The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations

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Abstract

We investigate the splitting of short exact sequences of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow E \longrightarrow 0,$$

where E is the dual of a Fréchet Schwartz space and X, Y are PLS-spaces, like the spaces of distributions or real analytic functions or their subspaces. In particular, we characterize pairs (E, X) as above such that $\mathrm{Ext}^1(E, X) = 0$ in the category of PLS-spaces and apply this characterization to many natural spaces X and E. We discover an extension of Vogt and Wagner's $(DN) - (\Omega)$ splitting theorem. These results are applied to parameter dependence of linear partial differential operators and surjectivity on spaces of vector valued distributions.

1 Introduction

The aim of this paper is to study the splitting of short exact sequences of PLS-spaces and its applications to parameter dependence of solutions of linear partial differential equations on spaces of distributions (see Section 5, Theorem 5.5). We study the functor Ext^1 for subspaces of $\mathscr{D}'(\Omega)$ and duals of Fréchet Schwartz spaces. This is considered in the framework of the so-called PLS-spaces (i.e., the smallest class of locally convex spaces containing all duals of Fréchet Schwartz spaces and closed with respect of taking countable products and closed subspaces). It contains important spaces appearing in analytic applications of linear functional analysis, like spaces of distributions, spaces of real analytic or quasi analytic functions, spaces of holomorphic or smooth functions; we refer the reader to the survey paper [9]. The crucial result of the present paper (Theorem 3.1) is a characterization of the pairs (F, X), X a PLS-space, F a Fréchet nuclear space, such that every short topologically exact sequence of PLS-spaces

$$0 \longrightarrow X \xrightarrow{\ j\ } Y \xrightarrow{\ q\ } F' \longrightarrow 0 \tag{1}$$

splits (i.e., q has a linear continuous right inverse) or equivalently, such that $\mathrm{Ext}^1_{PLS}(F', X) = 0$. Topological exactness of (1) means that j is a topological embedding onto the kernel of the

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continuous and open surjection q. The characterization is given in terms of some inequality. The proof is technical, complicated and based on the method of the functor Proj^1 for spectra of LB-spaces. The case when both X and F' are substituted by Fréchet spaces was characterized long ago under assumptions that one space is nuclear or one space is a suitable sequence space. In fact, necessity in that case is due to Vogt [38]; he also introduced a useful sufficient condition. Sufficiency for both spaces being sequence spaces is due to Krone and Vogt [19]. Sufficiency in other cases was an open problem for some time. A breakthrough was made by Frerick [13] who proved the case of all nuclear Fréchet spaces and, finally, Frerick and Wengenroth proved sufficiency in all cases for Fréchet spaces in [15]. The condition they all used was slightly different from ours - a characterization even more similar to ours is given in [43, 5.2.5]. There have been very few splitting results for PLS-spaces so far, see [11], [42], [20], [10, Theorem 2.3], [41], [4], comp. [14] and [43, Sec. 5.3]. However, this is considered as an important problem in the modern theory of locally convex spaces and their analytic applications; see [41].

In [4] we investigated the vanishing of $\mathrm{Ext}^1_{PLS}(F, X)$ for a nuclear Fréchet space F, while in the present paper we attack the same question for the dual F'. This is a different, much more difficult problem. For instance, the reduction to the vanishing of the derived functor Proj^1 for spectra of LB-spaces was standard in [4], but now it requires several new ideas and ingredients, among them a key observation due to Vogt in [41], see Lemma 3.3, proof of Theorem 3.4 (ii)$\Leftrightarrow$(iii). To avoid problems with local splitting we have to dualize the considered short exact sequences and to study sequences of LFS-spaces (see the proof of Theorem 3.4).

Although our condition looks complicated it turns out to be evaluable. Indeed, we characterize (Theorem 4.4, Cor. 4.5) PLS-spaces X such that $\mathrm{Ext}^1_{PLS}(\Lambda_r(\alpha)', X) = 0$, where $\Lambda_r(\alpha)$ is a stable power series space (like $H(\mathbb{D}^d)$) or even $C^\infty(U) \simeq \prod \Lambda_r(\alpha)$). The characterizing condition is of (Ω) type and is called (PA) . On the other hand, it turns out that if X has (PA) and a nuclear Fréchet space F has (Ω) then $\mathrm{Ext}^1_{PLS}(F', X) = 0$ (Theorem 4.1), this is the proper extension of the (DN) -(Ω) splitting theorem [25, 30.1]. That is why the discovery of (PA) as a suitable generalization of the condition (Ω) seems to be one of the main achievements of the paper. It is even more striking if one looks at Proposition 5.4 and compare it with earlier results on the property (Ω) of kernels of hypoelliptic operators (comp. [30], [35], [44, 2.2.6]).

The parameter dependence problem considers whether, for every linear partial differential operator with constant coefficients $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ convex open, and every family of distributions $(f_\lambda)_{\lambda\in U} \subseteq \mathscr{D}'(\Omega)$ depending smoothly C^∞ (or holomorphically etc.) on the parameter λ running through an arbitrary C^∞-manifold U (or Stein manifold U etc.), there is an analogous family $(u_\lambda)_{\lambda\in U}$ with the same type of dependence on $\lambda \in U$ such that

$$P(D)u_\lambda = f_\lambda \qquad \forall\ \lambda \in U. \tag{2}$$

Let us recall that (f_λ) depends holomorphically (smoothly) on $\lambda \in U$ if for every test function φ, $\lambda \mapsto \langle f_\lambda, \varphi\rangle$ is holomorphic (C^∞-smooth). This problem has been extensively studied, even in a much more general setting (for instance, if $P(D)$ depends on λ as well); see [21], [22], [33], [3], [2], comp. introduction of [4]. Using tensor product techniques the parameter dependence is equivalent to the problem of surjectivity of $P(D)$ on the spaces of vector valued distributions $\mathscr{D}'(\Omega, F)$, where, e.g., $F = C^\infty(U)$ (for smooth dependence) or $F = H(U)$ (for holomorphic dependence). Our splitting results implies a positive solution for any Fréchet space F with property (Ω) (Theorem 5.5), for instance, $F \simeq H(U), C^\infty(U), \Lambda_r(\alpha), C^\infty[0, 1]$, etc., see [25, 29.11]. Our method is potentially applicable to arbitrary surjective linear continuous operators $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ and even to more general spaces than $\mathscr{D}'$.

In this applications of splitting results, the crucial point is whether ker $P(D)$ has (PA) , which we prove by some trick (see Proposition 5.4 and Theorem 5.1). For more applications of our splitting result for spaces of real analytic functions and Roumieu quasianalytic classes of ultradifferentiable functions see the forthcoming paper [5].

The paper is organized as follows. Section 2 contains preliminaries and notation. In Section 3 we prove the main splitting theorem. In Section 4 we apply it for some natural spaces, especially, sequence spaces, we introduce conditions (PA) and $(P\underline{A})$ and give examples and applications. In Section 5 we apply our theory to the parameter dependence problem.

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2 Preliminaries

In the present section we collect some basic notation which is very similar to the one used in [4].

By an *operator* we mean a linear continuous map. By $L(Z, Y)$ we denote the set of all operators $T : Z \longrightarrow Y$. If $A \subseteq Z$ and $B \subseteq Y$, then $W(A, B) := \{T \in L(Z, Y) : \ T(A) \subseteq B\}$.

A locally convex space X is a *PLS-space* if it is a projective limit of a sequence of strong duals of Fréchet -Schwartz spaces (i.e., LS-spaces), see survey paper [9]. If we take strong duals of nuclear Fréchet spaces instead (i.e., LN-spaces) then X is called a *PLN-space.* Every closed subspace and every Hausdorff quotient of a PLS-space is a PLS-space, [11, 1.2 and 1.3]. Every PLS-space is automatically complete and Schwartz, PLN-spaces are even nuclear. Every Fréchet-Schwartz space is a PLS-space and every strongly nuclear Fréchet space is a PLN-space.

Every PLS-space X satisfies $X = \mathrm{proj}_{N\in\mathbb{N}} \ \mathrm{ind}_{n\in\mathbb{N}} \ X_{N,n}$, $X_{N,n}$ are Banach spaces, $X_N := \mathrm{ind}_{n\in\mathbb{N}} \ X_{N,n}$ denotes the locally convex inductive limit with compact linking maps, and $\mathrm{proj}_{N\in\mathbb{N}} \ X_N$ denotes the topological projective limit of a sequence $(X_N)_{N\in\mathbb{N}}$. The linking maps will be denoted by $i_N^K : X_K \to X_N$ and $i_N : X \longrightarrow X_N$. If $\overline{i_N X} = X_N$ for each N sufficiently big then we call the spectrum (X_N) *reduced.* We denote the closed unit ball of $X_{N,n}$ by $B_{N,n}$ and its polar in X_N' by $U_{N,n}$. In $E = \mathrm{ind}_{n\in\mathbb{N}} \ E_n$ we always denote by B_n the unit ball of the Banach space $(E_n, ||.||_n)$, by U_n its polar in E_n' and by $j_m^n : E_n \to E_m$ the injective compact linking map. Without loss of generality we assume that for every $M \geq N$, $m \geq n$

$$i_N^M(B_{M,n}) \subseteq B_{N,n}, \quad B_{N,n} \subseteq B_{N,m}, \quad B_n \subseteq B_m.$$

This notation will be kept throughout the paper. We will use in the category of PLS-spaces the notions of pull-back and push-out as described, for instance, in [43, Def. 5.1.2], comp. [11].

