Weighted (LB)-spaces of holomorphic functions: $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ and completeness of $\mathcal{V}_0H(G)$

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Dedicated to Richard Aron and Reinhold Meise on the occasion of their 60th birthdays

Abstract

In this article we show that algebraic equalities between weighted inductive limits of spaces of holomorphic functions and/or between their projective hulls sometimes have strong consequences for the locally convex properties of the spaces involved in these equalities. For these results we impose the typical conditions which imply biduality between the spaces with the o- and O-growth conditions and use interpolating sequences for the step spaces $Hv_n(G)$, $n \in \mathbb{N}$. In Section 1 we show that $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ holds algebraically if and only if this space is (DFS) and that $H\overline{V}(G) = H\overline{V}_0(G)$ is true if and only if the space is semi-Montel. In Section 2 we provide a new characterization of the semi-Montel property of $H\overline{V}(G)$, which is much simpler than the one given before (in [10]). Section 3 proves that the completeness of $\mathcal{V}_0H(G)$ sometimes implies that the inductive limit $\mathrm{ind}_n\, H(v_n)_0(G)$ must be boundedly retractive.

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0 Introduction, notation and preliminaries

0.1 Introduction

There is quite a vast amount of literature on weighted inductive limits of spaces of continuous and holomorphic functions and on the projective description of such inductive limits. These spaces have important applications in functional analysis (spectral theory, functional calculus), complex analysis, partial differential equations and convolution equations, as well as distribution theory. But many articles on weighted inductive limits were written for the main reason that these spaces are also interesting objects of mathematical research and that still, after a development of more than 25 years, several important questions have remained open.

In some sense the theory really started with the seminal article [11] of Bierstedt, Meise and Summers in which the general framework was established and in which several fundamental results were proved; there the importance of condition (S) and of the regularly decreasing condition was also clarified. That is why still today [11] is quoted so often; e.g., see Bonet, Meise, Melikhov [18], Taskinen [28], Jasiczak [22], Boiti, Nacinovich [12], and Mattila, Saksman, Taskinen [24]. After [11], new incentives came mainly from the article [6] of Bierstedt and Meise, in which condition (D) was introduced – later on, this led to our characterization of the distinguished Köthe echelon spaces in [3] –, and from our articles [4], with which biduality came into the play as an important tool, as well as [9], which explained the special situation for radial weights on balanced domains. While the first articles mainly dealt with spaces of continuous functions, for which the situation is much easier technically and for which everything was understood very well by the end of the 80s, the research then focused on spaces of holomorphic functions. The results for this case were surveyed in 2001 in [2]; then also seven open problems were formulated which have influenced the more recent developments.

Condition (S) implies that the weighted inductive limits $\mathcal{V}H(G)$ with O-growth conditions and $\mathcal{V}_0H(G)$ with o-growth conditions coincide, and it also implies projective description, see [11], Theorem 1.6. It seemed for a long time that condition (S), while easily verified in applications, was too strong from a theoretical point of view because then $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ is even a (DFS)-space, while the property semi-Montel is already enough to apply the Baernstein Lemma (an open mapping result) to get projective description. The main result in Section 1 of the present paper (Theorem 4), however, shows that in many cases the algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ already forces this space to be (DFS) and that similarly the equality $H\overline{V}(G) = H\overline{V}_0(G)$ of the projective hulls forces the space to be semi-Montel. Using associated weights, it was possible to give a (rather complicated) characterization when the projective hull $H\overline{V}(G)$ is semi-Montel, see [10], Theorem 2.1.(b); under some extra conditions we will give a simpler characterization as the main result (Theorem 15.) of the present Section 2. But note that Bonet and Taskinen [13] gave an example that projective description may fail even if $H\overline{V}(G)$ is a semi-Montel space.

It is one of the aims of the present article to point out that, under strong enough assumptions, the weighted inductive limits of spaces of holomorphic functions behave in a very similar way as the corresponding spaces of continuous functions. Seemingly weak conditions only on algebraic equalities between weighted inductive limits and/or their projective hulls have strong consequences for the locally convex properties of the spaces involved in these equalities, above all in the case of spaces with o-growth conditions. This case is particularly interesting because the problem if $\mathcal{V}_0H(G)$ is a topological subspace of $H\overline{V}_0(G)$ is still wide open (and remains so after this article): Positive results were recently obtained for radial weights v on the open unit disc $\mathbb{D}$, see our article [5] and the paper [16] by Bonet, Engliš and Taskinen, but no single counterexample is known even for more general weights v on arbitrary domains G.

The organization of the present article is as follows: After recalling the notation and some of the known results, we start in Section 1 with the treatment of the algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$. The main tool in this section are interpolating sequences for the step spaces $Hv_n(G)$. In Theorem 4. it is proved that, if for each $n \in \mathbb{N}$ every discrete sequence in G contains an interpolating sequence for $Hv_n(G)$, then $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ implies that the inductive limit is a (DFS)-space and that similarly $H\overline{V}(G) = H\overline{V}_0(G)$ implies that the projective hull $H\overline{V}(G)$ is semi-Montel. The converse also holds whenever the typical conditions for the biduality of the spaces with the o- and O-growth conditions are satisfied. In the rest of the section we establish that the condition on interpolating sequences is satisfied for weights on $\mathbb{D}$ (Corollary 5.) or, more generally, on open connected subsets G of $\mathbb{C}$ for which all the components of the complement on the Riemann sphere contain more than one point (Proposition 7.), for weights on absolutely convex and bounded subsets of $\mathbb{C}^N$ (Proposition 12.), as well as for certain weights on $\mathbb{C}$ (Proposition 9.).

Next, our Section 2 is devoted to the consequences of the algebraic equality $H\overline{V}(G) = H\overline{V}_0(G)$ only under the biduality conditions. We prove in Proposition 14. that this equality is then equivalent to the semireflexivity of $H\overline{V}_0(G)$. In Theorem 15. we give another characterization of $H\overline{V}(G)$ semi-Montel which is much simpler than the one obtained in [10] and which is reformulated in Proposition 16. in a form similar to condition $(\overline{\mathrm{M}})$ of [6].

Section 3 deals with the completeness of the space $\mathcal{V}_0H(G)$. The main result (Theorem 20.) is that, for a decreasing sequence of weights on $\mathbb{D}$ for which $\mathcal{V}_0H(\mathbb{D})$ is a dense topological subspace of $H\overline{V}_0(\mathbb{D})$, completeness of $\mathcal{V}_0H(\mathbb{D})$ already implies that the inductive limit $\mathrm{ind}_n\, H(v_n)_0(\mathbb{D})$ must be boundedly retractive. – In an Appendix we give an example of a sequence of radial weights on $\mathbb{D}$ which is regularly decreasing, but does not satisfy condition (S), thus solving a question which several people had asked us.

Let G be an open subset of $\mathbb{C}^N$. $H(G)$ denotes the space of all holomorphic functions on G. It is usually endowed with the compact-open topology co; i.e., the topology of uniform convergence on the compact subsets of G. A *weight* v on G is a strictly positive continuous function on G. For such a weight, the *weighted Banach spaces of holomorphic functions on* G are defined by

$$Hv(G) := \{f \in H(G);\ ||f||_v = \sup_{z\in G} v(z)|f(z)| < +\infty\},$$
$$Hv_0(G) := \{f \in H(G);\ vf \text{ vanishes at } \infty \text{ on } G\},$$

endowed with the norm $||.||_v$. $Hv(G)$ is a Banach space with a topology stronger than co. The closed unit ball B_v of this space is co-compact by Montel's theorem; hence $Hv(G)$ has a predual. $Hv_0(G)$ is a closed subspace of $Hv(G)$. If B_v is contained in the co-closure of the closed unit ball C_v of $Hv_0(G)$, then one has the canonical biduality $Hv_0(G)'' = Hv(G)$, see [8], Corollary 1.2.

Now let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of weights on G. Then the *weighted inductive limits of spaces of holomorphic functions on* G are defined by

$$\mathcal{V}H(G) := \mathrm{ind}_n\, Hv_n(G),$$
$$\mathcal{V}_0H(G) := \mathrm{ind}_n\, H(v_n)_0(G);$$

that is, $\mathcal{V}H(G)$ is the increasing union of the Banach spaces $Hv_n(G)$ with the strongest locally convex topology (or, equivalently, with the strongest topological vector space topology) for which all the injections $Hv_n(G) \to \mathcal{V}H(G)$ become continuous, $n \in \mathbb{N}$, and similarly for $\mathcal{V}_0H(G)$. These are locally convex inductive limits. It is clear that $\mathcal{V}_0H(G)$ is continuously embedded in $\mathcal{V}H(G)$, and it is not clear a priori if it is even a topological subspace (but this holds in certain good cases, see [4]).

In an effort to describe the inductive limit topologies by a system of weighted sup-seminorms, Bierstedt, Meise and Summers [11] defined the *associated system*

$$\overline{V} = \overline{V}(\mathcal{V}) := \{\overline{v} \text{ weight on } G;\ \forall n:\ \sup_G \frac{\overline{v}}{v_n} < +\infty\};$$

that is, $\overline{V}$ consists of all weights on G which are dominated by a function of the form $\inf_n C_n v_n$ with constants $C_n > 0$ for each n. Then the *projective hulls of the weighted inductive limits* are the complete locally convex spaces

$$H\overline{V}(G) := \{f \in H(G);\ \forall \overline{v} \in \overline{V}:\ p_{\overline{v}}(f) = \sup_G \overline{v}|f| < +\infty\},$$
$$H\overline{V}_0(G) := \{f \in H(G);\ \overline{v}f \text{ vanishes at } \infty \text{ on } G\ \forall \overline{v} \in \overline{V}\},$$

endowed with the topology given by the system $\{p_{\overline{v}};\ \overline{v} \in \overline{V}\}$ of seminorms. $H\overline{V}_0(G)$ is a closed topological subspace of $H\overline{V}(G)$. By definition we have continuous linear embeddings $\mathcal{V}H(G) \to H\overline{V}(G)$ and $\mathcal{V}_0H(G) \to H\overline{V}_0(G)$.

It was proved in [11] and [7], Section 3 that $\mathcal{V}H(G) = H\overline{V}(G)$ holds algebraically and that $\mathcal{V}H(G)$ is always complete. If the sequence $\mathcal{V} = (v_n)_n$ is regularly decreasing in the sense of [11], then also the algebraic equality $\mathcal{V}_0H(G) = H\overline{V}_0(G)$ holds and $\mathcal{V}_0H(G)$ is complete. The well known *projective description problem* asks: When do we have $\mathcal{V}H(G) = H\overline{V}(G)$ topologically, and when is $\mathcal{V}_0H(G)$ a topological subspace of $H\overline{V}_0(G)$? The first counterexamples to projective description are due to Bonet and Taskinen [13]. All known counterexamples so far are in the case of O-growth conditions.

