EXTENSION OF VECTOR VALUED HOLOMORPHIC AND HARMONIC FUNCTIONS

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Abstract. We present a unified approach to study extensions of vector valued holomorphic or harmonic functions from the existence of weak or weak*-holomorphic or harmonic extensions. Several recent results due to Arendt, Nikolski, Bierstedt, Holtmanns and Grosse-Erdmann are extended. An open problem by Grosse-Erdmann is solved in the negative. Using the extension results we prove existence of Wolff type representations for the duals of certain function spaces.
2000 Mathematics Subject Classification: 46E40, 46A04
Key Words: vector valued holomorphic functions, topological sheaves, Fréchet Schwartz spaces, (DFS)-spaces, locally complete spaces

1. Introduction

The purpose of this paper is to present a unified treatment of the extension of holomorphic or harmonic vector valued functions, including the several variables case. Vector-valued holomorphic functions are useful in the theory of topological algebras [18], in the theory of one-parameter semigroups [3, 17], in infinite dimensional holomorphy [14, Chapter 3], and also in operator theory [1, 19]. Composition operators of spaces of this type have been investigated recently [9, 10, 31, 30]. The topic we consider is closely related to the investigation of conditions to ensure that a weakly holomorphic function with values in a locally convex space is holomorphic. In fact, it is much easier to show that a function is weakly holomorphic and conclude that the original function is holomorphic as a consequence of an abstract theorem. The classical theorem of Dunford and Grothendieck shows that a function f defined on an open set $\Omega \subseteq \mathbb{C}$ in the complex plane with values in a complete locally convex space E is holomorphic if $u \circ f$ is holomorphic for every $u \in E'$ in the topological dual of E. Several authors presented extensions of this result and related it to the extension of holomorphic functions; see Bogdanowicz [7], Colombeau [12] and Gramsch [20, 21]. Weak conditions for holomorphy of a vector valued function have found renewed interest recently. Grosse-Erdmann [22] showed that it is enough to test weak holomorphy of a locally bounded function with values in a locally complete locally convex space on the elements of a separating subset of the dual of the range space, solving a problem posed by Wrobel [41]. Arendt and Nikolski [2] gave a short proof of this result if the range space is Fréchet; and Grosse-Erdmann [23] shortened his original proof with a more functional analytic approach. He also treated holomorphic extension, and stated several open problems which we

The research of the first and the third authors is supported by FEDER and MCYT, Proyecto no. MTM2004-02262, AVCIT Grupo 03/050 and the research net MTM2004-21420-E.
The research began during a stay of the second author at the Universidad Politécnica de Valencia and it has been finished during the stay of the third author at the Bergische Universität Wuppertal. We would like to thank both Universities for the hospitality and support.

treat in our article.

The basic problem we consider can be stated as follows: Let Ω be an open subset of $\mathbb{R}^N$ (or a smooth manifold), let $\mathscr{F}$ be a sheaf of smooth functions on Ω, and let $f : M \to E$ be a function acting from a subset M of Ω into a locally convex space E such that $u \circ f$ has a unique extension $f_u \in \mathscr{F}(\Omega)$ for each u in a separating subset of E', does f have an extension $\hat{f}$ belonging to the space $\mathscr{F}(\Omega, E)$ of vector valued $\mathscr{F}(\Omega)$-functions? We present theorems which simultaneously extend results due to Gramsch [21], Arendt, Nikolski [2] and Grosse-Erdmann [23]. Our approach using sheaves of smooth functions permits us to treat, not only spaces defined on open subsets of the complex plane as in [2] and [23], but also holomorphic or harmonic functions of several variables and kernels of linear partial differential operators, thus including consequences about the work of Bierstedt, Holtmanns [6] and Enflo, Smithies [16]. Besides the positive results, we solve a problem of Grosse-Erdmann in the negative; see the Example 20. Finally, we give representations of $\mathscr{F}'(\Omega)$ and of $\mathscr{F}'(\Omega, E)$ in the spirit of Wolff description of the dual of the space one variable holomorphic functions on a domain [40], used by Grosse-Erdmann to obtain the extension result [23, Theorem 2].