Let $A = (a_{N,n}(j))$ be a matrix of non negative elements satisfying the following conditions: (i) $a_{N,n+1}(j) \leq a_{N,n}(j) \leq a_{N+1,n}(j)$; (ii) $\forall \ j \ \exists \ N \ \forall \ n \ a_{N,n}(j) > 0$; (iii) $\lim_{j\to\infty} \frac{a_{N,n+1}(j)}{a_{N,n}(j)} = 0$.

We define the Köthe type PLS-sequence spaces $\Lambda^p(A)$ for $1 \leq p < \infty$,

$$\Lambda^p(A) := \{x = (x(j)) : \ \forall \ N \in \mathbb{N} \ \exists \ n \in \mathbb{N} : \quad \|x\|_{N,n} < \infty\},$$

where $\|x\|_{N,n} := \left(\sum_j |x(j)|^p a_{N,n}(j)\right)^{1/p}$. The definition for $p = \infty$ is analogous. Clearly, $\Lambda^p(A) = \mathrm{proj}_{N\in\mathbb{N}} \ \mathrm{ind}_{n\in\mathbb{N}} \ l_p(a_{N,n})$, where $l_p(a_{N,n})$ denotes the weighted l_p-space equipped with the norm $\| \cdot \|_{N,n}$. The condition (iii) implies that $\Lambda^p(A)$ is a PLS-space. Every PLS-sequence space $\Lambda^p(A)$ is isomorphic to a countable product of spaces for a matrix with strictly positive elements. $\Lambda^p(A)$ is even a PLN-space if instead of (iii), we assume (iv) $\sum_j \frac{a_{N,n+1}(j)}{a_{N,n}(j)} < \infty$.

If $a_{N,n}(j) := \exp(r_N\alpha_j - s_n\beta_j)$ where $\alpha_j, \beta_j > 0$ such that $\alpha_j + \beta_j \to \infty$ and $r_N \nearrow r$, $s_n \nearrow s$ then we call the corresponding Köthe type space $\Lambda(A)$ to be PLS-type power series space and denote by $\Lambda_{r,s}(\alpha, \beta)$. It suffices to consider only $r, s = 0, \infty$, comp. [39]. For $\Lambda_r(\alpha)$ see [25].

For further information from functional analysis see [25] ((DN) - (Ω) invariants are explained there) and [18], for the theory of PDE see [16]. For the modern theory of locally convex inductive limits see [1]. More details about notation can be seen in [4].

3 Splitting of short exact sequences

We will consider pairs (E, X) satisfying one of the following standard assumptions:

(a) X is a PLN-space;

(b) X is a Köthe type PLS-space, $X = \Lambda^\infty(A)$;

(c) E is an LN-space;

(d) E is a Köthe coechelon LS-space of order 1, $X = k_1(v)$.

Now, we formulate the main theorem (known for E, X both DFS-spaces see [43, 5.2.5]):

Theorem 3.1 *Let X be an ultrabornological PLS-space, a reduced projective limit $X = \text{proj}_{N\in\mathbb{N}} X_N$ of LS-spaces $X_N = \text{ind}_{n\in\mathbb{N}} X_{N,n}$. Let $E = \text{ind}\,{}_\nu E_\nu$ be an LS-space (an injective inductive limit). Assume that the pair (E, X) satisfies assumptions (b) or (c) or (d) above, then the following assertions are equivalent:*

(1) $\text{Ext}^1_{PLS}(E, X) = 0$*;*

(2) the pair (E, X) satisfies the condition (G), i.e.,

$$\forall\ N, \nu\ \exists\ M \geq N, \mu \geq \nu\ \forall\ K \geq M, \kappa \geq \mu\ \exists\ n\ \forall\ m \geq n\ \exists\ k \geq m, S$$
$$\forall\ y \in X'_N, x \in E_\nu : \quad \|y \circ i_N^M\|^*_{M,m} \|j_\mu^\nu x\|_\mu \leq S \left(\|y\|^*_{N,n}\|x\|_\nu + \|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa\right);$$

(3) the pair (E, X) satisfies the condition (G_ε), i.e.,

$$\forall\ N, \nu\ \exists\ M \geq N, \mu \geq \nu\ \forall\ K \geq M, \kappa \geq \mu\ \exists\ n\ \forall\ m \geq n, \varepsilon > 0\ \exists\ k \geq m, S$$
$$\forall\ y \in X'_N, x \in E_\nu : \quad \|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu \leq \varepsilon\|y\|^*_{N,n}\|x\|_\nu + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa;$$

Ultrabornologicity of X follows from (G). We conjecture that 3.1 holds also in case (a).

We recall tools from the homological theory of locally convex spaces; a nice presentation of the theory is contained in lecture notes [43], comp. [4]. If (X_N, i_N^K) is a projective spectrum of locally convex spaces, the so-called *fundamental resolution* is defined as an exact sequence:

$$0 \longrightarrow X \longrightarrow \prod_{N\in\mathbb{N}} X_N \xrightarrow{\ \sigma\ } \prod_{N\in\mathbb{N}} X_N,$$

where X is the projective limit of the spectrum and $\sigma((x_N)) = (i_N^{N+1}x_{N+1} - x_N)$. We define

$$\text{Proj}^1\,(X_N) := \prod_{N\in\mathbb{N}} X_N / \text{im}\ \sigma.$$

The value of Proj^1 does not depend on the choice of a reduced spectrum of LS-spaces representing X. Moreover, for PLS-spaces the following conditions are equivalent: (i) $\text{Proj}^1 X = 0$; (ii) X is ultrabornological; (iii) X is barreled; (iv) X is reflexive (see [43, 3.3.10]).

We apply the functor Proj^1 to various spectra of spaces of operators. For example, if $X = \text{proj}_{N\in\mathbb{N}} X_N$, then in the spectrum $L(F, X_N)$ linking maps are defined as $I_N^K : L(F, X_K) \to L(F, X_N)$, $I_N^K(T) = i_N^K \circ T$ and $I_N : L(F, X) \to L(F, X_N)$, $I_N(T) := i_N \circ T$. For other cases the linking maps are defined analogously.

Lemma 3.2 *If X is a PLS-space,* $\text{Proj}^1 X = 0$ *and Z is a Banach space, then*

(1) $\mathrm{Proj}^1\ L(Z, X_N) = 0$ *if* $X = \Lambda^\infty(A)$*;*

(2) $\mathrm{Proj}^1\ L(X'_N, Z) = 0$ *if* $X = \Lambda^\infty(A)$*;*

(3) $\mathrm{Proj}^1\ L(X'_N, Z) = 0$ *if* $Z = l_\infty$*;*

(4) $\mathrm{Proj}^1\ L(X'_N, Z) = 0$ *if* X *is a PLN-space.*

Proof: Since $\Lambda^\infty(A)$ is isomorphic to a countable product of spaces of the same type for a strictly positive matrix, we may assume that all the elements in A are strictly positive.

(1): By [43, 3.2.18], $\mathrm{Proj}^1\ X = 0$ implies:

$$\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m \geq n, \varepsilon > 0\ \exists\ k \geq m, S\ \forall\ i: \\ a_{M,m}(i) \geq \min\left(\varepsilon^{-1} a_{N,n}(i), S^{-1} a_{K,k}(i)\right).$$

Since X_N is a coechelon Köthe sequence space $k_\infty(v)$, we may treat elements of $L(Z, X_N)$ as sequences of functionals $(f_i) \subseteq Z'$ and after that identification

$$W(B, B_{N,n}) = \{(f_i) : \sup_i \|f_i\| a_{N,n}(i) \leq 1\},$$

where B and $B_{N,n}$ denote as usual the unit balls in Z and $X_{N,n}$ respectively. We will show that

$$W(B, B_{M,m}) \subseteq \varepsilon W(B, B_{N,n}) + S W(B, B_{K,k}).$$

Let $(f_i) \in W(B, B_{M,m})$. We take $g_i := f_i$ if $S^{-1} a_{K,k}(i) \geq \varepsilon^{-1} a_{N,n}(i)$ and 0 otherwise. Then

$$\frac{\|g_i\| a_{N,n}(i)}{\varepsilon} \leq \|g_i\| a_{M,m}(i) \leq 1.$$

Therefore $(g_i) \in \varepsilon W(B, B_{N,n})$ and analogously $(f_i - g_i) \in S W(B, B_{K,k})$. Apply [43, 3.2.14].

(2): The proof is analogous to that of (1).

(3): $L(X'_N, Z) = l_\infty(X_N)$ and the result follows from [43, 3.3.11 and 3.3.16].

(4): This is [2, Lemma 3.5]. □

Lemma 3.3 *(see [41, Lemma 3.1]) Let* X *be a PLS-space and* E *be an LS-space satisfying one of the assumptions (a) — (d). If* $H = E'$ *and*

$$0 \longrightarrow H \xrightarrow{\ j\ } F \xrightarrow{\ q\ } G \longrightarrow 0 \tag{3}$$

is a short exact sequence of Fréchet spaces, then we have the following exact sequence:

$$0 \longrightarrow L(X', H) \longrightarrow L(X', F) \longrightarrow L(X', G) \longrightarrow \\ \longrightarrow \mathrm{Proj}^1\ L(X'_N, H_N) \longrightarrow \mathrm{Proj}^1\ L(X'_N, F_N) \longrightarrow \mathrm{Proj}^1\ L(X'_N, G_N) \longrightarrow 0.$$

Proof: This is [43, 3.1.5] applied to spectrum of short exact sequences

$$0 \longrightarrow L(X'_N, H_N) \longrightarrow L(X'_N, F_N) \longrightarrow L(X'_N, G_N) \longrightarrow 0.$$

□

Now, we are ready to reduce the splitting problem to the vanishing of Proj^1.