Associated weights were introduced (in the general case) and studied by Bierstedt, Bonet, Taskinen [10]. For a weight v on G, the *associated weight* $\tilde{v}$ is defined by

$$\tilde{v}(z) := \frac{1}{||\delta_z||} = \frac{1}{\sup\{|f(z)|;\ f \in B_v\}},\ z \in G,$$

where $\delta_z \in Hv(G)'$ denotes the point evaluation in $z \in G$, $\delta_z(f) = f(z)$ for $f \in Hv(G)$, the norm $||.||$ is taken in $Hv(G)'$, and B_v is again the unit ball of $Hv(G)$; i.e., each $f \in B_v$ satisfies $|f| \leq 1/v$ on G. $1/\tilde{v}$ is always continuous and plurisubharmonic, and we have $v \leq \tilde{v}$, but in general it may happen that $\tilde{v}$ takes the value $+\infty$ at some points of G. With the usual conventions (in defining $H\tilde{v}(G)$), however, $Hv(G) = H\tilde{v}(G)$ holds isometrically.

Our notation on locally convex spaces is standard; e.g., see [21] and [25]. For notation concerning inductive limits see [1]. We follow the notation for weighted inductive limits introduced in [11] throughout this article.

1 The algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$

Let G be an open subset of $\mathbb{C}^N$. Before we can treat the equality in the title of this section, we will first deal with the purely set theoretic inclusion $Hv(G) \subset Hw_0(G)$, where v and w denote weights on G.

Lemma 1 *(a) Let $(f_k)_k$ be a sequence in $Hv(G)$ which converges to 0 in co. If*

$$(*)\ \forall \varepsilon > 0\ \exists k(\varepsilon) \in \mathbb{N},\ K_\varepsilon \subset G \text{ compact } \forall k \geq k(\varepsilon)\ \forall z \in G \setminus K_\varepsilon : \\ w(z)|f_k(z)| < \varepsilon,$$

then $f_k \to 0$ in $Hw_0(G)$.

(b) If $(*)$ holds for each bounded sequence $(f_k)_k$ in $Hv(G)$ which tends to 0 in

co, then the unit ball B_v of $Hv(G)$ is relatively compact in $Hw_0(G)$; that is, the inclusion map $Hv(G) \to Hw_0(G)$ is compact.

PROOF. (a) Fix $\varepsilon > 0$ and set $K := K_\varepsilon$. If $z \in G \backslash K$, one has $w(z)|f_k(z)| < \varepsilon$ for each $k \geq k(\varepsilon)$. On the other hand, $f_k \to 0$ holds uniformly on K; thus we find $k_0 \in \mathbb{N}$, without loss of generality $k_0 \geq k(\varepsilon)$, such that for any $k \geq k_0$ and any $z \in K$,

$$|f_k(z)| < \frac{\varepsilon}{\sup_{z \in K} w(z)}.$$

It follows that $f_k \to 0$ in $Hw_0(G)$ since for $k \geq k_0$ and $z \in K$ one easily concludes that $w(z)|f_k(z)| < \varepsilon$, while if $z \in G \setminus K$, then again $w(z)|f_k(z)| < \varepsilon$ since $k_0 \geq k(\varepsilon)$.

(b) To show that B_v is relatively compact in $Hw_0(G)$, we fix a sequence $(f_k)_k \subset B_v$ and will find a subsequence which converges in $Hw_0(G)$. Since B_v is compact in $(H(G), \mathrm{co})$ by Montel's theorem, there is a subsequence $(f_{k_j})_j$ which converges to $f_0 \in B_v$ in co. Now $(f_{k_j} - f_0)_j \subset 2B_v$ and $f_{k_j} - f_0 \to 0$ in co as $j \to \infty$. By (a) we then get $f_{k_j} - f_0 \to 0$ in $Hw_0(G)$; i.e., $f_{k_j} \to f_0$ in $Hw_0(G)$. $\square$

In the proofs of most of the results which follow in this section the main tool will be interpolating sequences, a notion which we will introduce now.

Definition 2 *A sequence $(z_j)_j \subset G$ is interpolating for $Hv(G)$ if for every sequence $(\alpha_j)_j \subset \mathbb{C}$ with $\sup_{j \in \mathbb{N}} v(z_j)|\alpha_j| < +\infty$ there is $g \in Hv(G)$ such that $g(z_j) = \alpha_j$ for each $j \in \mathbb{N}$. Let*

$$\ell_\infty(v) = \ell_\infty((v(z_j))_j) := \{(\alpha_j)_j \in \mathbb{C}^{\mathbb{N}};\ \sup_j v(z_j)|\alpha_j| < +\infty\}.$$

Then $(z_j)_j$ is interpolating for $Hv(G)$ precisely if the map $R : Hv(G) \to \ell_\infty(v)$, defined by $R(g) := (g(z_j))_j$, is surjective.

Proposition 3 *Assume that the set-theoretic inclusion $Hv(G) \subset Hw_0(G)$ holds. If every discrete sequence $(z_j)_j$ in G contains an interpolating subsequence for $Hv(G)$, then the inclusion map $Hv(G) \to Hw_0(G)$ is compact.*

PROOF. By Lemma 1.(b) it suffices to show that $(*)$ holds for each bounded sequence $(f_k)_k$ in $Hv(G)$ which converges to 0 with respect to co, and we fix such a sequence $(f_k)_k$. Suppose that $(*)$ does not hold. Then, for a fundamental sequence $(K_k)_k$ of compact subsets of G,

$$\exists \varepsilon_0 > 0\ \forall k \in \mathbb{N}\ \exists z_k \in G \setminus K_k,\ z_k \neq z_j \text{ for } j < k :\ w(z_k)|f_k(z_k)| \geq \varepsilon_0.$$

Since $(f_k)_k$ is bounded in $Hv(G)$,

$$\sup_k v(z_k)|f_k(z_k)| \leq \sup_k \sup_{z\in G} v(z)|f_k(z)| < +\infty$$

holds. By construction $(z_k)_k$ is discrete; hence by our hypothesis we can find a subsequence $(z_{k_j})_j$ of $(z_k)_k$ which is interpolating for $Hv(G)$. Since $(z_{k_j})_j$ is interpolating, there exists $g \in Hv(G)$ with $g(z_{k_j}) = f_{k_j}(z_{k_j})$ for each $j \in \mathbb{N}$. By assumption g belongs to $Hw_0(G)$; thus, $\lim_{j\to\infty} w(z_{k_j})|g(z_{k_j})| = 0$ which implies $\lim_{j\to\infty} w(z_{k_j})|f_{k_j}(z_{k_j})| = 0$, a contradiction. $\square$

In the sequel let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of weights on G. The closed unit ball of $Hv_n(G)$ will be denoted by B_n and the closed unit ball of $H(v_n)_0(G)$ by C_n, $n \in \mathbb{N}$.

Theorem 4 *We assume that for each $n \in \mathbb{N}$ every discrete sequence in G contains a subsequence which is interpolating for $Hv_n(G)$. Then the following implications are true:*

(a) The algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ implies that $\mathcal{V}H(G)$ and $\mathcal{V}_0H(G)$ are (DFS)-spaces.

(b) Similarly, the algebraic equality $H\overline{V}(G) = H\overline{V}_0(G)$ implies that $H\overline{V}(G)$ is semi-Montel; i.e., in this space each bounded subset is relatively compact.

(c) If, in addition, B_n is contained in the co-closure of C_n for each n, then the converse of (a) and (b) is also true. That is, then $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ is equivalent to $\mathcal{V}H(G)$ (DFS), and $H\overline{V}(G) = H\overline{V}_0(G)$ is equivalent to $H\overline{V}(G)$ semi-Montel.

PROOF. (a) The hypothesis and Grothendieck's factorization theorem (see [25], 24.33) imply that for each n there is $m > n$ with $Hv_n(G) \subset H(v_m)_0(G)$. Proposition 3 then yields that the inclusion map is compact. Restricting to $H(v_n)_0(G)$ we get that the inclusion map $H(v_n)_0(G) \to H(v_m)_0(G)$ is compact. Thus, $\mathcal{V}_0H(G)$ is a (DFS)-space. Also, composing with the inclusion $H(v_m)_0(G) \subset Hv_m(G)$, we obtain that the mapping $Hv_n(G) \to Hv_m(G)$ is compact; thus, $\mathcal{V}H(G)$ is a (DFS)-space, too.

(b) Let B be a bounded subset of $H\overline{V}(G)$. Since $H\overline{V}(G)$ and $\mathcal{V}H(G)$ have the same bounded sets and since the inductive limit $\mathcal{V}H(G) = \mathrm{ind}_n Hv_n(G)$ is regular (see [11], Remarks before Example 1.12), there exists n such that B is contained and bounded in $Hv_n(G)$. Without loss of generality, we may assume that $B \subset B_n$. Since $H\overline{V}(G)$ is the projective limit of the spaces $H\overline{v}(G)$, $\overline{v} \in \overline{V}$ strictly positive and continuous (see [11], Proposition on page 112), it is enough to show that B_n is relatively compact in $H\overline{v}(G)$ for each $\overline{v} \in \overline{V}$ strictly positive and continuous. By the hypothesis we have

$$Hv_n(G) \subset H\overline{V}(G) = H\overline{V}_0(G) \subset H\overline{v}_0(G).$$

Again applying Proposition 3, it follows that the inclusion $Hv_n(G) \to H\overline{v}_0(G)$ is compact. Therefore B_n is relatively compact in $H\overline{v}_0(G) \subset H\overline{v}(G)$, as desired.

(c) Let $\mathcal{V}H(G)$ be a (DFS)-space and fix $f \in \mathcal{V}H(G)$. Without loss of generality we may assume that $f \in B_n$ for some n. By our hypothesis there is a sequence $(f_j)_j \subset C_n$ with $f_j \to f$ in co. Since $\mathcal{V}H(G)$ is a (DFS)-space, co and the norm topology induced by some $Hv_m(G)$, $m > n$, coincide on B_n. Thus we obtain $f_j \to f$ in $Hv_m(G)$. But $f_j \in H(v_m)_0(G)$ for each $j \in \mathbb{N}$, and it follows that $f \in H(v_m)_0(G) \subset \mathcal{V}_0H(G)$, from which we get the conclusion.