Our proofs are functional analytic. They are based on properties of Fréchet Schwartz spaces, the local completion of a locally convex space [35, Chapter 5], a theorem of Raikov about (DFS)-spaces, see e.g. [35, 8.5.28], and the theory of ε-products of Schwartz [38]. In fact all the spaces of holomorphic or harmonic functions we are interested in are Fréchet Schwartz spaces and their duals are (DFS)-spaces. These powerful abstract techniques have not been exploited before in connection with the present research. They permit us to derive many results with relatively smooth proofs.

2. Preliminaries and notation

2.1. **ε-products and locally complete spaces.** Our notation for locally convex spaces and functional analysis is standard. We refer the reader to [26, 28, 32, 35], and we recall some terminology. For a locally convex space E, which we assume to be Hausdorff, $E^\star$ and E' stand for its algebraic dual and topological dual, respectively. We denote by $\beta(E, F)$ the strong topology and by $\sigma(E, F)$ the weak topologies on E with respect to a dual pair $\langle E, F\rangle$. As usual, if E' is the topological dual of a locally convex space E, the topology $\sigma(E', E)$ is called the weak* topology. We denote by $co = co(E', E)$ the topology of uniform convergence on the compact and absolutely convex subsets of the locally convex space E. The polar in E of a subset A of F in the dual pair $\langle E, F\rangle$ is $A^\circ := \{x \in E : |f(x)| \leq 1 \text{ for all } f \in A\}$. A subspace G of E' is called *separating* if $u(x) = 0$ for each $u \in G$ implies $x = 0$. Clearly this is equivalent to G being weak*-dense (or dense in the co-topology). If E, F are locally convex spaces, then $L(E, F)$ denotes the vector space of all continuous linear maps from E to F. Given $T \in L(E, F)$ we denote by $T^t \in L(F', E')$ its transpose defined by $T^t(u) = u \circ T \in E'$ for each $u \in F'$. $E\varepsilon F := L_e(E'_{co}, F)$ is called the Schwartz's ε-product of E and F [28, 38]; here e denotes the topology of uniform convergence on the equicontinuous subsets of E'. The map $T \mapsto T^t$ is an isomorphism between $E\varepsilon F$ and $F\varepsilon E$. We refer the reader for more information, especially for the representation of spaces of vector valued functions, to [4, 28, 37, 38]. In case Y is a Fréchet

Schwartz (or (FS)) space, i.e. a Fréchet space which has a defining spectrum of Banach spaces with compact linking maps, $Y\varepsilon E = L_\beta(Y'_\beta, E)$, since Y is in particular a Montel space, i.e. a locally convex space which is barrelled and such that all its bounded sets are relatively compact. Montel spaces are reflexive and Fréchet Schwartz spaces have even a fundamental system of reflexive Banach spaces; c.f. [28, 32]. A (DFS)-space is the strong dual of a Fréchet Schwartz space and can be represented as a countable inductive limit of a sequence of (reflexive) Banach spaces with compact linking maps.

A locally convex space E is said to be *locally complete* whenever every absolutely convex, closed, bounded subset B of E spans a Banach space E_B endowed with the Minkowski gauge of B. A linear subspace F of E is said to be *locally closed* if for every continuously embedded normed space $(X, \|\cdot\|)$ and every sequence $(x_n)_n \subseteq F \cap X$ which converges to some x in the normed space X, we have $x \in F$. The local closure of a linear subspace $F \subseteq E$ is defined as the smallest locally closed subspace of E which contains F. For a locally convex space E, if $\widehat{E}$ denotes the completion of E, the local completion E^{lc} of E is defined as the local closure of E in $\widehat{E}$. Every locally complete subspace of E is locally closed and a locally closed subspace of a locally complete space is locally complete, [35, 5.1.20].

