in some neighborhood of x_0. The point x_0 is said to be *singular* for f if f is not analytic at x_0. It is easy to see that the set $S(f) := \{x \in I\colon f \text{ is singular at } x\}$ is closed in I. If $x_0 \in S(f)$, then there are two possibilities: Either $T(f, x_0)$ converges in no neighborhood of x_0 (a *Pringsheim singularity*) or $T(f, x_0)$ converges in some neighborhood of x_0 but to a function which is different from f (a *Cauchy singularity*). The respective set will be denoted by $PS(f)$ and $CS(f)$ (so $S(f) = PS(f) \cup CS(f)$, a disjoint union). It follows that a Pringsheim singularity is "worse" than a Cauchy one. We have that $x_0 \in PS(f)$ if and only if $\rho(f, x_0) = 0$, where $\rho(f, x_0)$ denotes the radius of convergence of $T(f, x_0)$, that is,

$$\rho(f, x_0) = \left(\limsup_{n \to \infty} \left| \frac{f^{(n)}(x_0)}{n!} \right|^{1/n} \right)^{-1}.$$

For instance, the classical function $g : I \to \mathbb{R}$ given by

$$g(x) = \begin{cases} \exp(-1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1}$$

satisfies $S(g) = CS(g) = \{0\}$, $PS(g) = \emptyset$. It is easy to see that $PS(f)$ is a G_δ subset of I, for every $f \in C^\infty(I)$. In 1893, Pringsheim [28] proved that $CS(f)$ is never very large; specifically, it cannot contain an interval. The exact structure of $CS(f)$ and $PS(f)$ was given by Zahorski [34] in 1947. He proved that, given two subsets $A, B \subset I$, then there exists an $f \in C^\infty(I)$ with $CS(f) = A$, $PS(f) = B$ if and only if the following holds:

(a) A is an F_σ subset of first category and B is a G_δ subset.
(b) $A \cup B$ is closed and $A \cap B = \emptyset$.

Let us denote by $\mathcal{S}$ the set of all smooth nowhere analytic functions—that is, $\mathcal{S} = \{f \in C^\infty(I)\colon S(f) = I\}$—and by $\mathcal{PS}$ the (smaller) set of all smooth functions with a Pringsheim singularity at every point, that is, $\mathcal{PS} = \{f \in C^\infty(I)\colon PS(f) = I\}$. Observe that $\{f \in C^\infty(I)\colon CS(f) = I\} = \emptyset$. By contrast, the above stated Zahorski result proves specially that $\mathcal{PS} \neq \emptyset$. In Zahorski's paper, it is established the following question posed by Steinhaus and Marczewski: Is $\mathcal{PS}$ not only a nonempty family, but even topologically generic? More specifically, is $\mathcal{PS}$ a residual subset in $C^\infty(I)$? A positive answer would imply, of course, the topological genericity of the (bigger) set $\mathcal{S}$ in $C^\infty(I)$.

By using the fact that $C^\infty(I)$ is a Baire space, Morgenstern [26] proved in 1954 the last assertion: The set $\mathcal{S}$ is residual in $C^\infty(I)$. And Salzmann and Zeller [32, Section 2] answered in 1955 Steinhaus–Marczewski's question in

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the affirmative: $\mathcal{PS}$ is residual in $C^\infty(I)$. Their papers are probably not well known (with the additional handicap that the proof in [26] contains two gaps, see [32, p. 356]), because several authors have published later similar results. Namely, Christensen [10] established in 1971 that the set $\mathcal{S}_0 := \{f \in C^\infty(I)$: there exists a residual subset $A_f \subset I$ such that $\rho(f,x) = 0$ for all $x \in A_f\}$ is residual in $C^\infty(I)$, and Darst [11] proved in 1973 the residuality of $\mathcal{S}$ in $C^\infty(I)$. Note that $\mathcal{PS} \subset \mathcal{S}_0 \subset \mathcal{S}$, where the last inclusion derives from the closedness of $S(f)$. It follows that Christensen's result implies Morgenstern–Darst's result, but does not imply the residuality of $\mathcal{PS}$. Finally, the author [6] in 1987 (for *complex* functions) and Ramsamujh [29] in 1991 (for real or complex functions) obtained that $\mathcal{PS}$ is residual, with proofs very different from that of Salzmann–Zeller.

Remarks 1.1.

1. Let $f \in C^\infty(I)$. The example (1) (note that $g^{(n)}(0) = 0$ for all $n \geqslant 0$) shows that even the condition $\rho(f,x_0) = +\infty$ is not enough for f to be analytic at x_0. Nevertheless, if there is a neighborhood U of x_0 such that $\inf_{x\in U} \rho(f,x) > 0$, then f is analytic at x_0 [28]. The exact condition is: f is analytic at x_0 if and only if $\sup_{n\in\mathbb{N}}(\frac{\|f^{(n)}\|_U}{n!})^{1/n} < +\infty$ for some neighborhood U of x_0 (see [25, Chapter 1]). Another sufficient condition is furnished by Bernstein's theorem [12, pp. 51–52]: If $x_0 \in I^0$ and there exists a neighborhood U of x_0 with either $f^{(n)}(x) \geqslant 0$ for all $(x,n) \in U \times \mathbb{N}_0$ or $(-1)^n f^{(n)}(x) \geqslant 0$ for all $(x,n) \in U \times \mathbb{N}_0$, then f is analytic at x_0.
2. We may have $f \in \mathcal{S}$ with $\rho(f,x) = +\infty$ at a dense set of points. For instance, in [1] it is exhibited a function $f \in \mathcal{S}$ such that $T(f,x_0)$ is a polynomial at each diadic point x_0 (see also [15]).
3. In [6], the author obtains the residuality of $\mathcal{PS}$ as a corollary of a more general statement, namely, given a pair of sequences $(a_n), (b_n) \subset (0,+\infty)$, the class of C^∞-smooth functions $f : I \to \mathbb{C}$ such that

$$\liminf_{n\to\infty}\bigl(a_n\bigl|f^{(n)}(x)\bigr|\bigr) = 0 \quad \text{and} \quad \limsup_{n\to\infty}\bigl(b_n\bigl|f^{(n)}(x)\bigr|\bigr) = +\infty \quad \text{for all } x \in I,$$

is residual. In other words, most smooth functions have sequences of derivatives that are "big and small everywhere." The result was inspired by Borel's theorem asserting that, given a point $x_0 \in \mathbb{R}$ and a sequence $(c_n) \subset \mathbb{R}$, there exists a function $f \in C^\infty(\mathbb{R})$ with $f^{(n)}(x_0) = c_n$ $(n \in \mathbb{N}_0)$ [12, pp. 50–51].
4. Smooth nowhere analytic functions with additional properties have been constructed. For instance, Kim and Kwon exhibited in 2000 [23] an increasing function $f_0 \in \mathcal{S}$. Incidentally, if we set $F(x) := \int_0^x f_0(t)\,dt$, then we obtain a convex function $F \in \mathcal{S}$.
5. An interesting, trivial property of the class $\mathcal{PS}$ is its invariance under derivatives: If $D : f \in C^\infty(I) \to f' \in C^\infty(I)$ is the derivative operator, then $D(\mathcal{PS}) = \mathcal{PS} = D^{-1}(\mathcal{PS})$. The same holds for $\mathcal{S}$.