Finally let $H\overline{V}(G)$ be a semi-Montel space and fix $f \in H\overline{V}(G)$. By the algebraic equality $H\overline{V}(G) = \mathcal{V}H(G)$ we get $f \in Hv_n(G)$ for some n and may assume that $f \in B_n$. By hypothesis there is $(f_j)_j \subset C_n \subset \mathcal{V}_0H(G) \subset H\overline{V}_0(G)$ with $f_j \to f$ in co. Since $H\overline{V}(G)$ is semi-Montel, co and the weighted topology of $H\overline{V}(G)$ coincide on the bounded subset B_n of $H\overline{V}(G)$. Hence $f_j \to f$ in $H\overline{V}(G)$ and $f \in H\overline{V}_0(G)$. $\square$

Clearly, the condition

$$(S) \quad \forall n\ \exists m > n : \ \frac{v_m}{v_n} \text{ vanishes at } \infty \text{ on } G$$

implies that $\mathcal{V}H(G) = \mathcal{V}_0H(G) = H\overline{V}_0(G) = H\overline{V}(G)$ holds algebraically. It is known that then this equality also holds topologically and that the space is a (DFS)-space, cf. [11], Theorem 1.6. It should be noted that using associated weights (cf. [10]) there is even a characterization of the (DFS)-spaces $\mathcal{V}H(G)$ and of the semi-Montel spaces $H\overline{V}(G)$, see [10], Theorem 2.1. (a) and (b); also see the discussion at the beginning of Section 3.B of that article. (We will come back to the second of these characterizations in the next section.) For open sets $G \subset \mathbb{C}$ and weights v_n satisfying an extra condition a characterization of the (DFS)-property of $\mathcal{V}H(G)$ which is easier to evaluate (viz., that the sequence $(\tilde{v}_n)_n$ of associated weights satisfies (S)) was derived in [17], Theorem 10. But that, under our present hypotheses, algebraic equalities alone (like in Theorem 4) characterize such locally convex properties of $\mathcal{V}H(G)$ and $H\overline{V}(G)$ is a somewhat surprising fact.

We next give some cases when the hypotheses of Proposition 3 and/or Theorem 4 are satisfied and start with the unit disc $\mathbb{D}$.

Corollary 5 *(a) Let v be a weight on the unit disc $\mathbb{D}$ with $Hv(\mathbb{D}) \neq \{0\}$. Then* *every discrete sequence in $\mathbb{D}$ contains an interpolating sequence for $H\tilde{v}(\mathbb{D})$, where $\tilde{v}$ is the associated weight, cf. [10].*

(b) Note that $Hv(\mathbb{D}) = H\tilde{v}(\mathbb{D})$ holds isometrically, cf. [10], Observation 1.12. Hence, if the inclusion $Hv(\mathbb{D}) \subset Hw_0(\mathbb{D})$ holds for some other weight w on $\mathbb{D}$, then the inclusion map $Hv(\mathbb{D}) \to Hw_0(\mathbb{D})$ is compact by Proposition 3.

(c) If all the weights in the decreasing sequence $\mathcal{V} = (v_n)_n$ on $\mathbb{D}$ satisfy $Hv_n(\mathbb{D}) \neq \{0\}$, then parts (a) and (b) of Theorem 4 apply. In fact, if the weights v_n are radial on $\mathbb{D}$ and vanish at the boundary of $\mathbb{D}$, then also Theorem 4.(c) applies.

PROOF. The condition $Hv(\mathbb{D}) \neq \{0\}$ implies that $\tilde{v}$ is a weight; i.e., that $\tilde{v}(z) < +\infty$ for each $z \in \mathbb{D}$. (See Remark 6. below for a more general statement.) Let $(z_k)_k$ be a discrete sequence in $\mathbb{D}$. We can choose a subsequence $(z_j)_j = (z_{k_j})_j$ of $(z_k)_k$ which is interpolating for $H^\infty(\mathbb{D})$ and a sequence $(\varphi_j)_j \subset H^\infty(\mathbb{D})$ such that $\varphi_j(z_i) = \delta_{ij}$ for all i and j and such that $\sum_j |\varphi_j| \leq M$ on $\mathbb{D}$ for some constant $M > 0$, see [29], III.E.4., b) implies c) or [15], page 141. Fix $(\alpha_j)_j \subset \mathbb{C}$ with $\sup_j \tilde{v}(z_j)|\alpha_j| =: m < +\infty$. For each $j \in \mathbb{N}$ one can find $f_j \in B_v =$ the unit ball of $Hv(\mathbb{D})$ with $f_j(z_j) = 1/\tilde{v}(z_j)$; cf. [10], 1.2.(iv). Now put $f := \sum_{j=1}^\infty \tilde{v}(z_j)\alpha_j\varphi_j f_j$. We will check that $f \in Hv(\mathbb{D}) = H\tilde{v}(\mathbb{D})$; clearly $f(z_j) = \alpha_j$ for each $j \in \mathbb{N}$. But for any $z \in \mathbb{D}$ we see that

$$\begin{aligned} v(z)|f(z)| &\leq v(z)\sum_{j=1}^{\infty} \tilde{v}(z_j)|\alpha_j||\varphi_j(z)||f_j(z)| \\ &\leq M(\sup_j \tilde{v}(z_j)|\alpha_j|)\left(\sup_j \sup_{z\in\mathbb{D}} v(z)|f_j(z)|\right) \leq Mm. \end{aligned}$$

Since the convergence of the series is uniform on compact subsets of $\mathbb{D}$, f is holomorphic on $\mathbb{D}$. Note that for radial weights v_n on $\mathbb{D}$ which vanish at the boundary it was proved in [9], Theorem 1.5.(c) that B_n is contained in the co-closure of C_n for each n. $\square$

Part of Corollary 5 still holds for more general domains $G \subset \mathbb{C}$, as we now show. To start with, let us prove the following fact.

Remark 6 *If for some weight v on an open connected set $G \subset \mathbb{C}$ one has $Hv(G) \neq \{0\}$, then the associated weight $\tilde{v}$ really is a weight; i.e., $\tilde{v}(z) < +\infty$ for each $z \in G$.*

PROOF. Fix $z_0 \in G$. Since $Hv(G) \neq \{0\}$, we find $g \in Hv(G)$, $g \neq 0$. In particular, g has only zeros of finite order. Without loss of generality we may assume that $g \in B_v$.

Case 1: $g(z_0) \neq 0$. Then $\tilde{w}(z_0) := \sup\{|f(z_0)|;\ f \in H(G),\ |f| \leq 1/v \text{ on } G\}$ is positive, which implies that $\tilde{v}(z_0) = 1/\tilde{w}(z_0) < +\infty$.

Case 2: $g(z_0) = 0$; that is, z_0 is a zero of order, say, $m \in \mathbb{N}$ of g. Then put $h(z) := g(z)/(z - z_0)^m$, $z \in G$; clearly, $h \in H(G)$ with $h(z_0) \neq 0$. We claim that also h is an element of $Hv(G)$ and are then reduced to Case 1. To see our claim, note first that the continuity of h in z_0 implies that there are $\varepsilon > 0$ with $\overline{D(z_0, \varepsilon)} \subset G$ (where $D(z_0, \varepsilon)$ is the open disc with center z_0 and radius ε) and some positive constant M with $|h(z)| \leq M$ for all $z \in \overline{D(z_0, \varepsilon)}$. Now for $z \in \overline{D(z_0, \varepsilon)}$ we have

$$v(z)|h(z)| \leq M \sup_{z \in \overline{D(z_0, \varepsilon)}} v(z) < +\infty,$$

while for all $z \notin \overline{D(z_0, \varepsilon)}$, $|z - z_0| > \varepsilon$ holds, and hence

$$v(z)|h(z)| = v(z)|g(z)||z - z_0|^{-m} < \varepsilon^{-m} \sup_G v|g|.$$

Note that if $\tilde{v}$ does not take the value $+\infty$, then $\tilde{v}$ is always strictly positive and continuous by [10], 1.2.(ii). $\square$

Proposition 7 *(a) Let G be an open connected subset of $\mathbb{C}$ such that, for the Riemann sphere $\mathbb{C}^*$, $\mathbb{C}^* \setminus G$ does not have a connected component consisting of only one point and such that $Hv(G) \neq \{0\}$ holds. Then every discrete sequence in G contains a subsequence which is interpolating for $H\tilde{v}(G)$, where $\tilde{v}$ again denotes the associated weight, cf. [10].*

(b) Hence part (b) and the first sentence of part (c) of Corollary 5 also hold with such a domain G instead of $\mathbb{D}$.

PROOF. Fix a discrete sequence $(z_n)_n \subset G$. We may assume without loss of generality that $z_n \to z_0$ for some z_0 in some closed connected component $L \subset \mathbb{C}^* \setminus G$ which, by hypothesis, consists of more than one point. $U := \mathbb{C}^* \setminus L$ is the union of G with all connected components of $\mathbb{C}^* \setminus G$ (if any) except L. Hence U is open and connected, and it is easy to see that U is conformally equivalent to $\mathbb{D}$.

From that point on, we can proceed as in the proof of Corollary 5. (a), noting that G is a subset of $U \sim \mathbb{D}$, where $\sim$ denotes conformal equivalence. $\square$

In particular, Proposition 7 applies to $G = U := \{z \in \mathbb{C};\ \mathrm{Im} z > 0\}$, the upper half plane. For the case that a weight v on U satisfies two natural assumptions, the unit ball B_v of $Hv(U)$ is contained in the co-closure of the unit ball C_v of $Hv_0(U)$ by Holtmanns [20], proof of Theorem 4.2.1. Thus we obtain from Theorem 4.(c):

Proposition 8 *Let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of weights on the upper half plane U such that $Hv_n(U) \neq \{0\}$ for each n. We also assume that for each $n \in \mathbb{N}$ the weight v_n satisfies the following two conditions:*

(i) $v_n(z) \to 0$ as $z \in U$ tends to a point on the real line,

(ii) there is $0 < r_0 < 1$ with $v_n(z) \leq v_n(z + ir)$ for all $z \in U$ and $0 < r \leq r_0$.

Then the algebraic equality $\mathcal{V}H(U) = \mathcal{V}_0H(U)$ is equivalent to $\mathcal{V}H(U)$ (DFS), and the algebraic equality $H\overline{V}(U) = H\overline{V}_0(U)$ is equivalent to $H\overline{V}(U)$ semi-Montel.

Note that Holtmanns [20], Theorem 4.2.3 and Proposition 4.2.5 also obtained similar results for strips and bounded starshaped open sets $G \subset \mathbb{C}$ with a central point (instead of the half plane U).

We now use the work of Marco, Massaneda, Ortega-Cerdà (see [23]) to give examples of weights v on $\mathbb{C}$ which satisfy the assumptions of Proposition 3. Let Φ be a (nonharmonic) subharmonic function on $\mathbb{C}$ whose Laplacian $\mu = \Delta\Phi$ (a nonnegative Borel measure, finite on compact sets) is a *doubling measure* in the sense of [23], Definition 5; that is, there is $C > 0$ so that $\mu(D(z, 2r)) \leq C\mu(D(z, r))$ for all $z \in \mathbb{C}$ and $r > 0$, where $D(z, r)$ again denotes the open disc with center z and radius r. By [23], $\Phi(z) = |z|^\beta$, $\beta > 0$, or, more generally, $\Phi(z) = |z|^\beta(\log(1+|z|^2))^\alpha$, $\alpha \geq 0$ and $\beta > 0$, yield functions Φ which satisfy this assumption, while $\Phi(z) = \exp|z|$ does not. – For each $z \in \mathbb{C}$ let $\rho(z)$ denote the positive radius with $\mu(D(z, \rho(z))) = 1$.