Theorem 3.4 *Let* X *be a PLS-space with* $\mathrm{Proj}^1\ X = 0$ *and let* E *be an LS-space satisfying one of the conditions (a) — (d) then the following assertions are equivalent:*

(i) $\mathrm{Ext}^1_{PLS}(E,X)=0$;

(ii) $\mathrm{Proj}^1\ L(X',E'_N)=0$;

(iii) $\mathrm{Proj}^1\ L(X'_N,E'_N)=0$.

Proof: (i)⇒(ii): Note, that X is ultrabornological, X' a complete LFS-space. For any operator $T : X' \to \prod E'_n$, we get twice the pull-back of the fundamental resolution of E':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & \prod E'_N & \xrightarrow{\sigma} & \prod E'_N & \longrightarrow & 0 \\ & & \uparrow \mathrm{id} & & \uparrow & & \uparrow T & & \\ 0 & \longrightarrow & E' & \xrightarrow{j} & Y & \xrightarrow{q} & X' & \longrightarrow & 0 \\ & & \uparrow \mathrm{id} & & \uparrow & & \uparrow i'_N & & \\ 0 & \longrightarrow & E' & \xrightarrow{j_N} & Y_N & \xrightarrow{q_N} & X'_N & \longrightarrow & 0. \end{array} \tag{4}$$

We will show in few steps that Y is a complete LFS-space. Completeness, metrizability and to be a Schwartz space are the three space properties (see[8, Th. 2.3.3], [32, Th. 3.7]), thus Y is complete and Y_N is a Fréchet Schwartz space. Since $X' = \bigcup X'_N$ also $Y = \bigcup Y_N$ and, by Grothendieck factorization theorem, every bounded set in X' (in Y) is bounded in some X'_N (Y_N, resp.). Since E' is a Fréchet Schwartz space, it is quasinormable. By [25, 26.17], q_N lifts bounded sets and, consequently, also q lifts bounded sets. We have proved that LFS-space $Y^u = \mathrm{ind}_{N\in\mathbb{N}} Y_N$ is the ultrabornological space associated to Y. Then

$$0 \longrightarrow E' \xrightarrow{j} Y^u \xrightarrow{q} X' \longrightarrow 0$$

is topologically exact since E' and X' are ultrabornological. By Roelcke's lemma [31], $Y = Y^u$ topologically, so Y is reflexive by [25, 24.19]. By duality and $\mathrm{Ext}^1_{PLS}(E,X)=0$, we get splitting of the middle row in (4) and lifting of T. Thus $\mathrm{Proj}^1\ L(X',E'_N)=0$.

(ii)⇒(i): Let us consider the following short topologically exact sequence of PLS-spaces:

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} E \longrightarrow 0. \tag{5}$$

Since $\mathrm{Proj}^1\ X=0$ and $\mathrm{Proj}^1\ E=0$, then [43, 3.1.5] implies that $\mathrm{Proj}^1\ Y=0$ and X, Y, E are reflexive. By [11, Lemma 1.5], q lifts bounded sets, thus we get by duality and the push-out the following diagram with topologically exact rows (E', Y', X' are LFS-spaces):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_N & \longrightarrow & Z & \xrightarrow{Q} & X' & \longrightarrow & 0 \\ & & \uparrow i'_N & & \uparrow & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & E' & \xrightarrow{q'} & Y' & \xrightarrow{j'} & X' & \longrightarrow & 0. \end{array} \tag{6}$$

If the upper rows splits then i'_N extends to Y' and we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & \prod E'_N & \xrightarrow{\sigma} & \prod E'_N & \longrightarrow & 0 \\ & & \uparrow \mathrm{id} & & \uparrow & & \uparrow T & & \\ 0 & \longrightarrow & E' & \xrightarrow{q'} & Y' & \xrightarrow{j'} & X' & \longrightarrow & 0. \end{array}$$

By $\mathrm{Proj}^1\ L(X',E'_N)=0$, T lifts. The lower row splits [11, 1.7], and, by duality, also (5) splits.

We prove that the upper row in (6) splits. This is evident for (a), (c) or (d). In case (b) X' is a direct sum of Köthe type LFS-spaces with l_1-type "norms". By [40, Prop. 5.1], every summand is a projective limit of l_1 Banach spaces and splitting of the upper row in (6) follows.

(ii)⇔(iii): The proof follows the idea of Vogt [41, Proposition 4.1]. We apply Lemma 3.3 to the canonical resolution of $H = E'$:

$$0 \longrightarrow H \xrightarrow{i} \prod_{n\in\mathbb{N}} H_n \xrightarrow{\sigma} \prod_{n\in\mathbb{N}} H_n \longrightarrow 0,$$

where $\sigma((x_n)_{n\in\mathbb{N}}) := (i_n^{n+1}x_{n+1} - x_n)_{n\in\mathbb{N}}$ and $i_n^{n+1} : H_{n+1} \to H_n$ are linking maps. We define

$$\begin{aligned}
\Sigma_1 : \prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n) \quad &\to \quad \prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n),\\
\Sigma_2 : \prod_{N\in\mathbb{N}} L(X'_N, \prod_{n\le N} H_n) \quad &\to \quad \prod_{N\in\mathbb{N}} L(X'_N, \prod_{n\le N} H_n);\\
\Sigma_1((T_{N,n})_{N\in\mathbb{N},n\in\mathbb{N}}) \quad &:= \quad (T_{N+1,n}\circ I^N_{N+1} - T_{N,n})_{N\in\mathbb{N},n\in\mathbb{N}},\\
\Sigma_2((T_{N,n})_{N\in\mathbb{N},n\le N}) \quad &:= \quad (T_{N+1,n}\circ I^N_{N+1} - T_{N,n})_{N\in\mathbb{N},n\le N}.
\end{aligned}$$

Here $I^N_{N+1} : X'_N \to X'_{N+1}$ are the natural embeddings. Clearly the following diagram commutes:

$$\begin{array}{ccc}
\prod_{N\in\mathbb{N}} L(X'_N, \prod_{n\le N} H_n) & \xrightarrow{\Sigma_2} & \prod_{N\in\mathbb{N}} L(X'_N, \prod_{n\le N} H_n)\\
{\scriptstyle A_1}\uparrow & & {\scriptstyle A_2}\uparrow\\
\prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n) & \xrightarrow{\Sigma_1} & \prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n),
\end{array}$$

where the vertical arrows are the natural projections. Let us observe that A_1 and A_2 are surjective, thus $A_2(\mathrm{im}\ \Sigma_1) = \mathrm{im}\ \Sigma_2$. Therefore A_2 induces a surjective map

$$\tilde{A}_2 : \left(\prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n)\right) / \,\mathrm{im}\ \Sigma_1 \to \left(\prod_{N\in\mathbb{N}} L(X'_N, \prod_{n\le N} H_n)\right) / \,\mathrm{im}\ \Sigma_2.$$

Hence $\mathrm{Proj}^1\ L(X'_N, \prod_{n\le N} H_n) = \left(\prod_{N\in\mathbb{N}} L(X'_N, \prod_{n\le N} H_n)\right) / \,\mathrm{im}\ \Sigma_2$ is a surjective image of $\left(\prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n)\right) / \,\mathrm{im}\ \Sigma_1$. Moreover, $\mathrm{im}\ \Sigma_1$ is a product of images of maps:

$$\prod_{N\in\mathbb{N}} L(X'_N, H_n) \to \prod_{N\in\mathbb{N}} L(X'_N, H_n), \quad (T_{N,n})_{N\in\mathbb{N}} \mapsto (T_{N+1,n}\circ I^N_{N+1} - T_{N,n})_{N\in\mathbb{N}},$$

thus

$$\left(\prod_{n\in\mathbb{N}}\prod_{N\in\mathbb{N}} L(X'_N, H_n)\right) / \,\mathrm{im}\ \Sigma_1 = \prod_{n\in\mathbb{N}} \mathrm{Proj}^1\ L(X'_N, H_n).$$

By Lemma 3.2, $\mathrm{Proj}^1\ L(X'_N, H_n) = 0$ and thus $\mathrm{Proj}^1\ L(X'_N, \prod_{n\le N} H_n) = 0$. Therefore, by Lemma 3.3, we have the following exact sequence where $\Sigma_0((T_n)_{n\in\mathbb{N}}) := (i_n^{n+1}T_{n+1} - T_n)_{n\in\mathbb{N}}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L(X', H) & \longrightarrow & \prod_{n\in\mathbb{N}} L(X', H_n) & \xrightarrow{\Sigma_0} & \prod_{n\in\mathbb{N}} L(X', H_n) & \longrightarrow & \\
& & & \longrightarrow & \mathrm{Proj}^1\ L(X'_N, H_N) & \longrightarrow & 0, & &
\end{array}$$

Thus

$$\mathrm{Proj}^1\ L(X', H_N) \simeq \prod_{n\in\mathbb{N}} L(X', X_n)/\,\mathrm{im}\ \Sigma_0 \simeq \mathrm{Proj}^1\ L(X'_N, H_N) \qquad \square$$

The proof of the next lemma follows from duality and [4, Lemma 4.5].