Once established the big topological size of $\mathcal{S}$, it is natural to wonder whether $\mathcal{S}$ possesses a big *algebraic* size. The fact that $\mathcal{S}$ is not a linear manifold increases the interest in this matter. Precisely, under the terminology of Gurariy and Quarta [21], we pose here the problem of the lineability of $\mathcal{S}$ in $C^\infty(I)$. If X is a topological vector space and A is a subset of X, then the *lineability* $\lambda(A)$ of A is defined as the maximum cardinality of the linear manifolds $M \subset X$ such that $M \setminus \{0\} \subset A$. A subset $A \subset X$ is called *lineable* if $A \cup \{0\}$ contains an infinite-dimensional linear manifold, that is, if $\lambda(A) \geqslant \operatorname{card}(\mathbb{N})$. Recall that $\dim(X) = \chi$ (:= the cardinality of the continuum) if X is a complete metrizable separable infinite-dimensional topological vector space (for instance, $X = C^\infty(I)$), so $\lambda(A) \leqslant \chi$ for every subset A in such a space. The two (stronger than mere lineability) notions of spaceability and algebraic genericity were introduced respectively in [21] and in Bayart's paper [4]. A subset $A \subset X$ is called *spaceable* (*algebraically generic*, respectively) in X if $A \cup \{0\}$ contains a linear manifold M such that M is closed and infinite-dimensional (such that M is dense in X, respectively). Examples of sets that are not linear manifolds but having some of the three latter properties can be found in [2–5,7,8,14,16–19,30]. Among these references, we emphasize specially that Fonf, Gurariy and Kadec [19] showed that the set of *nowhere* differentiable functions is spaceable (see [16] and [18] for the weaker property of lineability). In fact, much more is true: L. Rodríguez-Piazza [30] proved that every separable Banach space is isometric to a space of continuous nowhere differentiable functions.

In this paper, we turn our attention to analogous results for smooth functions. Our main aim is to establish the algebraic genericity of the class $\mathcal{PS}$ (so of $\mathcal{S}$) in $C^\infty(I)$, see Section 2. In Section 3, we also state that $\mathcal{S}$ has maximal lineability, that is, $\lambda(\mathcal{S}) = \chi$. In the complex case, it is even obtained that $\lambda(\mathcal{PS}) = \chi$. Furthermore, we focus our interest on the "algebraic status" of the class of smooth functions within the space of real continuous functions. The set $\mathcal{D}(I)$ of *everywhere* differentiable functions on I is linear and hence lineable (in fact, $\mathcal{D}(I)$ is algebraically generic,

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because it contains all polynomials). But Gurariy proved in [16] that this cannot be improved: $\mathcal{D}(I)$ is not spaceable in $C(I)$. This implies, trivially, that $C^\infty(I)$ is not spaceable in $C(I)$. Nevertheless, the situation is this time very different if we replace I by $I^0 = (0,1)$. In fact, we will prove in Section 4 that the space $C^\infty(I^0)$ is spaceable in $C(I^0)$.

2. Algebraic genericity of $\mathcal{PS}$

In order to establish the existence of large linear manifolds in $\mathcal{PS}$, we need the following auxiliary result, which asserts the existence of smooth functions with successive derivatives as big as desired. In fact, we obtain topological genericity by using a Baire category approach.

Lemma 2.1. *Let $(c_n) \subset (0,+\infty)$ be a sequence of positive real numbers, and M be an infinite subset of $\mathbb{N}_0$. Then the set*

$$\mathcal{M}\big((c_n), M\big) := \Big\{ f \in C^\infty(I)\text{: there are infinitely many } n \in M \text{ such that } \max\big\{\big|f^{(n)}(x)\big|, \big|f^{(n+1)}(x)\big|\big\} > c_n \\ \text{for all } x \in I \Big\}$$

is residual in $C^\infty(I)$.

Proof. Let (c_n), M be as in the hypothesis. The assertion of the lemma is equivalent to say that the set $\mathcal{A} := C^\infty(I) \setminus \mathcal{M}((c_n), M)$ is of first category. To prove this, observe that $\mathcal{A} = \bigcup_{N\in M} A_N$, where $A_N := \bigcap_{k>N,\, k\in M} B_k$ and

$$B_k := \Big\{ f \in C^\infty(I)\text{: there exists } x = x(f) \in I \text{ such that } \max\big\{\big|f^{(k)}(x)\big|, \big|f^{(k+1)}(x)\big|\big\} \leqslant c_k \Big\}.$$

Now, each map

$$\Phi_k : (f,x) \in C^\infty(I) \times I \mapsto \max\big\{\big|f^{(k)}(x)\big|, \big|f^{(k+1)}(x)\big|\big\} \in [0,+\infty) \quad (k \in \mathbb{N}_0)$$

is continuous. Therefore the set $\Phi_k^{-1}([0,c_k])$ is closed in $C^\infty(I) \times I$. Consequently, its projection on $C^\infty(I)$ is closed, because it is a projection that is parallel to I, which is compact. But such projection is precisely B_k, so B_k is closed. Since A_N is an intersection of certain sets B_k, we obtain that each A_N is also closed. It follows that $\mathcal{A}$ is an F_σ set.