2.2. **Holomorphic, harmonic and $\mathscr{C}^\infty$ functions.** Our notation for spaces of (vector valued) differentiable or holomorphic functions is standard. We refer the reader to [3, 26, 36, 37, 38].

A function $f : \Omega \subseteq \mathbb{R}^N \to E$ from an open connected subset (a *domain*) Ω of $\mathbb{R}^N$ into a locally convex space is said to be of class $\mathscr{C}^1$ if, for all $1 \leq i \leq N$, there is a continuous function $\frac{\partial f}{\partial x_i} : \Omega \to E$ such that

$$\frac{\partial f(x)}{\partial x_i} = \lim_{t\to 0} \frac{1}{t}(f(x+te_i) - f(x)),\ x \in \Omega.$$

Here e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^N$. For smooth functions on $\Omega \subseteq \mathbb{R}^N$ we use standard multi-index notation. Thus, if $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, then

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} f,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_N$. The space of all the functions $f : \Omega \to E$ such that $\frac{\partial^{|\alpha|} f}{\partial x^\alpha} : \Omega \to E$ is a well defined continuous function for $|\alpha| \leq k$ is denoted by $\mathscr{C}^k(\Omega, E)$. Whenever $f : \Omega \to E$ is infinitely differentiable, $P(\partial, x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ is a linear partial differential operator with smooth coefficients, then $P(\partial, x)f$ is also an infinitely differentiable function.

A function $f : \Omega \to E$ defined on an open subset Ω of the complex plane $\mathbb{C}$ is said to be *holomorphic* if, for each $z_0 \in \Omega$, there exists $r > 0$ and a sequence $(a_n)_n \in E$ such that $f(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ for each $z \in B(z_0, r)$. The space of holomorphic functions with values in E is denoted by $\mathscr{H}(\Omega, E)$. If the space E is locally complete, a function $f \in \mathscr{C}^\infty(\Omega, E)$ belongs to $\mathscr{H}(\Omega, E)$ if and only if f satisfies the Cauchy Riemann equations. Analogously one can define the space of vector-valued harmonic functions $\mathscr{h}(\Omega, E)$ as the vector valued kernel of the Laplacian. Several variables vector-valued holomorphic and harmonic functions are defined in a natural way.

If Ω is a domain and E is locally complete, then the spaces $\mathscr{C}^\infty(\Omega, E)$ (resp. $\mathscr{H}(\Omega, E)$, $\hbar(\Omega, E)$) and $\mathscr{C}^\infty(\Omega)\varepsilon E$ (resp. $\mathscr{H}(\Omega)\varepsilon E$, $\hbar(\Omega)\varepsilon E$) can be canonically identified via the map $f \to T_f(u) := u \circ f, u \in E'$. This is a consequence of [11, Prop. 2]. The result was well-known when E is quasicomplete, see e.g. [24, 26, 34]

2.3. **Sheaves.** We recall now the definition of topological sheaf. We refer the reader to [5, Section 1] or [34, Chapter V, Section 2] for more details and examples.

Definition 1. *Let Ω be an open set in $\mathbb{R}^N$. $\mathscr{F}$ is called a sheaf over Ω of locally convex spaces if it satisfies the following properties:*

(a) *For each open subset U of Ω there is a locally convex space $\mathscr{F}(U)$ such that $\mathscr{F}(\emptyset) := \{0\}$ and there are continuous linear maps (called restrictions) $\rho_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ if $V \subseteq U$ such that $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ whenever $W \subseteq V \subseteq U$.*
(b) *If $\omega \subseteq \Omega$ is open and ω is the union of open subsets $\{U : U \in \mathscr{U}\}$, then $\mathscr{F}(\omega)$ is the projective limit of $(\mathscr{F}(U), \rho_{\omega,U})_{U\in\mathscr{U}}$. In particular, for each family $\{f_U \in \mathscr{F}(U), U \in \mathscr{U}\}$ satisfying $\rho_{U,U\cap V}(f_U) = \rho_{V,U\cap V}(f_V)$, $U, V \in \mathscr{U}$, there is a unique $f \in \mathscr{F}(\omega)$ with $\rho_{\omega,U}(f) = f_U$ for each $U \in \mathscr{U}$.*