Lemma 3.5 *(a) Let E be an arbitrary LS-space, $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$. Suppose that $a, c \geq 0$, $b > 0$, $n \leq m \leq k$ and*

$$\forall\, x \in E_n \qquad a\|j_m^n x\|_m \leq b\|x\|_n + c\|j_k^n x\|_k \tag{7}$$

then

$$a(j_m^n)'(B_m^\circ) \subseteq 3bB_n^\circ + 2c(j_k^n)'(B_k^\circ).$$

(b) Let X be an arbitrary PLS-space, $X = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$, with a reduced spectrum. Suppose that $N \leq M \leq K$, $n \leq m \leq k$, $a, b, c \geq 0$ and

$$\forall\, y \in X_N' \qquad a\|y \circ i_N^M\|_{M,m}^* \leq b\|y\|_{N,n}^* + c\|y \circ i_N^K\|_{K,k}^* \tag{8}$$

then

$$ai_N^M(B_{M,m}) \subseteq 2bB_{N,n} + 2ci_N^K(B_{K,k}).$$

Proof of Theorem 3.1 (1)⇒(2): Let us observe that $L(X_N', E_N') = \operatorname{ind}_{n\in\mathbb{N}} L(X_{N,n}', E_N')$ algebraically thus it has a natural LB-space topology. Then, by Theorem 3.4 and [43, 3.2.18, 1. implies 3.] (the needed implication does not require LS-topology), we get

$$\begin{aligned} &\forall\, N,\nu \;\exists\, M \geq N, \mu \geq \nu \;\forall\, K \geq M, \kappa \geq \mu \;\exists\, n \;\forall\, m \geq n \;\exists\, k \geq m, S \\ &\qquad I_{N,\nu}^{M,\mu} W(U_{M,m}, U_\mu) \subseteq S(I_{N,\nu}^{K,\kappa} W(U_{K,k}, U_\kappa) + W(U_{N,n}, U_\nu)), \end{aligned} \tag{9}$$

where $I_{N,\nu}^{M,\mu} f := (j_\mu^\nu)' \circ f \circ (i_N^M)'$, $U_\mu = B_\mu^\circ$.

Fix $y \in X_N'$ and $x \in E_\nu$, $x \neq 0$. Since j_μ^ν is injective, $\|j_\mu^\nu x\|_\mu > 0$. There is $\varphi \in U_\mu$:

$$\varphi(j_\mu^\nu x) > (1/2)\|j_\mu^\nu x\|_\mu. \tag{10}$$

Take an arbitrary element $\xi \in B_{M,m} \subseteq X_M$ and define

$$\xi \otimes \varphi \in W(U_{M,m}, U_\mu) \subseteq L(X_M', E_\mu'), \qquad (\xi \otimes \varphi)(u) := \langle u, \xi\rangle \varphi \quad \text{for } u \in X_M'.$$

By (9),

$$I_{N,\nu}^{M,\mu}(\xi \otimes \varphi) = SI_{N,\nu}^{K,\kappa} P + SQ, \tag{11}$$

where $P \in W(U_{K,k}, U_\kappa), Q \in W(U_{N,n}, U_\nu)$. For y chosen before we have

$$\begin{aligned} I_{N,\nu}^{M,\mu}(\xi \otimes \varphi)(y) = \left[(j_\mu^\nu)' \circ (\xi \otimes \varphi) \circ (i_N^M)'\right](y) &= (j_\mu^\nu)'((\xi \otimes \varphi)(y \circ i_N^M)) = y(i_N^M \xi)(\varphi \circ j_\mu^\nu) \\ SI_{N,\nu}^{K,\kappa} P(y) = S\left[(j_\kappa^\nu)' \circ P \circ (i_N^K)'\right](y) &= SP(y \circ i_N^K) \circ j_\kappa^\nu. \end{aligned}$$

Evaluating both sides of (11) at fixed $y \in X_N'$ and applying it to fixed $x \in E_\nu$ we obtain

$$y(i_N^M \xi)\varphi(j_\mu^\nu x) = SP(y \circ i_N^K)(j_\kappa^\nu x) + SQ(y)(x).$$

Since $P \in W(U_{K,k}, U_\kappa)$ and $Q \in W(U_{N,n}, U_\nu)$, by (10), we have:

$$\begin{aligned} (1/2)\|j_\mu^\nu x\|_\mu |y(i_N^M \xi)| &\leq S(|P(y \circ i_N^K)(j_\kappa^\nu x)| + |Q(y)(x)|) \leq \\ &\leq S(\|P(y \circ i_N^K)\|_\kappa^* \|j_\kappa^\nu x\|_\kappa + \|Q(y)\|_\nu^* \|x\|_\nu) \leq \\ &\leq S(\|y \circ i_N^K\|_{K,k}^* \|j_\kappa^\nu x\|_\kappa + \|y\|_{N,n}^* \|x\|_\nu). \end{aligned}$$

Taking supremum over all $\xi \in B_{M,m}$ we get the conclusion for $2S$ instead of S.

(2)⇒(3): Since E is a reflexive LS-space and E' is quasinormable, we get from [24, Th. 7],

$$\forall\ \tilde{\nu}\ \exists\ \nu \geq \tilde{\nu}\ \forall\ \kappa, \rho > 0\ \exists\ D(\rho)\ \forall\ x \in E \quad \|x\|_\nu \leq \rho\|x\|_{\tilde{\nu}} + D(\rho)\|x\|_\kappa. \tag{12}$$

Moreover, since $\mathrm{Proj}^1\ X = 0$ and X is a PLS-space, we can apply [43, 3.2.18] to get

$$\begin{aligned} &\forall\ N\ \exists\ \tilde{M} \geq N\ \forall\ K\ \exists\ \tilde{n}\ \forall\ m \geq \tilde{n}, \gamma > 0\ \exists\ \tilde{k}, C\ \forall\ y \in X'_N \\ &\qquad\qquad \|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},m} \leq C\|y \circ i_N^K\|^*_{K,\tilde{k}} + \gamma\|y\|^*_{N,\tilde{n}}. \end{aligned} \tag{13}$$

Then, by (G) we get:
(14)

$$\begin{aligned} &\forall\ \tilde{M}, \nu\ \exists\ M \geq \tilde{M}, \mu \geq \nu\ \forall\ K \geq M, \kappa \geq \mu\ \exists\ n\ \forall\ m \geq n\ \exists\ k \geq m, S \\ &\quad \forall\ y \in X'_{\tilde{M}}\ \forall\ x \in E_\nu : \quad \|y \circ i_{\tilde{M}}^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu \leq S\left(\|y\|^*_{\tilde{M},n}\|x\|_\nu + \|y \circ i_{\tilde{M}}^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa\right). \end{aligned}$$

We choose quantifiers as follows. For every $\tilde{\nu}$ find $\nu \geq \tilde{\nu}$ according to (12). Then for arbitrary N find $\tilde{M} \geq N$ from (13), apply (14) and find $M \geq \tilde{M}$, $\mu \geq \nu$. Take arbitrary K, κ, then find n by (14) and $\tilde{n} \geq n$ by (13). Take arbitrary $m \geq \tilde{n}$ and find k, S according to (14). Then choose $\varepsilon > 0$ arbitrary and γ so small that $S\gamma \leq \varepsilon/2$. Using (13) find $\tilde{k} \geq k$ and C. Choose ρ so small that $SC\rho \leq \varepsilon/2$ and $S\rho \leq \varepsilon$. Now, we prove (G_ε). For a given $y \in X'_N$ we consider two cases: (1) $\|y \circ i_N^{\tilde{M}}\|_{\tilde{M},n} \leq \|y \circ i_N^K\|_{K,\tilde{k}}$; (2) otherwise.