It is enough to show that each A_N has empty interior. By way of contradiction, let us assume that $A_N^0 \neq \emptyset$. Then there would exist a basic neighborhood $U(g,\alpha,m) := \{h \in C^\infty(I)\text{: } p_j(h-g) < \alpha \text{ for all } j = 0,1,\dots,m\}$ such that $U(g,\alpha,m) \subset A_N$, for certain $g \in C^\infty(I)$, $\alpha > 0$ and $m \in \mathbb{N}$. By the density of the set of polynomials in $C^\infty(I)$, there are a polynomial P, a number $\varepsilon \in (0,1)$ and a positive integer n with $U(P,\varepsilon,n) \subset U(g,\alpha,m)$, so

$$U(P,\varepsilon,n) \subset A_N. \tag{2}$$

Let us choose $k \in M$ with $k > \max\{n, N, \text{ degree } (P)\}$, and let

$$b := (1 + c_k)\left(\frac{4}{\varepsilon}\right).$$

So $b > 1$. Now, we define the function

$$f(x) := P(x) + \frac{\varepsilon \sin(bx)}{2b^n}.$$

Note that the absolute value of the mth-derivative of the function $\varphi(x) := \sin(bx)$ is $b^m|\sin(bx)|$ (if m is even) or $b^m|\cos(bx)|$ (if m is odd). Then we have, for every $j \in \{0,1,\dots,n\}$, that

$$p_j(f-P) = \sup_{x\in I}\left|\frac{\varepsilon\varphi^{(j)}(x)}{2b^n}\right| \leqslant \frac{\varepsilon}{2} b^{j-n} \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $f \in U(P,\varepsilon,n)$. On the other hand, we have for any $x \in I$ that either $|\sin(bx)| \geqslant 1/2$ or $|\cos(bx)| \geqslant 1/2$, because of the basic law $\sin^2 t + \cos^2 t = 1$. Fix $x \in I$. Then we can select for each $j \in \mathbb{N}_0$ one number $m(j) \in \{j, j+1\}$ such that $|\varphi^{(m(j))}(x)| \geqslant b^{m(j)}/2$. Consequently,

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$$\max\{|f^{(k)}(x)|, |f^{(k+1)}(x)|\} \geqslant |f^{(m(k))}(x)| = \left|P^{(m(k))}(x) + \frac{\varepsilon}{2b^n}\varphi^{(m(k))}(x)\right| = \left|\frac{\varepsilon}{2b^n}\varphi^{(m(k))}(x)\right|$$
$$\geqslant \frac{\varepsilon}{4}b^{k-n} \geqslant \frac{\varepsilon}{4}b = 1 + c_k > c_k.$$

To summarize, we have found $k \in M$ with $k > N$ such that $\max\{|f^{(k)}|, |f^{(k+1)}|\} > c_k$ on I, a contradiction with (2). □

Remarks 2.2.

1. Lemma 2.1 holds in both cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If, specially, $\mathbb{K} = \mathbb{C}$, then the last proof works by replacing $\varphi(x) := \sin(bx)$ by $\varphi(x) := \exp(ibx)$. Hence we obtain a slightly stronger result in this case, namely, for each sequence $(c_n) \subset (0, +\infty)$, the set $\{f \in C^\infty(I)$: there are infinitelymany $n \in M$ such that $|f^{(n)}(x)| > c_n$ for all $x \in I\}$ is residual in $C^\infty(I)$. It is even possible to construct an explicit function $f \in C^\infty(I)$ satisfying $|f^{(n)}(x)| > c_n$ ($n \in \mathbb{N}$, $x \in I$) if $\mathbb{K} = \mathbb{C}$: Take $f(x) = \sum_{k=1}^\infty b_k^{1-k}\exp(ib_k x)$, where $b_k = 2 + c_k + \sum_{j=1}^{k-1} b_j^{k+1-j}$ (the last sum is defined as 0 if $k = 1$), see [6, Lemma]. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then with the same approach of this reference one can obtain an explicit function $F \in C^\infty(I)$ satisfying $\max\{|F^{(n)}|, |F^{(n+1)}|\} > c_n$ ($n \in \mathbb{N}$) on I: It is enough to take $F(x) = \sum_{k=1}^\infty b_k^{1-k}\sin(b_k x)$, where $b_k = 2(2 + c_k + [c_{k-1} + \sum_{j=1}^{k-1} b_j^{k+1-j}])$ (the term within the square brackets is defined as 0 if $k = 1$). This function could be used to furnish an alternative second part of the proof of Lemma 2.1. Indeed, if $\mathcal{Q}$ is the set of all polynomials, then $\mathcal{Q}$ is dense in $C^\infty(I)$, so the set $F + \mathcal{Q}$ of its F-translates is also dense. But the last set is contained in the G_δ set $\mathcal{M}((c_n), M)$. Thus, $\mathcal{M}((c_n), M)$ is a dense G_δ set, so residual.

2. The first result of the preceding remark completely fails to hold for $\mathbb{K} = \mathbb{R}$. In fact, for certain sequences (c_n), the set $A := \{f \in C^\infty(I)$: there are infinitely many $n \in \mathbb{N}$ such that $|f^{(n)}(x)| > c_n$ for all $x \in I\}$ can even be empty. To see this, fix $f \in C^\infty(I)$, $n \in \mathbb{N}_0$ and $c \in (0, +\infty)$, and suppose that $|f^{(n)}(x)| > c$ for any $x \in I$. We claim that there exists an interval $J \subset I$ of length $1/4^n$ such that $|f(x)| \geqslant c/4^n$. Let us prove this fact by induction on n. Observe that the result is clear for $n = 0$. Suppose that it has been proved for $n - 1$, and let us prove it for n. Without less of generality, we may suppose $f^{(n)}(x) > c$ for any $x \in I$. Integrating, we get $f^{(n-1)}(x) - f^{(n-1)}(y) \geqslant c(x - y)$ for any $x \geqslant y$ in I. On the one hand, if $f^{(n-1)}$ does not vanish on I, we may suppose it is positive. Thus, when $x \geqslant 1/2$, we obtain $f^{(n-1)}(x) \geqslant f^{(n-1)}(x) - f^{(n-1)}(0) \geqslant c(x - 0) \geqslant c/2$. On the other hand, if $f^{(n-1)}$ vanishes on I, say at x_0, then there exists an interval $J \subset I$ of size 1/4 such that $|x_0 - x| \geqslant 1/4$ for all $x \in J$. Hence $|f^{(n-1)}(x)| = |f^{(n-1)}(x) - f^{(n-1)}(x_0)| \geqslant c|x - x_0| \geqslant c/4$ ($x \in J$). The claim now follows by induction hypothesis. In particular, for $c_n = 8^n$, the set A is empty.

3. To demonstrate Lemma 2.1, we had primarily tried to follow the elegant approach of the proof of Theorem 1 in [29], where it is asserted the residuality of $\mathcal{PS}$ in $C^\infty(I)$. But there is a gap in the final part of it (with the notation of [29], it is there needed to exhibit for *every* $x_0 \in I$ a number m with $|g^{(m)}(x_0)| > M^m m!$, not only for a point x_0 furnished by a function $f \in F(M)$). Nevertheless, the residuality of $\mathcal{PS}$ (already proved in [32], as mentioned in Section 1) is, of course, true: Choose $M = \mathbb{N}$ and $c_n = (n + 1)!(n + 1)^{n+1}$ in Lemma 2.1.