By [23], Definition 3, a sequence $\Lambda \subset \mathbb{C}$ is *ρ-separated* if there is some positive number δ such that $|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda'))$ for all $\lambda, \lambda' \in \Lambda$, $\lambda \neq \lambda'$. By Definition 4 in the same article, the *upper uniform density* of Λ with respect to $\mu = \Delta\Phi$ is

$$D^+_{\Delta\Phi}(\Lambda) := \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#(\Lambda \cap \overline{D(z, r\rho(z))})}{\mu(D(z, r\rho(z)))}.$$

For Φ as above we now define the weight v_Φ by $v_\Phi(z) := \exp(-\Phi(z))$, $z \in \mathbb{C}$. Theorem B of [23] proves that the sequence Λ is interpolating for $Hv_\Phi(\mathbb{C})$ if and only if Λ is ρ-separated and $D^+_{\Delta\Phi}(\Lambda) < 1/2\pi$.

Proposition 9 *Let Φ be a subharmonic function on $\mathbb{C}$ such that its Laplacian $\mu = \Delta\Phi$ is a doubling measure. Then every discrete sequence in $\mathbb{C}$ has a subsequence which is interpolating for $Hv_\Phi(\mathbb{C})$.*

PROOF. (We thank X. Massaneda for providing us with the idea of this proof.) Fix a discrete sequence Λ' in $\mathbb{C}$. Select z_1 in the sequence with $|z_1| > 1$. Now for each $j > 1$ choose $z_j \in \Lambda'$ such that $|z_j| > (e^j + 1)|z_{j-1}|$. Recall from [23], (5) on page 869 that $\rho(z) \leq C_0|z|^\beta \leq C_0|z|$ for $|z| > 1$, where $\beta \in (0,1)$ and $C_0 > 0$ are fixed.

We first show that the sequence $\Lambda := (z_j)_j$ is ρ-separated: Fix $1 \leq i < j$. Clearly $|z_j - z_i| \geq |z_i| \geq C_0^{-1}\rho(z_i)$. On the other hand,

$$|z_j - z_i| \geq \frac{1}{2}|z_j| + (\frac{1}{2}|z_j| - |z_i|) \geq \frac{1}{2}|z_j| \geq \frac{1}{2C_0}\rho(z_j).$$

At this point, we claim that the upper uniform density $D^+_{\Delta\Phi}(\Lambda)$ is 0, which implies that Λ is interpolating for $Hv_\Phi(\mathbb{C})$ by [23], Theorem B. First, by [23], Remark 1, there are $\varepsilon > 0$ and $c > 0$ such that $\mu(D(z, r\rho(z))) \geq cr^\varepsilon$ for all $z \in \mathbb{C}$ and all $r > 1$. Let us now estimate the number $n(r,z)$ of points in $\Lambda \cap \overline{D(z, r\rho(z))}$, where $r > 1$. Fix $z \in \mathbb{C}$. Note that it follows from [23], Corollary 3 that

(*) there are $S > 0$ and $s > 0$ such that, for $r > 1$ and $\zeta \in D(z, r\rho(z))$, $\frac{\rho(z)}{\rho(\zeta)} \leq Sr^s$.

(Also note that S and s do not depend on r, as follows from the proof of [23], Corollary 3.) Let j denote the largest index such that $z_j \in \overline{D(z, r\rho(z))}$. If this the only point in this closed disc, then $n(z,r) = 1$. Otherwise there is $i < j$ with z_i, $z_j \in \overline{D(z, r\rho(z))}$, which implies $|z_j - z_i| \leq 2r\rho(z)$. On the other hand,

$$|z_j - z_i| \geq |z_j| - |z_i| \geq |z_j| - |z_{j-1}| \geq e^j|z_{j-1}| \geq e^j|z_i| \geq e^jC_0^{-1}\rho(z_i).$$

Both estimates imply

$$e^j \leq 2C_0r\frac{\rho(z)}{\rho(z_i)}.$$

Since $z_i \in D(z, r\rho(z))$, we can apply (*) to conclude $e^j \leq 2C_0Sr^{s+1}$, which certainly yields $n(z,r) = j \leq (s+1)\log(r) + \log(2C_0S)$. Finally, for $r > 1$ we have

$$\frac{\#(\Lambda \cap \overline{D(z, r\rho(z))})}{\mu(D(z, r\rho(z)))} \leq \frac{(s+1)\log(r) + \log(2C_0S)}{cr^\varepsilon},$$

which tends to 0 as r tends to ∞. This proves our claim. $\Box$

Corollary 10 *(a) Let v, w be weights on $\mathbb{C}$ such that $Hv(\mathbb{C}) \subset Hw_0(\mathbb{C})$ holds. If the weight v equals $\exp(-\Phi)$ for a subharmonic function Φ such that $\Delta\Phi$ is a doubling measure, then the inclusion map $Hv(\mathbb{C}) \to Hw_0(\mathbb{C})$ is compact.*

(b) Let $(\Phi_n)_n$ be an increasing sequence of subharmonic functions on $\mathbb{C}$ such that $\Delta\Phi_n$ is a doubling measure for each n. Let $\mathcal{V} = (v_n)_n$ with $v_n = \exp(-\Phi_n)$ for each $n \in \mathbb{N}$. Then we have:

(1) $\mathcal{V}H(\mathbb{C}) = \mathcal{V}_0H(\mathbb{C})$ algebraically implies that $\mathcal{V}H(\mathbb{C})$ is a (DFS)-space.

(2) $H\overline{V}(\mathbb{C}) = H\overline{V}_0(\mathbb{C})$ *algebraically implies that* $H\overline{V}(\mathbb{C})$ *is semi-Montel.*

(3) The converse of (1) and (2) also holds if all v_n *are radial and rapidly decreasing at infinity (that is,* $Hv_n(\mathbb{C})$ *or, equivalently,* $H(v_n)_0(\mathbb{C})$ *contains all the polynomials).*

PROOF. This is clear by Proposition 3. Theorem 4. and Proposition 9. Note that the unit ball B_n of $Hv_n(\mathbb{C})$ is in the co-closure of the unit ball C_n of $H(v_n)_0(\mathbb{C})$ for radial weights v_n on $\mathbb{C}$ which are rapidly decreasing at infinity by [9], Theorem 1.5.(c). $\square$

So far all our examples were for the case $N = 1$; i.e., for $G \subset \mathbb{C}$. But we can now also treat some domains in $\mathbb{C}^N$, $N > 1$. We consider absolutely convex open and bounded subsets $G \subset \mathbb{C}^N$ and leave it to the reader to formulate the consequences of the following proposition along the lines of Corollary 5.(b) and (c). Of course, examples of such domains are all polydiscs and the open Euclidean unit ball in $\mathbb{C}^N$, but more generally the case of the open unit ball with respect to any norm on $\mathbb{C}^N$ is covered.

Lemma 11 *Let* G *be an absolutely convex open subset of* $\mathbb{C}^N$, $N \geq 1$, *and* $z_0 \in \partial G$. *Then there is a linear form* w *on* $\mathbb{C}^N$ *such that* $w(G) = \mathbb{D}$ *and* $w(z_0) = 1$.

PROOF. Since G is absolutely convex, open and $z_0 \notin G$, a form of the Hahn-Banach separation theorem (e.g., see [26], Proposition 5 on page 29/30) yields a linear form u on $\mathbb{C}^N$ such that $u(z_0) \notin u(G)$; but since $z_0 \in \overline{G}$ and u is continuous, we have $u(z_0) \in \overline{u(G)}$. Applying [26], Lemma 4, page 30, we get that $u(G)$ is open and absolutely convex in $\mathbb{C}$ so that we must have $|u(z_0)| > |u(z)|$ for all $z \in G$. If we put $w := u/u(z_0)$, then w is a linear form on $\mathbb{C}^N$ with $w(G) \subset \mathbb{D}$ and $w(z_0) = 1$. But $1 \in \overline{w(G)}$ now implies $w(G) = \mathbb{D}$. $\square$

Proposition 12 *Let* G *be an absolutely convex open and bounded subset of* $\mathbb{C}^N$ *and let* v *be a weight on* G *such that the associated weight* $\tilde{v}$ *satisfies* $\tilde{v}(z) < +\infty$ *for each* $z \in G$ *(which holds, in particular, if* v *is bounded). Then every discrete sequence in* G *contains a subsequence which is interpolating for* $H\tilde{v}(G)$.

PROOF. By assumption $\tilde{v}$ is a weight and $H\tilde{v}(G) = Hv(G)$ holds isometrically; if v is bounded, $Hv(G)$ contains the constants. We modify the proof of Corollary 5.(a).

Let $(z_k)_k$ be a discrete sequence in G. Without loss of generality we may assume that there exists $z_0 \in \partial G$ with $z_k \to z_0$. Now we apply Lemma 11. to find a linear functional w on $\mathbb{C}^N$ such that $w(G) = \mathbb{D}$ and $w(z_0) = 1$. Since all $w(z_k) \in \mathbb{D}$ and $w(z_k) \to w(z_0) = 1 \in \partial\mathbb{D}$, there is a subsequence of $(w(z_k))_k$ which is discrete in $\mathbb{D}$; we may assume without loss of generality that the sequence $(w(z_k))_k$ itself is discrete. As in the proof of Corollary 5.(a), we can find a subsequence $(w(z_j))_j = (w(z_{k_j}))_j$ which is interpolating for $H^\infty(\mathbb{D})$ and a sequence $(\varphi_j)_j \subset H^\infty(\mathbb{D})$ such that $\varphi_j(w(z_i)) = \delta_{ij}$ for all i and j and $\sum_j |\varphi_j| \le M$ on $\mathbb{D}$ for some constant $M > 0$. Now fix a sequence $(\alpha_j)_j \subset \mathbb{C}$ with $\sup_j \tilde{v}(z_j)|\alpha_j| =: m < +\infty$. For each j one can find $f_j \in B_v =$ closed unit ball of $Hv(G)$ with $f_j(z_j) = 1/\tilde{v}(z_j)$. At this point put $f(z) := \sum_{j=1}^\infty \tilde{v}(z_j)\alpha_j\varphi_j(w(z))f_j(z)$ for each $z \in G$. It is clear that $f \in H(G)$ and that $f(z_j) = \alpha_j$ for each j. We check that $f \in Hv(G) = H\tilde{v}(G)$. But for each $z \in G$ we see that

$$\begin{aligned} v(z)|f(z)| &\le v(z)\sum_{j=1}^\infty \tilde{v}(z_j)|\alpha_j||\varphi_j(w(z))||f_j(z)| \\ &\le M(\sup_j \tilde{v}(z_j)|\alpha_j|)\left(\sup_j \sup_{z\in G} v(z)|f_j(z)|\right) \le Mm. \qquad \Box \end{aligned}$$

2 The algebraic equality $H\overline{V}(G) = H\overline{V}_0(G)$

Under the hypotheses that (1) for each $n \in \mathbb{N}$ every discrete sequence in G contains a subsequence which is interpolating for $Hv_n(G)$ and that (2) the closed unit ball B_n of $Hv_n(G)$ is contained in the co-closure of the unit ball C_n of $H(v_n)_0(G)$ for each $n \in \mathbb{N}$, we have characterized in Theorem 4.(c) when the algebraic equality in the title of this section holds. Here we will deal with this equality once more, but assuming only hypothesis (2). Thus, in this section let G be an open subset of $\mathbb{C}^N$ and $\mathcal{V} = (v_n)_n$ a decreasing sequence of weights on G such that B_n is contained in the co-closure of C_n for each n. Then it follows from [4], Propositions 10 and 13 that $H(v_n)_0(G)'' = Hv_n(G)$ holds isometrically for each n and that $((\mathcal{V}_0H(G))'_b)'_i = \mathcal{V}H(G)$ topologically, where $'_i$ denotes the inductive dual (e.g., see [1]).