Remark 2. The Open Mapping Theorem for (LB) spaces yields that if $\mathscr{F}$ is a sheaf over Ω of distinguished Fréchet spaces and $\{U_n : n \in \mathbb{N}\}$ is a countable covering by open subsets of an open subset ω of Ω then $\mathscr{F}(\omega)' = \mathrm{ind}_n \mathscr{F}(U_n)'$, the inductive limit taken with respect to the transpose of the restrictions.

In the sequel we will omit *of locally convex spaces* when we refer to a sheaf. Our main example is the sheaf $\mathscr{C}^\infty(\Omega)$ of infinitely differentiable functions over a domain Ω: for every open set $U \subseteq \Omega$ the vector space $\mathscr{C}^\infty(U)$ is the vector space of all infinite differentiable functions defined on U. Here the $\rho_{U,V}$ are simply the restrictions. By a closed subsheaf of $\mathscr{C}^\infty$ over Ω we mean a sheaf $\mathscr{F}$ satisfying that each $\mathscr{F}(\omega) \subseteq \mathscr{C}^\infty(\omega)$ is closed for each $\omega \subseteq \Omega$ open. The restriction maps are the same. All these sheaves are Fréchet Schwartz sheaves; by this we mean with this that $\mathscr{F}(\omega)$ is a Fréchet Schwartz space for each $\omega \subseteq \Omega$ open. We remark that some of the results which we offer could be formulated in a more abstract way in order to include other Fréchet-Schwartz sheaves of functions over Ω, like the ultradifferentiable functions of Beurling type.

Vector valued sheaves are defined using ε-products e.g. in [5, 1.4]. Let Ω be a domain in $\mathbb{R}^N$ and $\mathscr{C}^\infty$ be the sheaf of the infinite differentiable functions. Let $P_1(\partial, x), ..., P_m(\partial, x)$ be linear partial differential operators with smooth coefficients on Ω. Let $P_\omega : \mathscr{C}^\infty \to (\mathscr{C}^\infty)^m$ defined by

$$P_\omega : \mathscr{C}^\infty(\omega) \to \mathscr{C}^\infty(\omega)^m, \ f \mapsto (P_1(\partial, x)f, \ldots, P_m(\partial, x)f), \ \omega \subseteq \Omega$$

open. For a locally complete space E and $\omega \subseteq \Omega$ we consider the maps

$$P_\omega \varepsilon id : \mathscr{C}^\infty(\omega)\varepsilon E \to \mathscr{C}^\infty(\omega)^m \varepsilon E.$$

These maps define a morphism in the category of sheaves of vector spaces over Ω. Moreover, the maps

$$\begin{array}{rcl} ker(P_\omega)\varepsilon E = ker(P_\omega \varepsilon id) & \to & \{f \in \mathscr{C}^\infty(\omega, E) : P_1(\partial, x)f = \cdots = P_m(\partial, x)f = 0\}, \\ T & \mapsto & f_T, \quad f_T(x) = T(\delta_x), x \in \omega, \end{array}$$

define an isomorphism of sheaves. These remarks lead to the following definition.

Definition 3. *Let Ω be a domain in $\mathbb{R}^N$, let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over Ω, and let E be a locally complete space. Then the sheaf defined by*

$$\mathscr{F}(\omega, E) := \{x \mapsto T(\delta_x) : T \in \mathscr{F}(\omega)\varepsilon E\},\ \omega \subseteq \Omega \text{ open,}$$

is called the sheaf of $\mathscr{F}-$functions with values in E.