We are now ready to state the main result of this section, namely, the existence of dense linear manifolds of smooth functions having Pringsheim singularities everywhere. As a matter of fact, the same holds for smooth functions having derivatives of large orders as big as desired at all points.

Theorem 2.3.

(a) *Let (c_n) be a sequence in $(0, +\infty)$. Then the set*

$$\mathcal{A}((c_n)) := \left\{f \in C^\infty(I)\colon \limsup_{n\to\infty} \frac{|f^{(n)}(x)|}{c_n} = +\infty \text{ for all } x \in I\right\}$$

is algebraically generic in $C^\infty(I)$.

(b) *The set $\mathcal{PS}$ is algebraically generic in $C^\infty(I)$.*

Proof. Part (b) derives from (a) simply by taking $c_n := n!n^n$.

Let us prove (a). Since the set of all polynomials is dense in $C^\infty(I)$, this metric space is separable, so second-countable. It follows that one can find a countable open basis $\{V_n\colon n \in \mathbb{N}\}$ for its topology. Let $M_0 := \mathbb{N}$ and $d_n :=$

$\max\{c_n, c_{n+1}\}$ $(n \geqslant 0)$. According to Lemma 2.1, the set $\mathcal{M}((n(1+d_n)), M_0)$ is residual, hence dense. This allows us to choose a function

$$f_1 \in \mathcal{M}\big(\big(n(1+d_n)\big), M_0\big) \cap V_1.$$

Then there is an infinite subset $M_1 \subset M_0$ such that $\max\{|f_1^{(n)}(x)|, |f_1^{(n+1)}(x)|\} > n(1+d_n)$ for all $n \in M_1$ and all $x \in I$. Again by Lemma 2.1, the set $\mathcal{M}((n(1+d_n)(1+\|f_1^{(n)}\|_I + \|f_1^{(n+1)}\|_I)), M_1)$ is dense, so we can pick a function

$$f_2 \in \mathcal{M}\big(\big(n(1+d_n)\big(1+\big\|f_1^{(n)}\big\|_I + \big\|f_1^{(n+1)}\big\|_I\big)\big), M_1\big) \cap V_2.$$

An induction procedure leads us to the construction of a sequence of functions $\{f_k\colon k \in \mathbb{N}\} \subset C^\infty(I)$ and of a nested sequence of infinite sets $M_0 \supset M_1 \supset M_2 \supset M_3 \supset \cdots$ satisfying, for all $k \in \mathbb{N}$, $x \in I$, $n \in M_k$, that

$$f_k \in V_k \tag{3}$$

and

$$\max\big\{\big|f_k^{(n)}(x)\big|, \big|f_k^{(n+1)}(x)\big|\big\} > n(1+d_n)\left(1+\sum_{j=1}^{k-1}\big(\big\|f_j^{(n)}\big\|_I + \big\|f_j^{(n+1)}\big\|_I\big)\right) \tag{4}$$

where the last sum is defined as 0 if $k = 1$.

Next, let us define

$$\mathcal{D} := \operatorname{span}\big(\{f_k\colon k \in \mathbb{N}\}\big).$$

It is derived form (3) that $\{f_k\colon k \in \mathbb{N}\}$ is dense, so $\mathcal{D}$ is dense linear submanifold of $C^\infty(I)$. It remains to prove that every function $f \in \mathcal{D} \setminus \{0\}$ belongs to $\mathcal{A}((c_n))$. To prove it, note that for such a function there exist $N \in \mathbb{N}$ and real constants a_k $(k = 1, \ldots, N)$ such that $a_N \neq 0$ and $f = a_1 f_1 + \cdots + a_N f_N$. Since $a_N \neq 0$, we can find $n_0 \in \mathbb{N}$ such that $n|a_N| \geqslant a_k$ for all $n \geqslant n_0$ and all $k = 1, \ldots, N-1$. If $x \in I$ is fixed, by (4) we can select for each $n \in M_N$ one value $m(n) \in \{n, n+1\}$ such that $|f_N^{(m(n))}(x)| > n(1+d_n)(1+\sum_{j=1}^{N-1}(\|f_j^{(n)}\|_I + \|f_j^{(n+1)}\|_I))$. It follows, for all $x \in I$ and all $n \in M_N$ with $n \geqslant n_0$, that

$$\begin{aligned}
\frac{|f^{(m(n))}(x)|}{d_n} &\geqslant \frac{1}{d_n}|a_N|\big|f_N^{(m(n))}(x)\big| - \frac{1}{d_n}\sum_{k=1}^{N-1}|a_k|\big|f_k^{(m(n))}(x)\big| \\
&\geqslant \frac{1}{d_n}\left[n|a_N|(1+d_n)\left(1+\sum_{k=1}^{N-1}\big(\big\|f_k^{(n)}\big\|_I + \big\|f_k^{(n+1)}\big\|_I\big)\right) - \sum_{k=1}^{N-1}|a_k|\big\|f_k^{(m(n))}\big\|_I\right] \\
&\geqslant \frac{1}{d_n}\left[nd_n|a_N| + \sum_{k=1}^{N-1}\big(n|a_N| - |a_k|\big)\big\|f_k^{(m(n))}\big\|_I\right] \geqslant n|a_N|.
\end{aligned}$$

Hence $\lim_{n\to\infty,\, n\in M_N} \frac{|f^{(m(n))}(x)|}{c_{m(n)}} = +\infty$ for each $x \in I$, because $c_{m(n)} \leqslant d_n$ $(n \geqslant 1)$. Consequently,

$$\limsup_{n\to\infty} \frac{|f^{(n)}(x)|}{c_n} = +\infty \quad (x \in I),$$

so $f \in \mathcal{A}((c_n))$. $\square$

Remark 2.4. In the case $\mathbb{K} = \mathbb{C}$, we can use the version of Lemma 2.1 given in Remark 2.2.1 together with the approach of the last proof to show a little more, namely, the set $\{f \in C^\infty(I)\colon \limsup_{n\to\infty} \inf_{x\in I} \frac{|f^{(n)}(x)|}{c_n} = +\infty\}$ is algebraically generic for any sequence $(c_n) \subset (0, +\infty)$. Consequently, the set

$$\left\{f \in C^\infty(I)\colon \limsup_{n\to\infty} \inf_{x\in I}\left(\frac{|f^{(n)}(x)|}{n!}\right)^{1/n} = +\infty\right\}$$

—which is smaller that $\mathcal{PS}$—is algebraically generic.

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We close this section by posing the following problem.

Question 1. Since each function in $\mathcal{PS}$ is everywhere differentiable, we have by [16] that $\mathcal{PS}$ is not spaceable in $C(I)$. But, is $\mathcal{PS}$ spaceable in $C^\infty(I)$?