Lemma 13 *Let v denote a weight on G such that the unit ball B_v of $Hv(G)$ is contained in the co-closure of the unit ball C_v of $Hv_0(G)$. (Note that then $Hv(G) = Hv_0(G)''$ holds canonically.) Then co and $\sigma(Hv(G), Hv_0(G)')$ coincide on the bounded subsets of $Hv(G)$, and each bounded set $B \subset Hv(G)$ is $\sigma(Hv(G), Hv_0(G)')$-relatively compact.*

PROOF. By the Hahn-Banach theorem and the Riesz representation theorem, for each $l \in Hv_0(G)'$ there is a bounded Radon measure μ on G such that $(*)$ $l(f) = \int_G fvd\mu$ for each $f \in Hv_0(G)$, cf. [27]. The formula on the right hand side of $(*)$ then extends l to an element $L \in Hv(G)'$. In this light, the proof of [8], Theorem 1.1.(b) (involving the inner regularity of μ) actually shows that the restriction of each such $L \in Hv(G)'$ to any bounded subset B of $Hv(G)$ is co-continuous. On the other hand, $\sigma(Hv(G), Hv_0(G)')_{|B}$ is the weakest topology which makes each $L_{|B}$ continuous. Hence we get that $\sigma(Hv(G), Hv_0(G)')_{|B} \leq \mathrm{co}_{|B}$. But each bounded set $B \subset Hv(G)$ is co-relatively compact by Montel's theorem. Hence co coincides on B with each weaker Hausdorff topology and thus, in particular, with $\sigma(Hv(G), Hv_0(G)')$. $\square$

Proposition 14 *(a) The algebraic equality $H\overline{V}(G) = H\overline{V}_0(G)$ holds if and only if $H\overline{V}_0(G)$ is semireflexive.*

(b) Similarly, $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ if and only if for each $n \in \mathbb{N}$ there exists $m > n$ such that the inclusion map $i_{n,m} : H(v_n)_0(G) \to H(v_m)_0(G)$ is weakly compact.

PROOF. (a) As a closer inspection of the last part in the proof of Theorem 7. in [4] reveals, each element $l \in \mathcal{V}_0H(G)'$ can be extended to an element $L \in \mathcal{V}H(G)'$ which is co-continuous on the bounded subsets of $\mathcal{V}H(G)$. Hence on each B_n, $n \in \mathbb{N}$, co induces a finer topology than $\sigma(H\overline{V}(G), \mathcal{V}_0H(G)')$. Therefore every B_n is not only co-compact, but also compact with respect to $\sigma(H\overline{V}(G), \mathcal{V}_0H(G)')$. Now we use the equality $H\overline{V}(G) = H\overline{V}_0(G)$. It follows that each B_n is a $\sigma(H\overline{V}_0(G), \mathcal{V}_0H(G)')$-compact subset of $H\overline{V}_0(G)$. Since the inclusion $\mathcal{V}_0H(G) \to H\overline{V}_0(G)$ is continuous, $\sigma(H\overline{V}_0(G), \mathcal{V}_0H(G)')$ is finer than $\sigma(H\overline{V}_0(G), H\overline{V}_0(G)')$. This yields that each B_n, and thus each bounded subset of $H\overline{V}(G) = H\overline{V}_0(G)$, is $\sigma(H\overline{V}_0(G), H\overline{V}_0(G)')$-relatively compact, and $H\overline{V}_0(G)$ is semireflexive.

Suppose now that $H\overline{V}_0(G)$ is semireflexive and fix $f \in H\overline{V}(G)$. Then f is contained in a multiple of a set B_n for some n, and without loss of generality we may assume $f \in B_n$. By our assumption there exists a sequence $(f_n)_n \in C_n \subset H(v_n)_0(G) \subset H\overline{V}_0(G)$ which converges to f with respect to co. $(f_n)_n$ is bounded in $H\overline{V}_0(G)$, and by the semireflexivity of this space it is $\sigma(H\overline{V}_0(G), H\overline{V}_0(G)')$-relatively compact and hence has a cluster point $f_0 \in H\overline{V}_0(G)$. Both co and $\sigma(H\overline{V}_0(G), H\overline{v}_0(G)')$ are stronger than pointwise convergence on G, and it follows that $f = f_0 \in H\overline{V}_0(G)$.

The algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ implies by Grothendieck's factorization theorem that for $n \in \mathbb{N}$ there is $m > n$ such that $Hv_n(G) \subset H(v_m)_0(G)$ with continuous inclusion. Now the inclusion $i_{n,m} : H(v_n)_0(G) \to H(v_m)_0(G)$

must be weakly compact according to Grothendieck's theorem (see [21], 17.2.7) since its bitranspose maps $Hv_n(G)$ into $H(v_m)_0(G)$.

On the other hand, if for each $n \in \mathbb{N}$ there is $m > n$ such that the linking map $i_{n,m} : H(v_n)_0(G) \to H(v_m)_0(G)$ is weakly compact, then Grothendieck's theorem on weak compactness implies that $Hv_n(G) = H(v_n)_0(G)''$ is contained in $H(v_m)_0(G)$ and hence that $\mathcal{V}H(G) = \mathcal{V}_0 H(G)$. $\square$

Let us compare Proposition 14 with Theorem 4. If one adds to the assumption (2) of this section the hypothesis (1) that for each n every discrete sequence in G contains a subsequence which is interpolating for $Hv_n(G)$, then clearly compactness of the linking maps of the inductive limit $\mathcal{V}H(G)$ is equivalent to weak compactness. And similarly in this case $H\overline{V}_0(G)$ is semi-Montel if and only if it is semireflexive. Incidentally, in the last equivalence $H\overline{V}_0(G)$ can be replaced by $H\overline{V}(G)$ since semireflexive and semi-Montel are properties which are inherited by closed subspaces (cf. [21], 11.4.5.(a) and 11.5.4.(b)). We do not know if these equivalences remain true without hypothesis (1). Note that, for a single weight v on G, the Banach space $Hv_0(G)$ is (semi-) reflexive if and only if it is finite dimensional, see Bonet and Wolf [14].

In [10], Theorem 2.1(b), it was proved that $H\overline{V}(G)$ is semi-Montel if and only if for each $n \in \mathbb{N}$ and for each $\overline{v} \in \overline{V}$ there exists a nonnegative continuous function φ on G with compact support such that

$$\left(\min(\frac{1}{v_n}, \frac{1}{\varphi})\right)^{\sim} \leq 1/\overline{v}.$$

It is hard to evaluate such a characterization involving the associated weight of a minimum of two functions of which one is not directly connected with the given sequence $(v_n)_n$ of weights. Here, however, under assumption (2) and the additional hypothesis that $\tilde{v}_1(z) < +\infty$ for all $z \in G$ (which is true if v_1 is bounded and which implies that all associated weights $\tilde{v}_n$ are weights; i.e., $\tilde{v}_n(z) < +\infty$ for each n and each $z \in G$), we can give a simpler characterization when $H\overline{V}(G)$ is semi-Montel, only in terms of the associated weights $\tilde{v}_n$, as follows.

Theorem 15 *In addition to the assumptions of this section we suppose that $\tilde{v}_1(z) < +\infty$ for each $z \in G$. Then $H\overline{V}(G)$ is semi-Montel if and only if for each $\overline{v} \in \overline{V}$ and each $n \in \mathbb{N}$ the quotient $\overline{v}/\tilde{v}_n$ vanishes at ∞ on G.*

PROOF. If $H\overline{V}(G)$ is semi-Montel, $H\overline{V}_0(G)$ will also be semi-Montel, and hence by Proposition 14.(a), $H\overline{V}(G) = H\overline{V}_0(G)$ holds. Fix $\overline{v} \in \overline{V}$ and $n \in \mathbb{N}$. The subset $A := \{\overline{v}(z)\delta_z;\ z \in G\} \subset H\overline{V}(G)'$ (where δ_z denotes the point evaluation at z) is contained in the polar of a 0-neighborhood of $H\overline{V}(G)$

and hence is equicontinuous in $H\overline{V}(G)'$. Therefore $\sigma(H\overline{V}(G)', H\overline{V}(G))$ and the topology $\lambda(H\overline{V}(G)', H\overline{V}(G))$ of uniform convergence on the precompact subsets of $H\overline{V}(G)$ coincide on A (by the strong form of the Alaoğlu-Bourbaki Theorem, cf. [21], 8.5.2). Since $H\overline{V}(G) = H\overline{V}_0(G)$, the function $z \to \overline{v}(z)\delta_z$ vanishes at ∞ on G for $\sigma(H\overline{V}(G)', H\overline{V}(G))$, hence this function also vanishes at ∞ for $\lambda(H\overline{V}(G)', H\overline{V}(G))$ and, in particular, it does this uniformly on each set B_n, which is compact in the semi-Montel space $H\overline{V}(G)$. This means that

$$z \to \overline{v}(z) \sup\{|\delta_z(f)|;\ f \in B_n\} = \frac{\overline{v}(z)}{\tilde{v}_n(z)}$$

vanishes at ∞ on G.