Case (1). By (14) applied to $y \circ i_N^{\tilde{M}} \in X'_{\tilde{M}}$, $x \in E_\nu$, using (12) we get:

$$\begin{aligned} \|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu &\leq S\left(\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n}\|x\|_\nu + \|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa\right) \leq \\ &\leq S\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n}(\rho\|x\|_{\tilde{\nu}} + D(\rho)\|j_\kappa^\nu x\|_\kappa) + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa \leq \\ &\leq \varepsilon\|y\|^*_{N,n}\|x\|_{\tilde{\nu}} + S(1 + D(\rho))\|y \circ i_N^K\|^*_{K,\tilde{k}}\|j_\kappa^\nu x\|_\kappa. \end{aligned}$$

Case (2). Again, by (14), using first (13) and then (12), we obtain:

$$\begin{aligned} \|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu &\leq S(\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n}\|x\|_\nu + \|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa) \leq \\ &\leq S\gamma\|y\|^*_{N,\tilde{n}}\|x\|_\nu + SC\|y \circ i_N^K\|^*_{K,\tilde{k}}\|x\|_\nu + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa \leq \\ &\leq S\gamma\|y\|^*_{N,\tilde{n}}\|x\|_\nu + SC\rho\|y \circ i_N^K\|^*_{K,\tilde{k}}\|x\|_{\tilde{\nu}} + SCD(\rho)\|y \circ i_N^K\|^*_{K,\tilde{k}}\|j_\kappa^\nu x\|_\kappa + \\ &\quad + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa. \end{aligned}$$

Since $\nu \geq \tilde{\nu}$, $\tilde{M} \geq N$, $\tilde{n} \geq n$, we have $\|x\|_\nu \leq \|x\|_{\tilde{\nu}}$, $\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n} \leq \|y\|^*_{N,n} \leq \|y\|^*_{N,\tilde{n}}$ and

$$\|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu \leq \varepsilon\|y\|^*_{N,\tilde{n}}\|x\|_{\tilde{\nu}} + (SCD(\rho) + S)\|y \circ i_N^K\|_{K,\tilde{k}}\|j_\kappa^\nu x\|_\kappa.$$

(3)⇒(1): By Theorem 3.4, it suffices to show that $\mathrm{Proj}^1\ L(X'_N, E'_N) = 0$. By [43, 3.2.14], it suffices to show that

$$\begin{aligned} &\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m \geq n, \varepsilon > 0\ \exists\ k \geq m, S \\ &\qquad I_N^M W(U_{M,m}, U_M) \subseteq S(I_N^K W(U_{K,k}, U_K)) + \varepsilon W(U_{N,n}, U_N), \end{aligned} \tag{15}$$

where $I_N^M f := (j_N^M)' \circ f \circ (i_N^M)'$. We will show it separately for the assumptions (b), (c) and (d).

Case (b): $X = \Lambda^\infty(A)$ a Köthe type PLS-space. We assume first that $a_{1,n}(i) > 0$ for each n.

Let e_i be the unit vector in X', then $\|e_i\|^*_{N,n} = 1/a_{N,n}(i)$. Thus, by ($G_\varepsilon$), for $N=\nu$, $K=\kappa$ and M, μ chosen as the maximum of those two and denoted by M and for $x \in E_N$, $y = e_i$:

$$\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m \geq n, \varepsilon > 0\ \exists\ k \geq m, S\ \forall\ i \in \mathbb{N}\ \forall\ x \in E_N$$
$$\frac{\|j^N_M x\|_M}{a_{M,m}(i)} \leq \varepsilon \frac{\|x\|_N}{a_{N,n}(i)} + S\frac{\|j^N_K x\|_K}{a_{K,k}(i)}.$$

By Lemma 3.5,

$$\frac{1}{a_{M,m}(i)}(j^N_M)'(B^\circ_M) \subseteq \frac{3\varepsilon}{a_{N,n}(i)} B^\circ_N + \frac{2S}{a_{K,k}(i)}(j^N_K)'(B^\circ_K). \tag{16}$$

Now, we identify $W(U_{M,m}, U_M) \subseteq L(X'_M, E'_M)$ a space of vector valued sequences:

$$L(X'_M, E'_M) \;=\; \{u = (u(i))_{i\in\mathbb{N}} \subseteq E'_M :\ \exists\ m\ \sup_i a_{M,m}(i)\|u(i)\|^*_M < \infty\}.$$

In particular, $u = (u(i))_{i\in\mathbb{N}} \in W(U_{M,m}, U_M)$ if and only if $u(i) \in (a_{M,m}(i))^{-1}U_M$ for every i. By (16), taking some $v(i) \in (a_{N,n}(i))^{-1}U_N$ and $w(i) \in (a_{K,k}(i))^{-1}U_K$ we have

$$(j^N_M)'(u(i)) = 3\varepsilon v(i) + 2S(j^N_K)'(w(i)) \quad \text{for each } i \in \mathbb{N}.$$

Define $v \in W(U_{N,n}, U_N) \subseteq L(X'_N, E'_N)$ and $w \in W(U_{K,k}, U_K) \subseteq L(X'_K, E'_K)$, by

$$v(x) := (v(i)x)_{i\in\mathbb{N}}, \quad w(z) := (w(i)z)_{i\in\mathbb{N}} \qquad \text{for } x \in X'_N, z \in X'_K.$$

Obviously, $I^M_N u = 3\varepsilon v + 2SI^K_N w$ which implies (15) with slightly changed S and ε.

In the general case, $X = \Lambda^\infty(A)$ is a countable product of spaces for which we have proved $\mathrm{Ext}^1_{PLS}(E, X_S) = 0$. This implies (1).

Case (c): E is an LN-space, i.e., a nuclear LS-space.

We assume that E_ν is Hilbert and $j^\nu_{\nu+1} : E_\nu \to E_{\nu+1}$ is nuclear for every $\nu \in \mathbb{N}$. By Lemma 3.5 and (G_ε) applied for $\nu = N+2$, $\kappa = K+2 > \nu$ and $M = \mu$ we get:

$$\begin{gathered}\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m \geq n, \varepsilon > 0\ \exists\ k \geq m, S\ \forall\ x \in E_{N+2}\\ \|j^{N+2}_M x\|_M i^M_N B_{M,m} \subseteq \varepsilon\|x\|_{N+2} B_{N,n} + S\|j^{N+2}_{K+2}x\|_{K+2} i^K_N B_{K,k}.\end{gathered} \tag{17}$$

Let us choose orthonormal systems $(e_i)_{i\in\mathbb{N}} \subseteq E_{N+1}$ and $(f_i)_{i\in\mathbb{N}} \subseteq E_{K+1}$ such that

$$j^{N+1}_{K+1}x = \sum_i a_i \langle x, e_i\rangle_{N+1} f_i \qquad \forall\ x \in E_{N+1}.$$

Let us fix $\varphi \in W(U_{M,m}, U_M) \subseteq L(X'_M, E'_M)$. For arbitrary $u \in U_{M,m}$, $i \in \mathbb{N}$ we have

$$|e_i \circ (j^{N+1}_M)' \circ \varphi(u)| = |\varphi(u)(j^{N+1}_M(e_i))| \leq \|j^{N+1}_M(e_i)\|_M.$$

We have proved that $i^M_N(e_i \circ (j^{N+1}_M)' \circ \varphi) \in \|j^{N+2}_M(j^{N+1}_{N+2}e_i)\|_M i^M_N B_{M,m}$. By (17),

$$i^M_N(e_i \circ (j^{N+1}_m)' \circ \varphi) = \chi_i + i^K_n \psi_i, \tag{18}$$

where $\chi_i \in \varepsilon\|j^{N+1}_{N+2}e_i\|_{N+2}B_{N,n}$, $\psi_i \in S\|j^{N+2}_{K+2}j^{N+1}_{N+2}e_i\|_{K+2}B_{K,k} = S\|j^{N+1}_{K+2}e_i\|_{K+2}B_{K,k}$. We define two maps: first,

$$\chi(u) := \sum_i \chi_i(u)(j^N_{N+1})'(e^*_i)$$

for $u \in X'_N$ where $e_i^*(x) := \langle x, e_i\rangle_{N+1}$, $x \in E_{N+1}$, second,

$$\psi(v) := \sum_i a_i^{-1}\psi_i(v)(j^K_{K+1})'(f_i^*),$$

where the sum runs over all i such that $a_i \neq 0$, $v \in X'_K$ and $f_i^*(x) := \langle x, f_i\rangle_{K+1}$ for $x \in E'_{K+1}$.

We will show that χ is a well-defined element of a multiple of $W(U_{N,n}, U_N)$. Fix $x \in B_N$ and $u \in U_{N,n}$. Then, by Schwartz inequality,

$$\begin{aligned}|\chi(u)(x)| &\leq \sum_i |\chi_i(u)||\langle j^N_{N+1}x, e_i\rangle_{N+1}| \leq \varepsilon \sum_i \|j^{N+1}_{N+2}e_i\|_{N+2}|\langle j^N_{N+1}x, e_i\rangle_{N+1}| \\ &\leq \varepsilon\sigma(j^{N+1}_{N+2})\|j^N_{N+1}x\|_{N+1} \leq \varepsilon\sigma(j^{N+1}_{N+2}),\end{aligned}$$

where σ denotes the Hilbert-Schmidt norm of operators. The above estimates imply that the series in the definition of χ is convergent and

$$\chi \in \varepsilon\sigma(j^{N+1}_{N+2})W(U_{N,n}, U_N). \tag{19}$$

Fix $v \in U_{K,k}$ and $z \in B_K$. Similarly as above we get

$$\begin{aligned}|\psi(v)(z)| &\leq \sum_i (a_i)^{-1}|\psi_i(v)||\langle j^K_{K+1}z, f_i\rangle_{K+1}| \leq S\sum_i (a_i)^{-1}\|j^{N+1}_{K+2}e_i\|_{K+2}|\langle j^K_{K+1}z, f_i\rangle_{K+1}| \leq \\ &\leq S\sum_i \|j^{K+1}_{K+2}f_i\|_{K+2}|\langle j^K_{K+1}z, f_i\rangle_{K+1}| \leq S\sigma(j^{K+1}_{K+2}).\end{aligned}$$

This implies that

$$\psi \in S\sigma(j^{K+1}_{K+2})W(U_{K,k}, U_K). \tag{20}$$

By (19) and (20), in order to prove (15) it suffices to show that $I^M_N\varphi = \chi + I^K_N\psi$. This follows from an easy consequence of (18):

$$(I^M_N\varphi)(u)(x) = \chi(u)(x) + (I^K_N\psi)(u)(x) \quad \text{for every } u \in X'_N \text{ and } x \in E_N.$$