3. Maximal lineability of $\mathcal{S}$

From Theorem 2.3 it is derived, of course, the algebraic genericity of the class $\mathcal{S}$. In particular, $\mathcal{S}$ is lineable: $\lambda(\mathcal{S}) \geqslant \operatorname{card}(\mathbb{N})$. The next theorem will prove that much more is true. Precisely, $\lambda(\mathcal{S}) = \chi$. In fact, an adequate manifold can be found to guarantee simultaneously both properties of algebraic genericity and maximal lineability.

Theorem 3.1. *There exists a linear submanifold $\mathcal{D}$ of $C^\infty(I)$ satisfying the following*:

(a) *$\mathcal{D}$ is dense in $C^\infty(I)$.*
(b) $\dim(\mathcal{D}) = \chi$.
(c) $\mathcal{D} \setminus \{0\} \subset \mathcal{S}$.

Proof. Let us fix a translation-invariant distance d defining the topology of $C^\infty(I)$. Fix also a function $\varphi \in \mathcal{S}$. Let $\{P_n\}_{n\geqslant 1}$ be an enumeration of the polynomials with coefficients in $\mathbb{Q}$ (if $\mathbb{K} = \mathbb{R}$) or in $\mathbb{Q} + i\mathbb{Q}$ (if $\mathbb{K} = \mathbb{C}$). Then $\{P_n\}_{n\geqslant 1}$ is a dense subset of $C^\infty(I)$. Consider, for each $\alpha \in \mathbb{R}$, the function $e_\alpha(x) := \exp(\alpha x)$. The continuity of the scalar multiplication in the topological vector space $C^\infty(I)$ allows to assign to each $\alpha > 0$ a number $\varepsilon_\alpha > 0$ such that $d(0, \varepsilon_\alpha e_\alpha \varphi) < 1/\alpha$. Denote $\varphi_\alpha := \varepsilon_\alpha e_\alpha \varphi$ and $f_{n,\alpha} := P_n + \varphi_\alpha$ ($\alpha > 0$, $n \in \mathbb{N}$). It follows that

$$d(P_n, f_{n,\alpha}) < \frac{1}{\alpha} \quad (\alpha > 0,\ n \in \mathbb{N}).$$

Now, let us define

$$\mathcal{D} := \operatorname{span}\bigl(\bigl\{f_{n,\alpha}\colon \alpha \in [n, n+1),\ n \in \mathbb{N}\bigr\}\bigr).$$

It is clear that $\mathcal{D}$ is a linear submanifold of $C^\infty(I)$. Our task is to show that $\mathcal{D}$ satisfies (a), (b) and (c).

Firstly, observe that $\mathcal{D} \supset \{f_{n,n}\}_{n\geqslant 1}$ and that the set $\{f_{n,n}\}_{n\geqslant 1}$ is dense because $\{P_n\}_{n\geqslant 1}$ is and $d(P_n, f_{n,n}) < 1/n \to 0$ ($n \to \infty$). Therefore $\mathcal{D}$ is also dense, which proves (a).

Before proving (b), we need to show that, for each nonempty subset $A \subset \mathbb{R}$, the functions e_α ($\alpha \in A$) are linearly independent. Suppose that this is not the case. Then there would exist a number $N \in \mathbb{N}$, scalars $c_1, \ldots, c_N$ with $c_N \neq 0$ and values $\alpha_1 < \cdots < \alpha_N$ in A such that $c_1 e_{\alpha_1} + \cdots + c_N e_{\alpha_N} = 0$ on I. From the Analytic Continuation Principle, we obtain that the last equality holds on the whole line $\mathbb{R}$. We can suppose that $N \geqslant 2$. This implies

$$0 \neq c_N = -\bigl(c_1 e_{\alpha_1 - \alpha_N}(x) + \cdots + c_N e_{\alpha_{N-1} - \alpha_N}(x)\bigr) \to 0 \quad \text{as } x \to +\infty,$$

which is absurd. This shows the claimed linear independence.

In order to demonstrate (b), it is enough to show that, for each polynomial P and any nonempty subset $A \subset \mathbb{R}$, the functions $P + \varphi_\alpha$ ($\alpha \in A$) are linearly independent. Indeed, since $\mathcal{D} \supset \{P_1 + \varphi_\alpha\colon \alpha \in [0,1)\}$, we would have $\dim(\mathcal{D}) \geqslant \operatorname{card}([0,1)) = \chi$, from which (b) follows. So, fix P and A as above. Assume, by way of contradiction, that the functions $P + \varphi_\alpha$ ($\alpha \in A$) are not linearly independent. Then there would exist $N \in \mathbb{N}$, $c_1, \ldots, c_N$ with $c_N \neq 0$ and $\alpha_1 < \cdots < \alpha_N$ in A such that $c_1(P + \varphi_{\alpha_1}) + \cdots + c_N(P + \varphi_{\alpha_N}) = 0$ on I. Let $\psi := c_1 \varepsilon_{\alpha_1} e_{\alpha_1} + \cdots + c_N \varepsilon_{\alpha_N} e_{\alpha_N}$. Due to the linear independence of the functions e_α and to the continuity of ψ, there is an open interval $J \subset I$ such that $\psi(x) \neq 0$ for all $x \in J$. Therefore

$$\varphi(x) = -\frac{(\sum_{j=1}^N c_j) P(x)}{\psi(x)} \quad (x \in J).$$

Hence φ would be analytic on J. This is the desired contradiction.

Finally, we prove (c). Fix a function $f \in \mathcal{D} \setminus \{0\}$. Suppose, again by way of contradiction, that $f \notin \mathcal{S}$. Then $S(f) \neq I$ and there exist numbers $N \in \mathbb{N}$, $m_1, \ldots, m_N \in \mathbb{N}$, scalars $c_{j,k}$ and values $\alpha(j,k) \in [j, j+1)$ ($k = 1, \ldots, m_j$; $j = 1, \ldots, N$) satisfying $\alpha(j,1) < \cdots < \alpha(j, m_j)$ for all $j = 1, \ldots, N$, $c_{N,m_N} \neq 0$ and $f =$

$\sum_{j=1}^{N}\sum_{k=1}^{m_j} c_{j,k}(P_j + \varphi_{\alpha(j,k)})$. The key point is that the values $\alpha(j,k)$ are pairwise distinct. Let us set $h := \sum_{j=1}^{N}\sum_{k=1}^{m_j} c_{j,k}\varepsilon_{\alpha(j,k)}e_{\alpha(j,k)}$. By the claim proved above, this function is not identically zero on I. Also, thanks to the Analytic Continuation Principle, the set Z of zeros of h in the compact interval I cannot be infinite. Then $I \setminus (S(f) \cup Z)$ is a nonempty relatively open subset of I. Consequently, there is an interval $J \subset I$ where f is analytic and h vanishes at no point. Moreover, we have $f = Q + h\varphi$, where Q is the polynomial $Q = \sum_{j=1}^{N}(\sum_{k=1}^{m_j} c_{j,k})P_j$. It is derived that

$$\varphi(x) = \frac{f(x) - Q(x)}{h(x)}$$

on J. But this would force the analyticity of φ on such interval, a contradiction. □

In the case $\mathbb{K} = \mathbb{C}$, it is possible to obtain maximal lineability for the class $\mathcal{PS}$.