Now assume that the condition on the associated weights is satisfied. For fixed $n \in \mathbb{N}$ we prove that $H\overline{V}(G)$ and co induce the same 0-neighborhoods on B_n, by which $H\overline{V}(G)$ is a semi-Montel space since each bounded subset of this space is contained in one of the sets B_n, $n \in \mathbb{N}$, since B_n is co-compact and since two topologies coincide on an absolutely convex set if they yield the same 0-neighborhoods, cf. [21], 9.2.4. Fix $\overline{v} \in \overline{V}$. By hypothesis there is a compact set $K \subset G$ with $\overline{v}(z) \leq \tilde{v}_n(z)$ for all $z \in G \setminus K$. At this point it suffices to check that

$$\{f \in B_n;\ \sup_K |f| < \min\left(1, \frac{1}{\max_K \overline{v}}\right)\} \subset \{f \in B_n;\ \sup_G \overline{v}|f| \leq 1\}.$$

But in fact for $z \in G \setminus K$ we have $\overline{v}(z)|f(z)| \leq \tilde{v}_n(z)|f(z)| \leq 1$ since $f \in B_n$ (note that $|f| \leq 1/v_n$ is equivalent to $|f| \leq 1/\tilde{v}_n$ by [10], 1.2.(iii)), while for $z \in K$ we get $\overline{v}(z)|f(z)| \leq (\max_K \overline{v})|f(z)| < 1$. $\square$

The condition in Theorem 15. can be reformulated in a different way so that, up to the use of the associated weight at one point, it resembles condition $(\overline{\mathrm{M}})$ in [6] (in particular, compare with Corollary 5.3 of that article). We thank Elke Wolf for communicating this to us.

Proposition 16 *Under the hypotheses of Theorem 15. the following conditions are equivalent:*

(1) For each $\overline{v} \in \overline{V}$ and each $n \in \mathbb{N}$ the quotient $\overline{v}/\tilde{v}_n$ vanishes at ∞ on G.

(2) For each $n \in \mathbb{N}$ and each subset Y of G which is not relatively compact there exists $n' = n'(n, Y) > n$ with

$$\inf_{y \in Y} \frac{v_{n'}(y)}{\tilde{v}_n(y)} = 0.$$

PROOF. (1)$\Rightarrow$(2) (indirect): If (2) is not satisfied, there are n and a subset Y of G which is not relatively compact such that for $m > n$ there exists $\varepsilon_m > 0$ with $v_m(y) \geq \varepsilon_m \tilde{v}_n(y)$ for each $y \in Y$. Put $\overline{v} := \inf \{v_{n+1}/\varepsilon_{n+1}, v_{n+2}/\varepsilon_{n+2}, ...\}$; we clearly have $\overline{v} \in \overline{V}$. Our estimates yield $\overline{v} \geq \tilde{v}_n$ on Y so that $\overline{v}/\tilde{v}_n$ does not vanish at ∞ on G, a contradiction to (1).

(2)$\Rightarrow$(1) (indirect): Let $(K_k)_k$ be a fundamental sequence of compact subsets of G. If (1) does not hold, there are $\overline{v} \in \overline{V}$, n and $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there exists $z_k \in G \setminus K_k$ with $\overline{v}(z_k)/\tilde{v}_n(z_k) \geq \varepsilon$. For arbitrary m there is $\alpha_m > 0$ with $\overline{v} \leq \alpha_m v_m$ on G. Hence for each $k \in \mathbb{N}$ and $m > n$ we get

$$v_m(z_k) \geq \alpha_m^{-1} \overline{v}(z_k) \geq \alpha_m^{-1} \varepsilon \tilde{v}_n(z_k).$$

With $Y := \{z_k;\ k \in \mathbb{N}\}$ we arrive at a contradiction to (2). $\square$

We close this section with the following observation.

Corollary 17 *(a) If the inductive limit $\mathcal{V}H(G)$ is a (semi-) Montel space, then the algebraic and topological equality $\mathcal{V}H(G) = H\overline{V}(G) = H\overline{V}_0(G)$ holds.*

(b) If, in addition, for each n every discrete sequence in G contains an interpolating sequence for $Hv_n(G)$, then the converse of (a) is also true; i.e., the algebraic and topological equality $\mathcal{V}H(G) = H\overline{V}_0(G)$ implies that $\mathcal{V}H(G)$ is (semi-) Montel.

PROOF. A well known application of the Baernstein lemma (an open mapping result), cf. [11], shows that $\mathcal{V}H(G)$ semi-Montel implies projective description; i.e., $\mathcal{V}H(G) = H\overline{V}(G)$ holds topologically. Since then $H\overline{V}(G)$ is semi-Montel, it follows from Proposition 14.(a) that also $H\overline{V}(G) = H\overline{V}_0(G)$ is true.

On the other hand, if there is an algebraic equality $\mathcal{V}H(G) = H\overline{V}_0(G)$, then clearly $H\overline{V}(G) = H\overline{V}_0(G)$ must also hold. Under our hypotheses we can then use Theorem 4.(b) to conclude that $H\overline{V}(G)$ is semi-Montel. Now the topological equality $\mathcal{V}H(G) = H\overline{V}_0(G)$ yields the Montel property for $\mathcal{V}H(G)$. $\square$

This may be the right place to remind that Bonet, Taskinen [13] constructed an example in which the topologies of $\mathcal{V}H(G)$ and $H\overline{V}(G)$ are different even though $H\overline{V}(G)$ is a semi-Montel space. But also note that in the case of this example the assumptions of the present section are not satisfied.

3 Completeness of $\mathcal{V}_0H(G)$

In this section let G again denote an open subset of $\mathbb{C}^N$ and $\mathcal{V} = (v_n)_n$ a decreasing sequence of weights on G. We would like to discuss now when the inductive limit $\mathcal{V}_0H(G)$ is complete and start with a few simple remarks.

Remark 18 *Again assume that the closed unit ball B_n of $Hv_n(G)$ is contained in the co-closure of the unit ball C_n of $H(v_n)_0(G)$ for each $n \in \mathbb{N}$. If the algebraic equality $\mathcal{V}_0H(G) = H\overline{V}_0(G)$ holds, then $\mathcal{V}_0H(G)$ is complete.*

PROOF. Under our assumption, $\mathcal{V}_0H(G)$ is a topological subspace of $\mathcal{V}H(G)$ by the last sentence of [4], Theorem 7. On the other hand, $\mathcal{V}H(G)$ is complete according to [7], Proposition 5.(2). Now let $(f_\alpha)_\alpha$ be a Cauchy net in $\mathcal{V}_0H(G)$. Then $(f_\alpha)_\alpha$ is a Cauchy net in $\mathcal{V}H(G)$ and converges in the topology of this space to some $f \in \mathcal{V}H(G)$; a fortiori $f_\alpha \to f$ in $H\overline{V}(G)$. But since all f_α are elements of $\mathcal{V}_0H(G) \subset H\overline{V}_0(G)$, we obtain $f \in H\overline{V}_0(G)$. By the algebraic equality $\mathcal{V}_0H(G) = H\overline{V}_0(G)$ it follows that $f \in \mathcal{V}_0H(G)$ and that $f_\alpha \to f$ in $\mathcal{V}_0H(G)$, which proves the completeness of $\mathcal{V}_0H(G)$. $\square$

If the sequence $\mathcal{V} = (v_n)_n$ is *regularly decreasing*, i.e.,

$$\forall n \; \exists m > n \; \forall Y \subset G : \; \sup_Y \frac{v_m}{v_n} > 0 \;\Rightarrow\; \sup_Y \frac{v_k}{v_n} > 0 \; \forall k \geq m,$$

then the algebraic equality $\mathcal{V}_0H(G) = H\overline{V}_0(G)$ does hold since it is true for the corresponding spaces of continuous functions, cf. [11], Theorem 2.3. In this case the inductive limit $\mathcal{V}H(G) = \mathrm{ind}_n\, Hv_n(G)$ is *boundedly retractive*; i.e., for each bounded set $B \subset \mathcal{V}H(G)$ there is n such that B is a bounded subset of $Hv_n(G)$ and such that the topologies of $\mathcal{V}H(G)$ and $Hv_n(G)$ coincide on B, cf. [7], beginning of the proof of Theorem 6.(2)(ii). Using associated weights [10], Theorem 2.1.(c) actually characterizes when $\mathcal{V}H(G)$ is boundedly retractive. Moreover, note that for regularly decreasing $\mathcal{V}$ also the inductive limit $\mathcal{V}_0H(G) = \mathrm{ind}_n\, H(v_n)_0(G)$ is boundedly retractive.

The algebraic equality in Remark 18. has another consequence:

Remark 19 *(a) For each $n \in \mathbb{N}$, $F_n := Hv_n(G) \cap H\overline{V}_0(G)$ is closed in $Hv_n(G)$, hence a Banach space with the induced norm.*

(b) The algebraic equality $\mathcal{V}_0H(G) = H\overline{V}_0(G)$ holds if and only if for each n there exists $m > n$ such that $F_n \subset H(v_m)_0(G)$.

PROOF. (a) Fix $(f_k)_k \subset F_n$ with $f_k \to f \in Hv_n(G)$ in $Hv_n(G)$; a fortiori $f_k \to f$ in $H\overline{V}(G)$. But since all the functions f_k are elements of $H\overline{V}_0(G)$, it follows that also $f \in Hv_n(G) \cap H\overline{V}_0(G) = F_n$.

(b) Define $F := \mathrm{ind}_n F_n$; by (a) this is an (LB)-space. If $\mathcal{V}_0 H(G) = H\overline{V}_0(G)$ holds, then we get $\mathcal{V}_0 H(G) = \mathrm{ind}_m H(v_m)_0(G) = F = \mathrm{ind}_m F_m$ algebraically. Grothendieck's factorization theorem then yields that for each n there exists $m > n$ such that $F_n \subset H(v_m)_0(G)$ with continuous inclusion.

On the other hand, if the condition on the spaces F_n in (b) is satisfied, fix $f \in H\overline{V}_0(G)$. Then also $f \in H\overline{V}(G) = \mathcal{V}H(G)$ so that there is n with $f \in Hv_n(G) \cap H\overline{V}_0(G) = F_n$. The condition in (b) implies $f \in H(v_m)_0(G)$; that is, $f \in \mathcal{V}_0 H(G)$. $\square$

For the main result of this section we have to restrict our attention to the unit disc $\mathbb{D}$ and have to suppose that $\mathcal{V}_0 H(\mathbb{D})$ is a dense topological subspace of its projective hull $H\overline{V}_0(\mathbb{D})$.

Theorem 20 *Let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of weights on $\mathbb{D}$ such that $\mathcal{V}_0 H(\mathbb{D})$ coincides with $H\overline{V}_0(\mathbb{D})$ algebraically and topologically. Then the inductive limit $\mathcal{V}_0 H(\mathbb{D}) = \mathrm{ind}_n H(v_n)_0(\mathbb{D})$ must actually be a boundedly retractive inductive limit.*

PROOF. The property boundedly retractive is equivalent (cf. [1]) to sequentially retractive. ($\mathcal{V}_0 H(\mathbb{D})$ is *sequentially retractive* if for each sequence $(f_k)_k \subset \mathcal{V}_0 H(\mathbb{D})$ with $f_k \to 0$ in $\mathcal{V}_0 H(\mathbb{D})$ there is n such that $f_k \to 0$ in $H(v_n)_0(\mathbb{D})$. The important notion of sequentially retractive inductive limit was introduced and studied by Klaus Floret [19].)