Case (d): $E = k_1(v)$ is a Köthe coechelon space, where $\|x\|_\nu := \sum_i v_\nu(i)|x_i|$ and $v_\nu(i) > 0$ for each $\nu, i \in \mathbb{N}$. Evaluating (G_ε) for $x = e_i \in E_N$, where $N = \nu$, $K = \kappa$ and $M = \mu$ we obtain:

$$\begin{gathered}\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m \geq n, \varepsilon > 0\ \exists\ k, S\ \forall\ y \in X'_N\ \forall\ i \in \mathbb{N}: \\ \|y \circ i^M_N\|^*_{M,m}v_M(i) \leq \varepsilon\|y\|^*_{N,n}v_N(i) + S\|y \circ i^K_N\|^*_{K,k}v_K(i).\end{gathered}$$

By Lemma 3.5 changing ε and S suitably we get:

$$v_M(i)i^M_N B_{M,m} \subseteq \varepsilon v_N(i)B_{N,n} + Sv_K(i)i^K_N B_{K,k}. \tag{21}$$

Let $f \in W(U_{M,m}, U_M)$. Observe $E'_M = l_\infty(1/v_M)$, then $f(z) = (f_i(z))_{i\in\mathbb{N}} \in U_M$ for every $z \in U_{M,m}$ and $|f_i(z)| \leq v_M(i)$. Thus $f_i \in v_M(i)B_{M,m}$ for every $i \in \mathbb{N}$. By (21), we get for $i \in \mathbb{N}$: $i^M_N f_i = \varepsilon v_N(i)g_i + Sv_K(i)i^K_N h_i$, where $g_i \in B_{N,n}$ and $h_i \in B_{K,k}$. Clearly $I^M_N f = \varepsilon g + SI^K_N h$ for

$$\begin{aligned}g &: X'_N \to E'_N = l_\infty(1/v_N), \qquad g(y) := (v_N(i)g_i(y))_{i\in\mathbb{N}}, \\ h &: X'_K \to E'_K = l_\infty(1/v_K), \qquad h(y) := (v_K(i)h_i(y))_{i\in\mathbb{N}}.\end{aligned}$$

Finally, $g \in W(U_{N,n}, U_N)$, $h \in W(U_{K,k}, U_K)$ which completes the proof by (15). □

4 Splitting results for special spaces

In the present section we obtain a more natural splitting result and apply it to sequence spaces. Let us define the condition (P$\underline{\text{A}}$) for a PLS-space X as follows:

$$\tag{22} \begin{aligned} &\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \exists\ \theta \in]0,1[\ \exists\ k, C\ \forall\ y \in X'_N;\\ &\qquad \|y \circ i_N^M\|^*_{M,m} \leq C \max\left(\|y \circ i_N^K\|^*_{K,k}{}^{(1-\theta)}, \|y\|^*_{N,n}{}^{(1-\theta)}\right) \|y\|^*_{N,n}{}^{\theta}\end{aligned}$$

or, equivalently (see the proof of [4, Lemma 5.1]),

$$\tag{23} \begin{aligned} &\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \exists\ \eta > 0\ \exists\ k, C, r_0 > 0\ \forall\ r < r_0\ \forall\ y \in X'_N;\\ &\qquad \|y \circ i_N^M\|^*_{M,m} \leq C\left(r^\eta \|y \circ i_N^K\|^*_{K,k} + \frac{1}{r}\|y\|^*_{N,n}\right).\end{aligned}$$

Changing the quantifier for θ, η to be $\forall \theta \in]0,1[$ (resp. $\forall \eta > 0$) one gets the condition (PA)

These conditions are PLS-versions of conditions $(\underline{A})$ and (A) (see [34]) which are dual to $(\underline{DN})$ and (DN) respectively [25, Sec. 29]. It is worth noting that $(P\underline{A})$ and $(P\Omega)$ differs only by inequality $r < r_0$ and $r > r_0$, respectively. The same analogy holds between (PA) and $(P\overline{\overline{\Omega}})$

We present now an analogue of the famous $(DN) - (\Omega)$ splitting theorem [25, 30.1].

Theorem 4.1 *Let E be an LS-space, X a PLS-space satisfying (b), (c) or (d), then* $\mathrm{Ext}^1_{PLS}(E, X) = 0$ *whenever E' has $(\overline{\overline{\Omega}})$ and X has $(P\underline{A})$ or E' has (Ω) and X has (PA) .*

Proof: By Theorem 3.1, it suffices to show that $(E, X) \in$(G). Recall that $E' \in (\overline{\overline{\Omega}})$ means

$$\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M, \theta \in]0,1[\ \exists\ D\ \forall\ x \in E \quad \|x\|_M \leq D\|x\|_N^\theta \|x\|_K^{1-\theta}.$$

Fix N and find M which is good for $(P\underline{A})$ and $(\overline{\overline{\Omega}})$. Then fix K, find n from $(P\underline{A})$ and fix m. Finally, fix k and η from $(P\underline{A})$. Take $x \in E_N$ and $r := \frac{\|x\|_M}{\|x\|_N}$. By $(\overline{\overline{\Omega}})$, $\theta := \frac{\eta}{\eta+1}$,

$$r^\eta = \left(\frac{\|x\|_M}{\|x\|_N}\right)^\eta \leq D\frac{\|x\|_K}{\|x\|_M}.$$

We substitute r into (23) to get

$$\|y \circ i_N^M\|^*_{M,m} \leq C\left(D\frac{\|x\|_K}{\|x\|_M}\|y \circ i_N^K\|^*_{K,k} + \frac{\|x\|_N}{\|x\|_M}\|y\|^*_{N,n}\right).$$

This completes the proof. The case $E' \in (\Omega)$ and $X \in (PA)$ is analogous. □

The following proposition summarize elementary facts concerning $(P\underline{A})$ and (PA) .

Proposition 4.2 *Every Fréchet Schwartz space has (PA) and (P$\underline{A}$). An LS-space has (PA) or (P$\underline{A}$) if and only if it has (A) or ($\underline{A}$) respectively. The conditions (PA) and (P$\underline{A}$) are inherited by complete quotients and countable products and (PA)⇒(P$\underline{A}$)⇒* $\mathrm{Proj}^1\ X = 0$.

The proof is so similar to the proof of [4, Cor. 5.2, Prop. 5.3 and Prop. 5.4] that we omit it. Observe that duals of power series spaces have always $(\underline{A})$ and they have (A) only for infinite type spaces [25, Sec. 29]. Thus products of such spaces have correspondingly $(P\underline{A})$ and (PA) .

Theorem 4.3 *(a) The Köthe type PLS-space $\Lambda^p(A)$ for $1 \leq p \leq \infty$ has (PA) if and only if*

$$\forall\ N\ \exists\ M\ \ \forall\ K\ \ \exists\ n\ \ \forall\ m, \theta \in]0,1[\ \ \exists\ k, C\ \ \forall\ i \in \mathbb{N}$$
$$a_{M,m}(i) \geq C \min\left(a_{K,k}(i)^{(1-\theta)}, a_{N,n}(i)^{(1-\theta)}\right) a_{N,n}(i)^{\theta}$$

The same condition holds for $(P\underline{A})$ with a suitable change of quantifiers.

(b) The PLS-type power series space $\Lambda_{r,s}(\alpha,\beta) \in (P\underline{A})$ iff either $s = \infty$ or it is isomorphic to a product of an LS-space and a Fréchet space (equivalently, $\mathrm{Proj}^1\ \Lambda_{r,s}(\alpha,\beta) = 0$*).*

(c) The PLS-type power series space $\Lambda_{r,s}(\alpha,\beta)$ satisfies condition (PA) iff either $s = \infty$ or the space is isomorphic to a Fréchet space.

It is worth noting that spaces of Beurling (ultra-)distributions $\mathscr{D}'_{(\omega)} \in (PA)$ are isomorphic to Köthe type PLS-spaces [37], see [7] for the definitions. The role of these new invariants and applications of our splitting result for spaces of real analytic functions and Roumieu (quasianalytic) classes of ultradifferentiable functions will be explained in [5]. The kernels of surjective convolution operators on $\mathscr{D}'_{(\omega)}(\mathbb{R})$, $\mathscr{E}_{\{\omega\}}(\mathbb{R})$ or $\mathscr{E}_{\{\omega\}}(]-1,1[)$ give examples of PLS-type power series spaces (see [12, Th. 2.10], [27, 2.11], [26, Satz 3.2, 3.18], [23], comp. [4, Th. 2.2]), in the first case they have (PA) in the other two $(P\underline{A})$.

Proof of 4.3: (a): Necessity follows by taking y as unit vectors. For the proof of sufficiency, translate the condition as in the definition of (PA) into the condition with the parameter r:

$$\forall\ N\ \exists\ M\ \ \forall\ K\ \ \exists\ n\ \ \forall\ m, \eta > 0\ \ \exists\ k, C, r_0 > 0\ \ \forall\ r < r_0\ \ \forall\ i \in \mathbb{N} : \text{such that } a_{N,l}(i) \neq 0 \text{ for all } l$$
$$\frac{1}{a_{M,m}(i)} \leq C \max\left(r^{\eta}\frac{1}{a_{K,k}(i)}, \frac{1}{r}\frac{1}{a_{N,n}(i)}\right).$$

Then prove that this condition holds for all vectors in X'_N instead of the unit vectors only.