Theorem 3.2. *Assume that $\mathbb{K} = \mathbb{C}$. Then $\lambda(\mathcal{PS}) = \chi$, that is, there exists a linear submanifold $\mathcal{D}$ of $C^\infty(I)$ satisfying* $\dim(\mathcal{D}) = \chi$ *and* $\mathcal{D} \setminus \{0\} \subset \mathcal{PS}$.

Proof. We follow the notation of the proof of Theorem 3.1. Fix a function $f \in \mathcal{PS}$ and consider

$$\mathcal{D} := \operatorname{span}\bigl(\{f e_\alpha\colon \alpha \in I\}\bigr).$$

Obviously, $\mathcal{D}$ is a linear submanifold of $C^\infty(I)$. Let us show that $\dim(\mathcal{D}) = \chi$. For this, it is enough to prove the linear independence of the functions $f e_\alpha$ ($\alpha \in I$). This follows from the following facts: the functions e_α are linearly independent, a finite linear combination of these functions is analytic, the set of zeros in I of an analytic function on I is finite and, finally, a Pringsheim singular function cannot vanish identically on an interval.

Therefore, our task is to select $f \in \mathcal{PS}$ such that $\mathcal{D} \setminus \{0\} \subset \mathcal{PS}$. Let us define inductively a pair of suitable sequences $(c_n), (b_n) \subset (0, +\infty)$. Firstly, set $c_1 := 4$, $b_1 := 2 + c_1$. Assume now that, for some integer $n \geqslant 2$, the numbers $c_1, \dots, c_{n-1}, b_1, \dots, b_{n-1}$ have already been determined. Then we define

$$c_n := 4 + 2\sum_{k=1}^{n-1} b_k^{n+1-k} + (2n)!(2n)^{2n} + (2n)!n\sum_{k=1}^{n-1} c_k,$$

$$b_n := 2 + c_n + \sum_{k=1}^{n-1} b_k^{n+1-k}.$$

Note that $b_n > 2$ for all $n \in \mathbb{N}$.

Secondly, we define f as in the beginning of Remark 2.2.1, that is,

$$f(x) := \sum_{k=1}^{\infty} b_k^{1-k}\exp(ib_k x).$$

Then $f \in C^\infty(I)$ and $|f^{(n)}(x)| > c_n$ ($n \in \mathbb{N}$, $x \in I$) [6, Lemma]. Moreover, since $c_n > n!n^n$, we get $\rho(f,x) = 0$ ($x \in I$), so $f \in \mathcal{PS}$. On the other hand, we have for $n \in \mathbb{N}$ and $x \in I$ that

$$\bigl|f^{(n)}(x)\bigr| \leqslant \bigl|b_n^{n+1-n} i^n \exp(ib_n x)\bigr| + \sum_{k \neq n}\bigl|b_k^{n+1-k} i^k \exp(ib_k x)\bigr| = b_n + \sum_{k=1}^{n-1} b_k^{n+1-k} + \bigl(1 + b_{n+2}^{-1} + b_{n+3}^{-2} + \cdots\bigr)$$

$$< b_n + \sum_{k=1}^{n-1} b_k^{n+1-k} + 2 = 4 + c_n + 2\sum_{k=1}^{n-1} b_k^{n+1-k} \leqslant 2c_n.$$

Finally, fix $g \in \mathcal{D} \setminus \{0\}$. Then there are $N \in \mathbb{N}$, complex constants $a_1, \dots, a_N$ and numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ in I with $h := \sum_{j=1}^{N} a_j e_{\alpha_j} \not\equiv 0$ and $g = fh$. Observe that

$$\bigl|h^{(n)}(x)\bigr| \leqslant e\sum_{j=1}^{N} |a_j| =: \beta \quad (n \in \mathbb{N}_0,\ x \in I).$$

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Let $x_0 \in I$, and let $p \in \mathbb{N}_0$ be the smallest integer such that $h^{(p)}(x_0) \neq 0$ (p exists because h is analytic). Denote $\gamma := |h^{(p)}(x_0)|$, $c_0 := \|f\|_I$, and fix $n > 2p + 2\beta\gamma^{-1}$ (so $n - p > 2\beta\gamma^{-1}$, $2n - 2p > n$ and $(2n-2p)! > n! \geqslant \binom{n}{k}$ for all $k \in \{0, 1, \ldots n\}$). From Leibniz's formula, we arrive to

$$\begin{aligned}
\left|g^{(n)}(x_0)\right| &= \left|\binom{n}{p} f^{(n-p)}(x_0) h^{(p)}(x_0) + \sum_{k=0}^{n-p-1} \binom{n}{k} f^{(k)}(x_0) h^{(n-k)}(x_0)\right| \\
&\geqslant \gamma c_{n-p} - 2\beta \sum_{k=0}^{n-p-1} \binom{n}{k} c_k \\
&\geqslant \gamma \left[c_{n-p} - (2n-2p)!(n-p) \sum_{k=1}^{n-p-1} c_k - (n-p)c_0 \right] \\
&\geqslant \gamma\left[(2n-2p)!(2n-2p)^{2n-2p} - (n-p)c_0\right] \geqslant \gamma\left[n!n^n - nc_0\right].
\end{aligned}$$

Then $\rho(g, x_0) = 0$. Consequently, $g \in \mathcal{PS}$, as desired. □

The following question arises naturally from the last two theorems.

Question 2. Is the conclusion of Theorem 3.1 true for $\mathcal{PS}$ instead of $\mathcal{S}$? Or, at least, does $\lambda(\mathcal{PS}) = \chi$ hold in the real case?

4. Spaceability of $C^\infty(I^0)$

Along this section, we will consider the case $\mathbb{K} = \mathbb{R}$. We have already mentioned that $\mathcal{D}(I)$ is not spaceable in $C(I)$ [16]. Consequently, $C^\infty(I)$ is not spaceable in $C(I)$ either. This might suggest that the same negative result holds if one replaces I by I^0. This is not the case, as Theorem 4.4 below shows. We point out—this time in a more positive direction, see also our Remark 4.5.2—that Gurariy [17] proved the spaceability in $C(I)$ of $C(I) \cap \mathcal{D}(I^0)$. This fact contributed to our interest in the subject.