We proceed by contradiction and assume that $\mathrm{ind}_n H(v_n)_0(\mathbb{D})$ is not sequentially retractive. Thus there exists a sequence $(f_k)_k \subset \mathcal{V}_0 H(\mathbb{D})$ with $f_k \to 0$ in $\mathcal{V}_0 H(\mathbb{D})$ such that $(f_k)_k$ does not converge to 0 in $H(v_m)_0(\mathbb{D})$ for each $m \in \mathbb{N}$. As a complete inductive limit $\mathrm{ind}_n H(v_n)_0(\mathbb{D})$ is regular and as the convergent sequence $(f_k)_k$ is bounded in $\mathcal{V}_0 H(\mathbb{D})$, we may assume without loss of generality that $(f_k)_k \subset C_1 =$ closed unit ball of $H(v_1)_0(\mathbb{D})$. The closure $\overline{C_1}$ of C_1 in $\mathcal{V}_0 H(\mathbb{D})$ is still bounded; by regularity there is $m > 1$ such that $\overline{C_1} \subset H(v_m)_0(\mathbb{D})$. By assumption $(f_k)_k$ does not converge to 0 in $H(v_m)_0(\mathbb{D})$. But as $f_k \to 0$ in co, we can find a discrete sequence $(z_k)_k \subset \mathbb{D}$ and $\varepsilon_0 > 0$ such that $v_m(z_k)|f_k(z_k)| \geq \varepsilon_0$ for each $k \in \mathbb{N}$.

At this point we choose a subsequence $(z_j)_j$ of $(z_k)_k$ which is interpolating for $H^\infty(\mathbb{D})$; for the corresponding subsequence $(f_j)_j$ of $(f_k)_k$ we have $v_m(z_j)|f_j(z_j)| \geq \varepsilon_0$ for all j. We can also find a sequence $(\varphi_j)_j \subset H^\infty(\mathbb{D})$ such that $\varphi_i(z_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} |\varphi_j| \leq M$ on $\mathbb{D}$ for some constant $M \geq 1$. It is clear that $g := \sum_{j=1}^{\infty} \varphi_j f_j$ is a holomorphic function on $\mathbb{D}$ which actually belongs to $MB_1 = M$ times the closed unit ball of $Hv_1(\mathbb{D})$ since for each $z \in \mathbb{D}$,

$$v_1(z)|\sum_{j=1}^{\infty} \varphi_j(z) f_j(z)| \leq \left(\sum_j |\varphi_j(z)|\right) (\sup_j \sup_{\mathbb{D}} v_1|f_j|) \leq M.$$

On the other hand $v_m(z_j)|g(z_j)| = v_m(z_j)|f_j(z_j)| \geq \varepsilon_0$ for each j, whence $g \notin H(v_m)_0(\mathbb{D})$.

Now let $g_s := \sum_{j=1}^{s} \varphi_j f_j$ for each $s \in \mathbb{N}$. We claim that each $g_s \in H(v_1)_0(\mathbb{D})$; by our previous estimate it then follows that $g_s \in MC_1$ for each s. To show our claim, fix s and let $\varepsilon > 0$ be given. There exists a compact set $K \subset \mathbb{D}$ such that $v_1(z)|f_j(z)| < \varepsilon/M$ for each $z \in \mathbb{D} \setminus K$ and $j = 1, ..., s$. We obtain that for any $z \in \mathbb{D} \setminus K$,

$$v_1(z)|g_s(z)| \leq v_1(z) \sum_{j=1}^{s} |\varphi_j(z)||f_j(z)| \leq M \max_{j=1,\ldots,s} v_1(z)|f_j(z)| < \varepsilon,$$

which does establish our claim.

Next we would like to show that $g \in \mathcal{V}_0 H(\mathbb{D})$; but by our hypothesis we have $\mathcal{V}_0 H(\mathbb{D}) = H\overline{V}_0(\mathbb{D})$ algebraically and topologically. Hence it suffices to show $g \in H\overline{V}_0(\mathbb{D})$. Fix $\overline{v} \in \overline{V}$ and $\varepsilon > 0$. Since $f_j \to 0$ in $H\overline{V}_0(\mathbb{D}) = \mathcal{V}_0 H(\mathbb{D})$, there is j_0 such that for all $j > j_0$, $\overline{v}(z)|f_j(z)| < \varepsilon/(2M)$ for each $z \in \mathbb{D}$. Since $f_1, ..., f_{j_0} \in H(v_1)_0(\mathbb{D}) \subset H\overline{V}_0(\mathbb{D})$, we can choose a compact set $K \subset \mathbb{D}$ such that $\overline{v}(z)|f_j(z)| < \varepsilon/(2M)$ for all $z \in \mathbb{D} \setminus K$ and $j = 1, ..., j_0$. Now we get for $z \in \mathbb{D} \setminus K$,

$$\begin{aligned}\overline{v}(z)|g(z)| &\leq \sum_{j=1}^{j_0} |\varphi_j(z)|\overline{v}(z)|f_j(z)| + \sum_{j=j_0+1}^{\infty} |\varphi_j(z)|\overline{v}(z)|f_j(z)| \\ &< \frac{\varepsilon}{2M} M + \frac{\varepsilon}{2M} M = \varepsilon,\end{aligned}$$

whence $g \in H\overline{V}_0(\mathbb{D})$.

Finally we prove that $g_s \to g$ in $\mathcal{V}_0 H(\mathbb{D})$. By our hypothesis, it is enough to show that $g - g_s = \sum_{j=s+1}^{\infty} \varphi_j f_j$ converges to 0 in $H\overline{V}_0(\mathbb{D})$ as $s \to \infty$. To do this, fix $\overline{v} \in \overline{V}$ and $\varepsilon > 0$. Since $f_j \to 0$ in $H\overline{V}_0(\mathbb{D})$, we find $s_0 \in \mathbb{N}$ such that for each $j > s_0$, $\sup_{\mathbb{D}} \overline{v}|f_j| < \varepsilon/M$. Then if $z \in \mathbb{D}$ and $s > s_0$, we obtain

$$\overline{v}(z)|\sum_{j=s+1}^{\infty} \varphi_j(z) f_j(z)| \leq \sum_{j=s+1}^{\infty} |\varphi_j(z)|\overline{v}(z)|f_j(z)| < \varepsilon.$$

Hence we have $\sup_{\mathbb{D}} \overline{v}|g - g_s| \leq \varepsilon$ for $s > s_0$.

Summing up, we have proved that $(g_s)_s \subset MC_1$ and that $g_s \to g \in \mathcal{V}_0 H(\mathbb{D})$ in $\mathcal{V}_0 H(\mathbb{D})$. From this we get $g \in M\overline{C_1} \subset H(v_m)_0(\mathbb{D})$, a contradiction to what was proved after the definition of g. $\square$

An inspection of the proof of Theorem 20. shows the following.

Remark 21 *Let $G \subset \mathbb{C}^N$ be an open set. Assume that (*) for each discrete sequence $(z_k)_k \subset G$ there exist a subsequence $(z_j)_j = (z_{k_j})_j$ and a sequence $(\varphi_j)_j \subset H^\infty(G)$ with $\varphi_j(z_i) = \delta_{ij}$ for all i and j and such that $\sum_j |\varphi_j| \leq M$ on G for some constant $M > 0$. If $\mathcal{V} = (v_n)_n$ is a decreasing sequence of weights on G such that $\mathcal{V}_0 H(G)$ is a dense topological subspace of $H\overline{V}_0(G)$, then completeness of the inductive limit implies that $\mathrm{ind}_n\, H(v_n)_0(G)$ must be boundedly retractive.*

Condition (*) in Remark 21. holds for open connected sets $G \subset \mathbb{C}$ such that $\mathbb{C}^* \setminus G$ does not have a connected component consisting of only one point, see the proof of Proposition 7. It also holds if $G \subset \mathbb{C}^N$ is open, bounded and absolutely convex, cf. the proof of Proposition 12. (In the notation of that proof it is enough to take $(\varphi_j \circ w)_j$ as the desired sequence.) We decided to state Theorem 20. in its present form; i.e., for $G = \mathbb{D}$ only, since otherwise there is a lack of examples of spaces $\mathcal{V}_0 H(G)$ which are dense topological subspaces of $H\overline{V}_0(G)$.

Corollary 22 *Let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of radial weights on the unit disc $\mathbb{D}$ with $\lim_{r\to 1-} v_n(r) = 0$ for each n and such that $\mathcal{V}_0 H(\mathbb{D})$ is a topological subspace of $H\overline{V}_0(\mathbb{D})$. Then the following conditions are equivalent:*

(1) $\mathcal{V}_0 H(\mathbb{D})$ is complete,

(2) the algebraic equality $\mathcal{V}_0 H(\mathbb{D}) = H\overline{V}_0(\mathbb{D})$ holds,

(3) the inductive limit $\mathcal{V}_0 H(\mathbb{D}) = \mathrm{ind}_n\, H(v_n)_0(\mathbb{D})$ is boundedly retractive.

PROOF. Our assumptions on the weights v_n imply that the polynomials are dense in both $\mathcal{V}_0 H(\mathbb{D})$ and $H\overline{V}_0(\mathbb{D})$, see [9], Theorems 1.5.(a) and 1.6.(a). Hence (1)⇒(2) is clear. (3)⇒(1) is trivial since each boundedly retractive (LB)-space must be complete. And (1)⇒(3) is a consequence of Theorem 20.

It remains to show (2)⇒(1). Since $\mathcal{V}_0 H(\mathbb{D})$ is a topological subspace of $\mathcal{V}H(\mathbb{D})$ by [4], last sentence of Theorem 7 and since $\mathcal{V}H(\mathbb{D})$ is complete by [7], Proposition 5.(2), it is enough to prove that $\mathcal{V}_0 H(\mathbb{D})$ is closed in $\mathcal{V}H(\mathbb{D})$. Take a net $(f_\alpha)_\alpha \subset \mathcal{V}_0 H(\mathbb{D})$ with $f_\alpha \to f \in \mathcal{V}H(\mathbb{D})$ in $\mathcal{V}H(\mathbb{D})$. Then we have $(f_\alpha)_\alpha \subset H\overline{V}_0(\mathbb{D})$ and $f_\alpha \to f$ in $H\overline{V}(\mathbb{D})$. Hence we get $f \in H\overline{V}_0(\mathbb{D})$ and then by (2) $f \in \mathcal{V}_0 H(\mathbb{D})$. $\square$

Note that by [5], Theorem 3.1. the hypotheses of Corollary 22. are satisfied if all v_n belong to a class $\mathcal{W}$ as in [5]. (Then the weights are normal in the sense of Shields and Williams and satisfy the conditions (L1) and (L2) of Lusky in a uniform way, cf. [5], pages 437 and 438. Also see [24].)