(b): By Proposition 4.2, $(P\underline{A})$ implies $\mathrm{Proj}^1 = 0$, apply [39, 4.3]. Sufficiency for $s < \infty$ follows from Prop. 4.2, since the LS-space factor must be a dual to a Fréchet power series space and it has $(\underline{A})$ (see [25, Sec. 29]). Sufficiency for $s = \infty$ follows from (c) below.

(c): Since LS-factor is $\Lambda'_0(\gamma)$, necessity for $s < \infty$ follows from (b)(see [25, Sec.29]). Sufficiency for $s < \infty$ follows from Prop. 4.2.

Assume that $s = \infty$. Let us take arbitrary N, choose $M := N + 1$ and take arbitrary K. Fix $n = 1$, take arbitrary m and $\theta \in]0,1[$. We choose k so big that

$$\theta \leq \frac{s_k - s_m}{s_k - s_n} \qquad \text{and} \qquad \frac{r_K - r_N}{r_M - r_N} < \frac{s_k - s_n}{s_m - s_n}.$$

Let us observe that if $\frac{r_K - r_M}{r_K - r_N} \leq \theta$ then

$$\exp(-r_M\alpha_i + s_m\beta_i) \leq \exp((-r_K\alpha_i + s_k\beta_i)(1-\theta)) \cdot \exp((-r_N\alpha_i + s_n\beta_i)\theta) \tag{24}$$

and

$$\|e_i\|^*_{M,m} \leq (\|e_i\|^*_{N,n})^{\theta}(\|e_i\|^*_{K,k})^{1-\theta}.$$

Now, assume that $\|e_i\|^*_{M,m} \geq \|e_i\|^*_{N,n}$ then

$$-r_M\alpha_i + s_m\beta_i \geq -r_N\alpha_i + s_n\beta_i \quad \text{and} \quad \alpha_i \leq \frac{s_m - s_n}{r_M - r_N}\beta_i < \left(\frac{s_k - s_n}{r_K - r_N}\right)\beta_i.$$

Observe that $f(\theta) := -r_K\alpha_i(1-\theta) + s_k\beta_i(1-\theta) - r_N\alpha_i\theta + s_n\beta_i\theta$ has negative derivative

$$f'(\theta) = (r_K - r_N)\alpha_i + (s_n - s_k)\beta_i < \left(\frac{s_k - s_n}{r_K - r_N}\right)(r_K - r_N)\beta_i + (s_n - s_k)\beta_i = 0.$$

Therefore, if the inequality (24) holds for big $\theta < 1$ then it holds for all $\theta \in]0,1[$ and either

$$\|e_i\|^*_{M,m} \le \|e_i\|^*_{N,n} \quad \text{or} \quad \|e_i\|^*_{M,m} \le (\|e_i\|^*_{N,n})^\theta (\|e_i\|^*_{K,k})^{1-\theta}.$$

We conclude by the same method as in (a). □

Theorem 4.4 *If α is stable, X is an ultrabornological PLS-space, then* $\mathrm{Ext}^1_{PLS}((\Lambda^\infty_r(\alpha))', X) = 0$ *if and only if X has (PA) .*

Remark. Clearly the same holds for $\prod_{n\in\mathbb{N}} \Lambda_r(\alpha^{(n)})$, for instance, $C^\infty(U) \simeq \prod_{n\in\mathbb{N}} \Lambda_\infty(\log j)$ for any smooth non-compact manifold U.

Proof: Sufficiency follows from Theorem 4.1 since $\Lambda_r(\alpha)$ has (Ω) .

Necessity. We may assume that $\alpha_0 = 0$ and that $\alpha_j \le d\alpha_{j-1}$ for some $d > 1$ and every $j \in \mathbb{N}$. We apply (G) for $x = e_j$. We fix N and find $M \ge N$ from (G), then we fix K. We choose η_0 such that $\frac{r_K - r_M}{r_M - r_N} \ge \eta_0 d$. There is n such that for every m there is $k(m)$ such that

$$\|y \circ i^M_N\|^*_{M,m} \le S\left(\exp\left((r_M - r_K)\alpha_j\right)\|y \circ i^K_N\|^*_{K,k(m)} + \exp\left((r_M - r_N)\alpha_j\right)\|y\|^*_{N,n}\right).$$

Let us take $r \le \exp\left((r_N - r_M)\alpha_0\right) = 1$. There is j such that

$$(r_N - r_M)\alpha_j \le \log r \le (r_N - r_M)\alpha_{j-1}.$$

Now, $\exp\left((r_M - r_N)\alpha_j\right) \le \exp\left(d(r_M - r_N)\alpha_{j-1}\right) \le \frac{1}{r^d}$. Clearly, for $\eta < \eta_0$ we have

$$\exp\left((r_M - r_K)\alpha_j\right) \le \exp\left((r_N - r_M)\frac{r_M - r_K}{r_N - r_M}\alpha_j\right) \le r^{\eta d}.$$

We have proved that

$$\forall\ N\ \exists\ M \ge N\ \forall\ K \ge M\ \exists\ n\ \exists\ \eta_0\ \forall\ m\ \exists\ k(m), S\ \forall\ \eta < \eta_0\ \forall\ r \in]0,1[:$$
$$\|y \circ i^M_N\|^*_{M,m} \le S_m\left(r^\eta \|y \circ i^K_N\|^*_{K,k(m)} + \frac{1}{r}\|y\|^*_{N,n}\right).$$

Then

$$\|y \circ i^K_N\|^*_{K,k(m)} \le \|y \circ i^M_N\|^*_{M,k(m)} \le S_{k(m)}\left(r^\eta \|y \circ i^N_K\|^*_{K,k(k(m))} + \frac{1}{r}\|y\|^*_{N,n}\right).$$

Combining the two inequalities above we get

$$\|y \circ i^M_N\|^*_{M,m} \le S_m(S_{k(m)} + 1)\left(r^{2\eta}\|y \circ i^K_N\|^*_{K,k(k(m))} + \frac{1}{r}\|y\|^*_{N,n}\right),$$

since $r^{\eta-1} < 1/r$. Repeating this procedure inductively we get

$$\|y \circ i^M_N\|^*_{M,m} \le S_{p,m}\left(r^{p\eta}\|y \circ i^K_N\|^*_{K,\tilde{k}(p)} + \frac{1}{r}\|y\|^*_{N,n}\right),$$

where $\tilde{k}(p) = k \circ k \circ \cdots \circ k(m)$, p-times composition, $p \in \mathbb{N}$. This completes the proof. □

Kunkle [20, Th.5. 14] proved $\mathrm{Ext}^1_{PLS}(\Lambda^1_{\infty,s}(\alpha,\beta), \Lambda^\infty_{p,\infty}(\gamma,\delta)) = 0$ for any p and s. We get:

Corollary 4.5 *If either $s = \infty$ or $\Lambda_{r,s}(\beta,\gamma)$ is a Fréchet space then*

$$\mathrm{Ext}^1_{PLS}((\Lambda^\infty_r(\alpha))', \Lambda_{r,s}(\beta,\gamma)) = 0.$$

Proof: The case $s = \infty$ follows from Theorem 4.3 (d), Theorem 4.1 and the propert (Ω) of $\Lambda_r(\alpha)$. The other case follows from [29, Th. 9.1]. □

5 Parameter dependence of solutions of differential equations

As explained in the introduction, the parameter dependence problem for linear PDO with constant coefficients is equivalent to the question if the partial differential operator

$$P(D) : \mathscr{D}'(\Omega, F) \to \mathscr{D}'(\Omega, F) \tag{25}$$

is surjective for suitably chosen Fréchet spaces F. We prove that this is the case for Ω convex and any nuclear Fréchet space F with property (Ω) (for instance, $F \simeq H(U), C^\infty(U), \Lambda_r(\alpha), C^\infty[0,1]$, etc., see [25, 29.11]). Our approach should be compared with [4, Section 3].

The positive solution for the holomorphic dependence was probably known to some specialists - Palamodov showed the authors the full proof without using splitting of short exact sequences. For the sake of completeness we give a full proof based on Palamodov's theory of systems of linear PDE and $(DN) - (\Omega)$ splitting result of Vogt and Wagner (see [25, 30.1]).

Theorem 5.1 *Let $\Omega \subseteq \mathbb{R}^d$ and U be any Stein manifold. For every linear partial differential operator with constant coefficients $P(D)$ the following map is surjective*

$$P(D) : \mathscr{D}'(\Omega, H(U)) \to \mathscr{D}'(\Omega, H(U)).$$

Proof: First, assume that $U \subseteq \mathbb{C}^d$ is a convex. To simplify notation we take $d = 1$.
We have the following differential complex obtained as in [28]:

$$\begin{array}{l} 0 \longrightarrow \ker \begin{pmatrix} \overline{\partial} \\ P(D) \end{pmatrix} \longrightarrow \mathscr{D}'(\Omega \times U) \xrightarrow{\begin{pmatrix} -\overline{\partial} \\ P(D) \end{pmatrix}} [\mathscr{D}'(\Omega \times U)]^2 \longrightarrow \\ \xrightarrow{(P(D)\,,\,\overline{\partial})} \mathscr{D}'(\Omega \times U) \longrightarrow 0 \end{array},$$

where $P(D)$ acts on Ω and $\overline{\partial}$ acts on $U \subseteq \mathbb{C} = \mathbb{R}^2$. This is a particular case of [28, VII, 7.2, Ex. 4]. Since $\Omega \times U$ is convex the complex is exact by [28, VII, 8.1, Th. 1].