Some background about the so-called *Müntz spaces* is needed. Let $\Lambda = \{\lambda_k\}_{k=0}^\infty$ be an increasing sequence of nonnegative numbers: $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Moreover, let $M(\Lambda) = \{t^{\lambda_k}\}_{k=0}^\infty$ be the sequence of the functions t^{λ_k} on I, and let $[M(\Lambda)]$ be the closed linear span of $M(\Lambda)$ in $C(I)$. We call $M(\Lambda)$ a Müntz sequence and $[M(\Lambda)]$ a Müntz space. Members of $\operatorname{span}(M(\Lambda))$ are called Müntz polynomials. The Müntz–Szasz theorem (see [31]) asserts that $[M(\Lambda)] = C(I)$ if and only if $\sum_{k=1}^\infty \lambda_k^{-1} = +\infty$. So if $\sum_{k=1}^\infty \lambda_k^{-1} < +\infty$, we obtain new Banach spaces $[M(\Lambda)]$ in $C(I)$. A sequence Λ is called entire if $\lambda_k \in \mathbb{N}$ ($k = 1, 2, \ldots$), while it is called lacunary if $\lambda_{k+1}/\lambda_k \geqslant q$ ($k = 1, 2, \ldots$) for some $q > 1$. Note that if Λ is lacunary, then $\sum_{k=1}^\infty \lambda_k^{-1} < +\infty$ and Λ satisfies the gap condition $\inf_k(\lambda_{k+1} - \lambda_k) > 0$.

For the proof of Theorem 4.4 the following two auxiliary results will be used. They contain, respectively, a Bernstein-type inequality and a Remez-type inequality, both of them in $C(I)$ for Müntz polynomials. They can be found, respectively, in Theorem 8.3.1 and Corollary 8.4.3 of [20].

Lemma 4.1. *If Λ is lacunary, then there exists a constant $c = c(\Lambda) \in (0, +\infty)$ such that, for any $f \in \operatorname{span}(M(\Lambda))$, we have*

$$\left|f'(t)\right| \leqslant \frac{c}{1-t}\|f\|_I \quad \text{for all } t \in [0, 1).$$

Lemma 4.2. *Assume that Λ is such that $\sum_{k=1}^\infty \lambda_k^{-1} < +\infty$ and $\inf_k(\lambda_{k+1} - \lambda_k) > 0$. Let $K \subset I$ be a compact set whose interior in I is nonempty. Then there is a constant $c = c(\Lambda, K) \in (0, +\infty)$ satisfying*

$$\|f\|_{[0, \inf K]} \leqslant c\|f\|_K \quad \text{for all } f \in \operatorname{span}\big(M(\Lambda)\big).$$

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Let us introduce the successive *derivatives* for an *entire* sequence $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$. The 1-derivative sequence of Λ is defined as the sequence $\Lambda^{(1)} = \{\mu_k\}_{k=0}^{\infty}$ given by $\mu_0 := 0$ and

$$\mu_k := \begin{cases} \lambda_k - 1 & \text{if } \lambda_1 > 1, \\ \lambda_{k+1} - 1 & \text{if } \lambda_1 = 1 \end{cases}$$

for $k \geqslant 1$. By induction, the N-derivative sequence of Λ is defined for $N \geqslant 2$ as $\Lambda^{(N)} := (\Lambda^{(N-1)})^{(1)}$. The terminology is motivated by the fact that if P is a Müntz polynomial for Λ, then its N-derivative $P^{(N)}$ is a Müntz polynomial for $\Lambda^{(N)}$. Observe that each $\Lambda^{(N)}$ is also an entire sequence. In addition, the inequality $(\lambda_{k+1} - 1)\lambda_k \geqslant \lambda_{k+1}(\lambda_k - 1)$ shows that if Λ is lacunary, then every $\Lambda^{(N)}$ is also lacunary.

The following result is in the core of the proof of Theorem 4.4.

Lemma 4.3. *If Λ is a lacunary entire sequence, then there are two mappings $\omega : \mathbb{N} \times \mathcal{K}(I^0) \to (0, +\infty)$, $\sigma : \mathbb{N} \times \mathcal{K}(I^0) \to \mathcal{K}(I^0)$ such that*

$$\left\| f^{(n)} \right\|_K \leqslant \omega(n, K) \| f \|_{\sigma(n,K)} \tag{5}$$

for all $n \in \mathbb{N}$, all $K \in \mathcal{K}(I^0)$ and all $f \in \operatorname{span}(M(\Lambda))$.

Proof. Fix a sequence Λ as in the hypothesis, and let $K \in \mathcal{K}(I^0)$. Then there are a, b such that $0 < a < b < 1$ and $K \subset [a, b]$. Choose any sequence $\{b_n\}_1^{\infty}$ with $b < b_1 < b_2 < \cdots < b_n < \cdots < 1$, and define the mapping σ by $\sigma(n, K) := [a, b_n]$. Recall that each sequence $\Lambda^{(n)}$ is also entire and lacunary, and that $f^{(n)} \in \operatorname{span}(M(\Lambda^{(n)}))$ whenever $f \in \operatorname{span}(M(\Lambda))$.

Observe that if $f \in \operatorname{span}(M(\Lambda))$, then $f_\alpha \in \operatorname{span}(M(\Lambda))$ for every $\alpha \in (0, 1)$, where $f_\alpha(t) := f(\alpha t)$. Indeed, if $f(t) = \sum_{k=0}^{\infty} a_k t^{\lambda_k}$, then $f_\alpha(t) = \sum_{k=0}^{\infty} (a_k \alpha^{\lambda_k}) t^{\lambda_k}$. By Lemma 4.1, there is a constant $c_1 \in (0, +\infty)$, not depending on g, such that

$$\left| g'(t) \right| \leqslant \frac{c_1}{1 - t} \| g \|_I \quad \bigl(t \in [0, 1),\ g \in \operatorname{span}\bigl(M(\Lambda)\bigr)\bigr).$$

By taking $g = f_{b_1}$, we get for all $t \in [0, 1)$ and all $f \in \operatorname{span}(M(\Lambda))$ that

$$b_1 \left| f'(b_1 t) \right| \leqslant \frac{c_1}{1 - t} \| f_{b_1} \|_I = \frac{c_1}{1 - t} \| f \|_{[0, b_1]}.$$

From Lemma 4.2, there is a constant $c_2 \in (0, +\infty)$, depending on a, b_1 (so on K) but not on f, such that

$$\| f \|_{[0,a]} \leqslant c_2 \| f \|_{[a, b_1]}.$$

Then $\| f \|_{[0, b_1]} \leqslant c_3 \| f \|_{[a, b_1]}$, where $c_3 := \max\{1, c_2\}$. Therefore

$$\left\| f'(b_1 t) \right\| \leqslant \frac{c_1 c_3}{b_1(1 - t)} \| f \|_{[a, b_1]} \quad \bigl(t \in [0, 1)\bigr).$$

By taking the supremum over $t \in [a/b_1, b/b_1]$, we obtain

$$\| f' \|_K \leqslant \| f' \|_{[a,b]} \leqslant \frac{c_1 c_3}{b_1 - b} \| f \|_{[a, b_1]} = \omega(1, K) \| f \|_{\sigma(1,K)},$$

where we have set $\omega(1, K) := c_1 c_3 / (b_1 - b)$.