Moreover, by Theorem 5 of [16] (and by [9], 1.6.(a)), the hypotheses of Corollary 22. are also given if $\mathcal{V}$ satisfies the condition (LOG) of [16]. Then the weights tend to 0 at the boundary of the disc logarithmically.

The case when the weights tend to 0 at the boundary exponentially is open. And there is still no single example known (not only for radial weights on $\mathbb{D}$) when $\mathcal{V}_0H(G)$ is not a topological subspace of $H\overline{V}_0(G)$, see Problem 3 of [2].

4 Appendix

It is the purpose of this appendix to give an example of a regularly decreasing sequence of radial weights on $\mathbb{D}$ which does not satisfy condition (S).

We first fix a decreasing sequence $\mathcal{W} = (w_n)_n$ of radial weights on $\mathbb{D}$ such that for each $n \in \mathbb{N}$ the following four conditions are satisfied:
(i) $w_n(0) = 1$, (ii) $w_n(s) < w_n(r)$ for $0 \le r < s < 1$, (iii) $\lim_{r\to 1-} w_n(r) = 0$, (iv) $\lim_{r\to 1-} w_{n+1}(r)/w_n(r) = 0$.
By (iv) the sequence $\mathcal{W}$ satisfies (S). To give a concrete example, one can take $w_n(r) = (1-r)^{3-1/n}$ for $n \in \mathbb{N}$ and $r \in [0, 1)$.

Next we fix a radial weight w_0 on $\mathbb{D}$ satisfying the conditions (i), (ii), and (iii) above and such that $w_0 \le w_n$ on $\mathbb{D}$ for each n. In the concrete example we may take $w_0(r) = (1-r)^3$ for $r \in [0, 1)$.

Claim. *A radial weight w on $\mathbb{D}$ can be constructed which satisfies (i), (ii), (iii) and $w_0 \le w \le w_1$ and such that there are two sequences $(r_k)_k$ and $(s_k)_k$ tending to 1 with $0 = r_1 < s_1 < r_2 < ... < r_k < s_k < ...$ and $w(r_k) = w_0(r_k)$, $w(s_k) = w_1(s_k)$ for each $k \in \mathbb{N}$.*

The claim will be proved later on. Assuming the claim, we first show how a sequence $\mathcal{V} = (v_n)_n$ with the desired properties can be found. Let us put $v_n(z) = \max\{w_n(z), w(z)\}$ for $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Clearly this sequence is decreasing, and each v_n is a radial weight on $\mathbb{D}$ which satisfies (i), (ii) and (iii). Note that since $v_n(s_k) = v_1(s_k)$ for each k and n and since $(s_k)_k$ tends to 1, *the sequence $\mathcal{V}$ does not satisfy condition (S).*

We will now prove that $\mathcal{V}$ *is regularly decreasing*, showing that for each n there is $m > n$ such that for each $k > n$ and each $\varepsilon \in (0, 1)$ there is $\delta > 0$ such that for any $z \in \mathbb{D}$ the inequality $v_m(z) \ge \varepsilon v_n(z)$ implies $v_k(z) \ge \delta v_n(z)$. Fix n and take $m = n + 1$. For $k > m$ and $\varepsilon \in (0, 1)$, (iv) allows us to find $r(\varepsilon) \in (0, 1)$ with $w_m(r) < \varepsilon w_n(r)$ for each $r \geq r(\varepsilon)$. Take $\delta := \min\{\varepsilon, v_k(r(\varepsilon))\} > 0$ and let z satisfy

$$\max\{w_m(z), w(z)\} = v_m(z) \geq \varepsilon v_n(z) = \varepsilon \max\{w_n(z), w(z)\}.$$

We have two cases:

Case 1: If $w_m(z) \geq w(z)$, then $v_m(z) = w_m(z)$, and a fortiori $v_n(z) = w_n(z)$. Hence we get $w_m(z) \geq \varepsilon w_n(z)$, which yields $|z| \leq r(\varepsilon)$. This then implies $v_k(z) \geq v_k(r(\varepsilon)) \geq \delta$. As all the weights w_n and v_n are ≤ 1 on $\mathbb{D}$, we conclude $v_k(z) \geq \delta v_n(z)$, as desired.

Case 2: Let now $w_m(z) < w(z)$; then we get $v_m(z) = w(z)$ and a fortiori also $v_k(z) = w(z)$.
Subcase 2.1: If now $w_n(z) \leq w(z)$, then $v_n(z) = w(z)$. Hence we can conclude the desired inequality as follows: $v_k(z) = w(z) \geq \varepsilon w(z) = \varepsilon v_n(z) \geq \delta v_n(z)$.
Subcase 2.2: If $w_n(z) > w(z)$, then we obtain $v_n(z) = w_n(z) > w(z)$. In this case, the assumption reads $w(z) = v_m(z) \geq \varepsilon v_n(z) = \varepsilon w_n(z)$, which implies

$$v_k(z) = w(z) = v_m(z) \geq \varepsilon v_n(z) \geq \delta v_n(z),$$

and the proof is finished: Our sequence $\mathcal{V}$ is regularly decreasing, but does not satisfy (S).

It remains to prove the claim. To make things easier, we will only give a proof in the case of our concrete example $w_0(r) = (1 - r)^3$ and $w_1(r) = (1 - r)^2$, $r \in [0, 1)$.

Define functions α and β from $[1, \infty)$ into itself by $\alpha(t) := 1/w_1(1-1/t)$ $(= t^2)$ and $\beta(t) := 1/w_0(1 - 1/t)$ $(= t^3)$, $t \in [1, \infty)$. α and β are convex functions; in fact, they have strictly positive derivatives which tend to ∞ as $t \to \infty$.

Let $a_1 := 1$ and consider the tangent $y = \beta(a_1) + \beta'(a_1)(t - a_1)$ $(= 1 + 3(t - 1))$ to $y = \beta(t)$ $(= t^3)$ at the point $(a_1, \beta(a_1))$ $(= (1, 1))$. This line intersects $y = \alpha(t)$ $(= t^2)$ at $((1, 1)$ and) $(b_1, \alpha(b_1))$ $(= (2, 4))$, hence we also obtain $\alpha(b_1) =$ $(4 =)$ $\beta(a_1) + \beta'(a_1)(b_1 - a_1)$. Set

$$\gamma(t) := \beta(a_1) + \beta'(a_1)(t - a_1) \; (= 1 + 3(t - 1)), \; t \in [a_1, b_1] \; (= [1, 2]).$$

By convexity $(1 + 3(t - 1) =)$ $\gamma(t) < \beta(t)$ $(= t^3)$ for $t \in (a_1, b_1]$ $(= (1, 2])$. By the definition of b_1 $(= 2)$ also $(t^2 =)$ $\alpha(t) < \gamma(t)$ $(= 1 + 3(t - 1))$ holds for $t \in [a_1, b_1)$ $(= [1, 2))$. Moreover, we have $\beta(a_1) =$ $(1 =)$ $\alpha(a_1)$ as well as $\alpha(b_1) =$ $(4 =)$ $\gamma(b_1)$.

Let us go to the next step in the induction. Find $a_2 > b_1$ $(= 2)$ such that

$$(3a_2^2 =)\ \beta'(a_2) > \frac{\beta(a_2) - \alpha(b_1)}{a_2 - b_1} \quad \left(= \frac{a_2^3 - 4}{a_2 - 2}\right).$$

(The last inequality amounts to $2a_2^3 - 6a_2^2 + 4 > 0$, which is satisfied for large enough a_2, say, $a_2 = 3$.) We define for $t \in (b_1, a_2](= (2, a_2])$,

$$\gamma(t) := \alpha(b_1) + \frac{\beta(a_2) - \alpha(b_1)}{a_2 - b_1}(t - b_1) \quad \left(= 4 + \frac{a_2^3 - 4}{a_2 - 2}(t - 2)\right).$$

If $t \in (b_1, a_2]$, since α is convex, we have

$$\begin{aligned}
(t^2 =)\ \alpha(t) &< \alpha(b_1) + \frac{\alpha(a_2) - \alpha(b_1)}{a_2 - b_1}(t - b_1) \quad \left(= 4 + \frac{a_2^2 - 4}{a_2 - 2}(t - 2)\right) \\
&< \left(4 + \frac{a_2^3 - 4}{a_2 - 2}(t - 2)\right) \\
&= \alpha(b_1) + \frac{\beta(a_2) - \alpha(b_1)}{a_2 - b_1}(t - b_1) = \gamma(t).
\end{aligned}$$

And since β is convex, our selection of a_2 leads to

$$\begin{aligned}
(t^3 =)\ \beta(t) &> \beta(a_2) + \beta'(a_2)(t - a_2) \quad \left(= a_2^3 + 3a_2^2(t - a_2)\right) \\
&> \left(a_2^3 + \frac{a_2^3 - 4}{a_2 - 2}(t - a_2)\right) \\
&= \beta(a_2) + \frac{\beta(a_2) - \alpha(b_1)}{a_2 - b_1}(t - a_2) = \gamma(t).
\end{aligned}$$

We continue with the point a_2 as we did before with a_1 and find $b_2 > a_2$ with $\beta(a_2) + \beta'(a_2)(b_2 - a_2) = \alpha(b_2)$. That is why we wrote both the abstract and the concrete formulation in the first step. Observe that the slope of the line $\beta(a_2) + \beta'(a_2)(t - a_2)$ is greater than the one of $\gamma(t)$ on $[b_1, a_2]$. Hence, if we set $\gamma(t) := \beta(a_2) + \beta'(a_2)(t - a_2)$ for $t \in (a_2, b_2]$, then γ is a convex function.

Continuing by induction, we can construct $\gamma : [1, \infty) \to [1, \infty)$ continuous, strictly increasing, convex with $\alpha(t) \leq \gamma(t) \leq \beta(t)$, $t \in [1, \infty)$, and sequences $(a_k)_k$ and $(b_k)_k$ tending to ∞ with $1 = a_1 < b_1 < a_2 < \ldots < a_k < b_k < \ldots$ such that $\gamma(a_k) = \beta(a_k)$ and $\gamma(b_k) = \alpha(b_k)$ for all k. Finally, the function $w(r) := 1/\gamma(1/(1-r))$, $r \in [0, 1)$, has the desired properties for $r_k := 1 - 1/a_k$ and $s_k := 1 - 1/b_k$ for all $k \in \mathbb{N}$.

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