This definition is coherent with the usual definition of the vector valued sheaves of holomorphic and harmonic functions with values in a locally complete locally convex space E by the remarks at the end of the former subsection. The coherence of this definition, which in principle could seem more restrictive, is also obtained as a direct consequence of Theorem 9 below. Observe that in the case of spaces of holomorphic or harmonic functions $\mathscr{F}$ on Ω, the spaces are closed subsheaves of the sheaf $\mathscr{C}$ of continuous functions on Ω. Equivalently, since all the spaces are Fréchet, the spaces $\mathscr{C}(\omega)$ and $\mathscr{C}^\infty(\omega)$ induce the same topology on $\mathscr{F}(\omega)$ for each $\omega \subseteq \Omega$ open. The same holds for sheaves defined by kernels of hypoelliptic linear partial differential operators with constant coefficients. The following definitions are needed to formulate precisely the first problem we want to deal with.

Definition 4. *A set $M \subseteq \Omega \times \mathbb{N}_0^N$ is called a set of uniqueness for $\mathscr{F}(\Omega)$ if $g \in \mathscr{F}(\Omega)$ vanishes whenever $\partial^\alpha g(x) = 0$ for all $(x, \alpha) \in M$, i.e. whenever span $\{\delta_x \circ \partial^\alpha : (x, \alpha) \in M\}$ is $\sigma(\mathscr{F}(\Omega)', \mathscr{F}(\Omega))$-dense.*

Definition 5. *If $M \subseteq \Omega \times \mathbb{N}_0^N$ is a set of uniqueness for $\mathscr{F}(\Omega)$ and $G \subseteq E'$ is a separating subspace, we define $\mathscr{F}_G(M, E)$ as the space of all $f : M \to E$ such that for each $u \in G$ there is $f_u \in \mathscr{F}(\Omega)$ with $\partial^\alpha f_u(x) = u \circ f(x, \alpha)$, $(x, \alpha) \in M$. Since M is supposed to be a set of uniqueness for $\mathscr{F}(\Omega)$ the functions f_u are unique.*

With the notation established so far, the first extension problem to be considered in this paper reads as follows: *When is the (injective) restriction map*

$$R_{M,G} : \mathscr{F}(\Omega, E) \to \mathscr{F}_G(M, E),\ f \mapsto (\partial^\alpha f(x))_{(x,\alpha)\in M}$$

surjective?

3. Extension of vector-valued functions

In the sequel E denotes a locally complete locally convex space, Ω an open and connected subset of $\mathbb{R}^N$, $\mathscr{F}$ a closed subsheaf of $\mathscr{C}^\infty$ over Ω and $\mathscr{F}(\Omega, E)$ the corresponding sheaf of functions with values in E. $\mathscr{F}(\Omega)$ is an (FS) space since it is supposed to be closed in $\mathscr{C}^\infty(\Omega)$. According to [2], a subspace $G \subseteq E'$ is said to *determine boundedness* if every $\sigma(E, G)$ bounded subset of E is also bounded in E. Clearly, if $G \subseteq E'$ determines boundedness in E', then G is separating, hence dense in $(E', \sigma(E', E))$. The following lemma is very important in the rest of the article. It states known results in a way which is suitable for the applications we have in mind.

Lemma 6. (a) *If $T \in L(E, F)$ there exists a (unique) extension $T^{lc} \in L(E^{lc}, F^{lc})$ of T to the local completions.*

(b) *If Y is a Fréchet Schwartz space and X is a subspace of the (DFS) space Y', then* $\overline{X}^{lc} = \overline{X}^{Y'}$.

(c) *If E is a locally convex space and t is an admissible topology, i.e. $\sigma(E, E') \leq t \leq \tau(E, E')$, then $(E, t)^{lc} = E^{lc}$ algebraically. In particular, if E is locally complete then E equipped with an admissible topology is also locally complete.*