Now, we proceed by induction. Since $\Lambda^{(1)}$ is lacunary, there is a constant $c_4 \in (0, +\infty)$ not depending on f such that

$$\| f'' \|_{[a,b]} \leqslant c_4 \| f' \|_{[a, b_1]} \quad \bigl(f \in \operatorname{span}\bigl(M(\Lambda)\bigr)\bigr),$$

because $D(\operatorname{span}(M(\Lambda))) \subset \operatorname{span}(M(\Lambda^{(1)}))$. Now, we can make the translation of roles $b \to b_1$, $b_1 \to b_2$, so obtaining for some constant $c_5 \in (0, +\infty)$ not depending of f that

$$\| f' \|_{[a, b_1]} \leqslant c_5 \| f \|_{[a, b_2]} \quad \bigl(f \in \operatorname{span}\bigl(M(\Lambda)\bigr)\bigr).$$

Let us define $\omega(2, K) := c_4 c_5$. By combining the last two inequalities, one arrives at

$$\| f'' \|_K \leqslant \| f'' \|_{[a,b]} \leqslant c_4 c_5 \| f \|_{[a, b_2]} = \omega(2, K) \| f \|_{\sigma(2,K)} \quad \bigl(f \in \operatorname{span}\bigl(M(\Lambda)\bigr)\bigr).$$

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It is evident that this process can be continued for every derivative $f^{(n)}$. Consequently, the mappings ω, σ can be constructed so that they satisfy the desired conclusion. $\square$

Next, we state the main result in this section, namely, $C^\infty(I^0)$ contains a closed (in $C(I^0)$) infinite-dimensional manifold.

Theorem 4.4. *The class $C^\infty(I^0)$ is spaceable in $C(I^0)$.*

Proof. Choose any lacunary entire sequence $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$. Define the set

$$\mathcal{F} := \text{closure}_{C(I^0)}\big(\text{span}\big(M(\Lambda)\big)\big).$$

Note that the closure is in $C(I^0)$, not in $C(I)$. It is clear that $\mathcal{F}$ is a closed linear submanifold of $C(I^0)$. Since the functions t^{λ_k} are linear independent, we have that $\dim(\mathcal{F}) = +\infty$. Hence our unique task is to prove that $\mathcal{F} \subset C^\infty(I^0)$.

From Lemma 4.3, there are two mappings $\omega : \mathbb{N} \times \mathcal{K}(I^0) \to (0, +\infty)$, $\sigma : \mathbb{N} \times \mathcal{K}(I^0) \to \mathcal{K}(I^0)$ such that (5) holds for all $f \in \text{span}(M(\Lambda))$. Fix a function $f \in \mathcal{F}$. Then there is a sequence (f_j) of Müntz polynomials such that $f_j \to f$ $(j \to \infty)$ uniformly on every set $L \in \mathcal{K}(I^0)$. Then (f_j) is a Cauchy sequence in the space $C(I^0)$. In particular, given $\varepsilon > 0$, $n \in \mathbb{N}$ and $K \in \mathcal{K}(I^0)$, there is $j_0 \in \mathbb{N}$ such that

$$\|f_j - f_k\|_{\sigma(n,K)} < \frac{\varepsilon}{\omega(n,K)} \quad (j, k \geqslant j_0).$$

Therefore, by Lemma 4.3,

$$\big\|f_j^{(n)} - f_k^{(n)}\big\|_K < \varepsilon \quad (j, k \geqslant j_0),$$

whence $(f_j^{(n)})$ is a Cauchy sequence in $C(I^0)$ for every n. It follows that for each n there exists a function $g_n \in C(I^0)$ such that

$$f_j^{(n)} \to g_n \quad (j \to \infty)$$

uniformly on compact subsets of I^0. From the uniform convergence of (f_j') to g_1 and of (f_j) to f on each compact subset of I^0, we get

$$f_j(x) = \int_{1/2}^{x} f_j'(t)\,dt + f_j(1/2) \to \int_{1/2}^{x} g_1(t)\,dt + f(1/2) \quad (j \to \infty).$$

But the pointwise convergence of (f_j) to f yields

$$f(x) = f(1/2) + \int_{1/2}^{x} g_1(t)\,dt \quad \big(x \in I^0\big).$$

Since $g_1 \in C(I^0)$, the fundamental theorem of calculus tells us that f is differentiable on I^0 and that $f' = g_1$. Finally, an induction procedure proves immediately the existence of all n-derivatives $f^{(n)}$ and that, in fact, $f^{(n)} = g_n$ for all $n \in \mathbb{N}$. Consequently, $f \in C^\infty(I^0)$, as desired. $\square$

Remarks 4.5.

1. Via adequate diffeomorphisms, Theorem 4.4 can be stated for any open interval of $\mathbb{R}$. In particular, by using the C^∞-smooth bijection $\varphi : x \in I^0 \mapsto \cot(\pi x) \in \mathbb{R}$, we easily obtain the spaceability of $C^\infty(\mathbb{R})$ in $C(\mathbb{R})$.

2. In [20, pp. 80–81] it is proved that, if $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ is a sequence with $\sum_{k=1}^\infty 1/\lambda_k < +\infty$ and $\inf_k(\lambda_{k+1} - \lambda_k) > 0$, then $[M(\Lambda)]$ is included in the class $C^\omega(I^0)$ of analytic functions on I^0. Therefore $C(I) \cap C^\omega(I^0)$ (hence $C(I) \cap C^\infty(I^0)$) is spaceable in $C(I)$. This improves the result of [17] and raises the following question, which finishes our paper.

Question 3. Is $C^\omega(I^0)$ spaceable in $C(I^0)$?

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Uncited references

Q1 [9]

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