If $f \in \mathscr{D}'(\Omega \times U)$, $\overline{\partial} f = 0$, then $\begin{pmatrix} 0 \\ f \end{pmatrix} \in \ker(P(D)\,,\,\overline{\partial}) \subseteq [\mathscr{D}'(\Omega \times U)]^2$, thus by exactness of the complex, there is $g \in \mathscr{D}'(\Omega \times U)$ such that $-\overline{\partial} g = 0$, $P(D)g = f$. We have proved that $P(D) : \ker \overline{\partial} \to \ker \overline{\partial}$ is surjective. By the very definition $\mathscr{D}(\Omega, H(U)) = L(\mathscr{D}(\Omega), H(U))$. Let us prove that

$$\{f \in \mathscr{D}'(\Omega \times U) : \overline{\partial} f = 0\} = L(\mathscr{D}(\Omega), H(U)).$$

Define a map $S_f : \mathscr{D}(\Omega) \to \mathscr{D}'(U)$, $\langle S_f(\varphi), \psi\rangle = \langle f, \varphi\psi\rangle$ for $\psi \in \mathscr{D}(U)$. Since $\langle \overline{\partial} S_f(\varphi), \psi\rangle = -\langle f, \varphi\overline{\partial}\psi\rangle = -\langle f, \overline{\partial}\psi\varphi\rangle = 0$, we have $S_f(\mathscr{D}(\Omega)) \subseteq H(U)$. On the other hand, for $\varphi \in \mathscr{D}(\Omega)$, $\psi \in \mathscr{D}(U)$, if $S : \mathscr{D}(\Omega) \to H(U)$ then we define $f_S \in \mathscr{D}'(\Omega \times U)$, $\langle f_S, \varphi\psi\rangle := \langle S(\varphi), \psi\rangle$. Clearly

$$\langle \overline{\partial} f_S, \varphi\psi\rangle = -\langle f_S, \varphi\overline{\partial}\psi\rangle = -\langle S(\varphi), \overline{\partial}\psi\rangle = \langle \overline{\partial} S(\varphi), \psi\rangle = 0, \quad \text{and } \overline{\partial} f_S = 0.$$

Let U be an arbitrary Stein manifold. By [17, 5.3.9], U embeds properly into $\mathbb{C}^d$ for suitable d as a submanifold. Clearly, we have the following short exact sequence of Fréchet spaces:

$$0 \longrightarrow I(U) \longrightarrow H(\mathbb{C}^d) \xrightarrow{q} H(U) \longrightarrow 0,$$

where $I(U) = \{f \in H(\mathbb{C}^d) : f|_U \equiv 0\}$. By [38] remark on page 195, $I(U)$ has (Ω) . Let $f \in H(U, \mathscr{D}'(\Omega))$ then it can be extended to $g \in H(\mathbb{C}^d, \mathscr{D}'(\Omega))$. Indeed,

$$H(U, \mathscr{D}'(\Omega)) \simeq H(U)\varepsilon\mathscr{D}'(\Omega) \simeq L(\mathscr{D}(\Omega), H(U))$$

and extendability is equivalent to the fact that every operator $T : \mathscr{D}(\Omega) \to H(U)$ lifts with respect to q. Since $\mathscr{D}(\Omega) \simeq \bigoplus_{N\in\mathbb{N}} s$ and s has (DN) the lifting follows from the $(DN) - (\Omega)$ splitting theorem [25, 30.1]. We get the conclusion combining extendability with surjectivity of

$$P(D) : \mathscr{D}'(\Omega, H(\mathbb{C}^d)) \to \mathscr{D}'(\Omega, H(\mathbb{C}^d)) \simeq H(\mathbb{C}^d, \mathscr{D}'(\Omega)). \quad \square$$

For the smooth dependence we cannot use the idea from the first part of the proof above but we can use the splitting theory as the following observation shows:

Proposition 5.2 *Let F be a Fréchet-Schwartz space, let $Y = \prod_{t\in\mathbb{N}} Y_t$ be a product of LS-spaces and let $T : Y \to Y$ be a surjective operator.*

(a) If $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$, then the map $T \otimes \mathrm{id} : Y\varepsilon F \to Y\varepsilon F$ is surjective.

(b) If $\mathrm{Ext}^1_{PLS}(F', Y_t) = 0$ for every t and $T \otimes \mathrm{id} : Y\varepsilon F \to Y\varepsilon F$ is surjective, then $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$.

(c) If either $Y_t \simeq \Lambda'_\infty(\beta_t)$ and F has (Ω) or $Y_t \simeq \Lambda'_0(\beta_t)$ and F has $(\overline{\overline{\Omega}})$, then

$$T \otimes \mathrm{id} : Y\varepsilon F \to Y\varepsilon F$$

is surjective if and only if $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$.

Proof: Use [4, Prop. 3.3, 3.4] and the fact that if F has (Ω) or F has $(\overline{\overline{\Omega}})$ then $\mathrm{Ext}^1_{PLS}(F', \Lambda'_\infty(\beta_t)) = 0$ or $\mathrm{Ext}^1_{PLS}(F', \Lambda'_0(\beta_t)) = 0$ respectively (see [38]). $\square$

Since for non-quasianalytic ω, $\mathscr{D}'_{(\omega)}(\Omega) \simeq [\Lambda'_\infty(\beta)]^{\mathbb{N}}$ and [37], we have:

Corollary 5.3 *Let F be Fréchet Schwartz with (Ω) and let $T : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ be surjective, then $T \otimes \mathrm{id} : \mathscr{D}'_{(\omega)}(\Omega, F) \to \mathscr{D}'_{(\omega)}(\Omega, F)$ is surjective if and only if $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$.*

The following result is crucial for the application of our splitting results from Sec. 3 and 4.

Proposition 5.4 *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open set, $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ a linear partial differential operator with constant coefficients. Then* $\ker P(D)$ *has the property (PA) .*

Proof: By Theorem 5.1, $P(D) : \mathscr{D}'(\Omega, H(\mathbb{D})) \to \mathscr{D}'(\Omega, H(\mathbb{D}))$ is surjective. By Cor. 5.3,

$$\mathrm{Ext}^1_{PLS}(H'(\mathbb{D}), \ker P(D)) = 0.$$

This completes the proof by Theorem 4.4. Observe that $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective and thus $\mathrm{Proj}^1 \ker P(D) = 0$ while $H(\mathbb{D}) \simeq \Lambda_0(\alpha)$ has (Ω) . $\square$

Theorem 5.5 *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open set, $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ a linear partial differential operator with constant coefficients, then for every Fréchet nuclear space F or Köthe sequence Fréchet-Schwartz space $F = \lambda_\infty(A)$ the map $P(D) : \mathscr{D}'(\Omega, F) \to \mathscr{D}'(\Omega, F)$ is surjective whenever $F \in (\Omega)$. In particular, it holds for $F = C^\infty(U)$, U an arbitrary smooth manifold.*

Proof: Apply Corollary 5.3, Proposition 5.4 and Theorem 4.1. $\square$

The property (Ω) is not a necessary condition in Theorem 5.5. This follows from the example in [36, p. 190] and the following result, which is a consequence of [2, Th. 36]. Recall that the condition LB_∞ is very restrictive see [36].

Proposition 5.6 *Let $F = \prod_{N\in\mathbb{N}} F_N$, F_N Fréchet spaces with property LB_∞ and $T : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ is surjective then the following map is surjective as well*

$$T \otimes \mathrm{id} : \mathscr{D}'_{(\omega)}(\Omega, F) \to \mathscr{D}'_{(\omega)}(\Omega, F).$$

Theorem 5.7 *If the convolution operator $T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}) \to \mathscr{D}'_{(\omega)}(\mathbb{R})$ is surjective, then*

$$T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}, F) \to \mathscr{D}'_{(\omega)}(\mathbb{R}, F)$$

is surjective for any Fréchet nuclear space F with property (Ω) or any Köthe sequence Fréchet-Schwartz space $F = \lambda_\infty(A)$ with property (Ω) .

Proof: By [12, Th. 2.10], ker $T_\mu \simeq \Lambda_{\infty,\infty}(\alpha, \beta)$. By Theorem 4.3, ker T_μ has (PA) . The result follows from Theorem 4.1 and Corollary 5.3. □

Similar results hold for $T_\mu : \mathscr{E}_{\{\omega\}}(\mathbb{R}) \to \mathscr{E}_{\{\omega\}}(\mathbb{R})$ or $T_\mu : \mathscr{E}_{\{\omega\}}(]-1,1[) \to \mathscr{E}_{\{\omega\}}(]-1,1[)$ and F with property $(\overline{\overline{\Omega}})$ (use [27, 2.11], [26, Satz 3.2, 3.18], [23] instead of [12]).

It is worth noting that for hypoelliptic operators one can drop the assumption of condition (Ω) in Theorem 5.5. Indeed, ker $P(D)$ is Fréchet, by [29, Th. 9.1], $\mathrm{Ext}^1_{PLS}(F', \ker P(D)) = 0$.

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