Proof. Part (a) is exactly [35, 5.1.25]. Part (b) is a consequence of Raikov's theorem [35, 8.5.28]. It is enough to show that $\overline{X}^{lc}$ is closed in Y'. Since Y' is a Fréchet Schwartz space, we have $Y' = \text{ind}_n G_n$, G_n a Banach space and the linking maps $G_n \to G_{n+1}$ are compact for each $n \in \mathbb{N}$. Since $\overline{X}^{lc}$ is locally closed by [35, 5.1.17], it intersects each G_n in a closed subset of G_n. We can apply Raikov's theorem [35, 8.5.28] to conclude that $\overline{X}^{lc}$ is closed in Y'. Part (c) follows from [35, 5.1.6 and 7], since

$$(E, \tau(E, E'))^\wedge \hookrightarrow (E, t)^\wedge \hookrightarrow (E, \sigma(E, E'))^\wedge$$

and all the admissible topologies have the same bounded sets. □

In the rest of the article we will make the following natural identification: *Suppose that X is a dense locally convex subspace of the dual Y' of a Fréchet Schwartz space Y. Since Y is reflexive, we consider Y as an algebraic subspace of X^*. In fact Y is the set of all elements of the dual X^* of X which are continuous on X for the topology induced by Y'.* In case no locally convex topology is mentioned on X, we endow it with the finest locally convex topology, which makes any linear mapping $T : X \to E$ continuous for an arbitrary locally convex space E. In this case T^t acts from E' into $X^\star$.

Proposition 7. *Let Y be a Fréchet Schwartz space, let $X \subseteq Y'$ be a dense subspace and let E be a locally complete space. If $T : X \to E$ is a linear map then the following conditions are equivalent:*

(i) *There is a (unique) extension $\widehat{T} \in L(Y', E)$ of T.*
(ii) $T^t(E') \subseteq Y \,(= Y'')$.
(iii) $(T^t)^{-1}(Y)$ *$(= \{u \in E' : u \circ T \in Y\})$ determines boundedness in E.*

Proof. Trivially (i) implies (ii) and (ii) implies (iii). Clearly, the map $T : (X, \sigma(X, Y)) \to (E, \sigma(E, (T^t)^{-1}(Y)))$ is always continuous. If we assume (iii), the space $(E, \sigma(E, (T^t)^{-1}(Y)))$ is locally complete. Lemma 6 yields that the local completion of $(X, \sigma(X, Y))$ is $(Y', \sigma(Y', Y))$ and hence we obtain a unique continuous linear extension $\widehat{T} : (Y', \sigma(Y', Y)) \to (E, \sigma(E, (T^t)^{-1}(Y)))$. Now, Y' endowed with its strong topology is a bornological space and $\widehat{T} : Y' \to E$ maps bounded sets in Y' into bounded sets in E. Hence $\widehat{T}$ is continuous. □

Corollary 8. *Let Y be a Fréchet Schwartz space, E a locally complete space, $T \in L(Y', E)$ and F a locally closed subspace of E. If there exists a separating subspace X of Y' such that $T(X) \subseteq F$ then $T \in L(Y', F)$.*

Proof. The restriction $T : X \to F$ is continuous, hence there is a unique extension $T^{lc} : X^{lc} \to F^{lc} = F$. Now $Y' = X^{lc}$ by Lemma 6 (b), so $T = T^{lc}$. □

Theorem 9. *Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over a domain $\Omega \subseteq \mathbb{R}^N$, let M be a set of uniqueness for $\mathscr{F}(\Omega)$, let G be a subspace of the dual of a locally complete space E which determines boundedness. Then the restriction map $R_{M,G}$ from $\mathscr{F}(\Omega, E)$ to $\mathscr{F}_G(M, E)$ is surjective.*

Proof. Let $f \in \mathscr{F}_G(M, E)$. The space $X := \operatorname{span}\{\delta_x \circ \partial^\alpha : (x, \alpha) \in M\}$ is a (weak*) dense subspace of the dual of the Fréchet Schwartz space $Y := \mathscr{F}(\Omega)$. Let the linear map $T : X \to E$ be determined by $T(\delta_x \circ \partial^\alpha) := f(x, \alpha),\ (x, \alpha) \in M$. Since G is $\sigma(E', E)$-dense, T is well-defined. Let $u \in G$ and f_u the unique element in $\mathscr{F}(\Omega)$ with $\partial^\alpha f_u(x) = u \circ T(\delta_x \circ \partial^\alpha),\ (x, \alpha) \in M$. We can consider f_u as a linear form on X, so we obtain $u \circ T \in \mathscr{F}(\Omega) = Y$ for all $u \in G$, hence $(T^t)^{-1}(Y)$ determines boundedness in E. By Proposition 7, there is an extension $\hat{T} \in \mathscr{F}(\Omega)\varepsilon E$ of T. Putting $\hat{f}(x) := \hat{T}(\delta_x), x \in \Omega$, we get that $R_{M,G}(\hat{f}) = f$. □

In particular, Theorem 9 shows that if $f : \Omega \to E$ is a function such that $u \circ f \in \mathscr{F}(\Omega)$ for each $u \in E'$ then there exists $T : \mathscr{F}(\Omega)' \to E$ such that $f(x) = T(\delta_x)$ for every $x \in \Omega$. Therefore, for $\Omega \subseteq \mathscr{C}^N$ open, one can obtain directly from Theorem 9 the representation $\mathscr{H}(\Omega, E) \simeq \mathscr{H}(\Omega)\varepsilon E$ valid for locally complete spaces E (cf. [11, 27]). Moreover, to illustrate the scope of Theorem 9, we mention that it gives a direct proof of the fact that weak-$\mathscr{C}^\infty$ implies $\mathscr{C}^\infty$: Let $f : \Omega \to E$ be a map into a locally complete space E such that $u \circ f \in \mathscr{C}^\infty(\Omega)$ for all $u \in E'$. Theorem 9 shows that $f(x) = T_f(\delta_x), x \in \Omega$, with $T_f \in \mathscr{C}^\infty(\Omega)\varepsilon E$. Using the Arzela-Ascoli Theorem the map $S : \Omega \to \mathscr{C}^\infty(\Omega)$, $x \mapsto \delta_x$, is infinite differentiable, hence $f = T_f \circ S$ is infinite differentiable. These also applies to holomorphic and harmonic functions. Thus, a general version of Dunford-Grothendieck Theorem is a simple consequence of the extension result Theorem 9. Also to illustrate the applicability of Theorem 9 we mention explicitly how to extend two results obtained in [2] for holomorphic functions with values in Banach spaces E; it is enough to take as $\mathscr{F}$ the sheaf of holomorphic functions.

Corollary 10. (a) *Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over a domain $\Omega \subseteq \mathbb{R}^N$ and let E be a locally complete locally convex space. If $G \subseteq E'$ determines boundedness in E then $\mathscr{F}(\Omega, E)$ is the space formed by the functions $f : \Omega \to E$ such that $u \circ f \in \mathscr{F}(\Omega)$ for all $u \in G$.*

(b) *(cf.* [2, Theorem 3.5, Lemma 3.6]*) Let $\Omega \subseteq \mathbb{C}$ be a domain, $A \subseteq \Omega$ a set with an accumulation point and $f : A \to E$ a map such that $u \circ f$ has an analytic extension to Ω for each u contained in a subspace G of E' which determines boundedness on E. Then there exists $\hat{f} \in \mathscr{H}(\Omega, E)$ which extends f.*

(c) *(cf.* [2, Theorem 2.2]*) Let $\Omega \subseteq \mathbb{C}$ a domain, let E be a locally complete space, let F be a locally closed subspace of E and let $f \in \mathscr{H}(\Omega, E)$. Assume that*

(i) *the set $\Omega_0 := \{z \in \Omega : f(z) \in F\}$ has an accumulation point in Ω; or*

(ii) *there exists $z_0 \in \Omega$ such that $\frac{\partial^k f(z_0)}{\partial z^k} \in F$ for $k = 0, 1, 2 \ldots$,*

then $f(z) \in F$ for all $z \in \Omega$.

Proof. Part (a) follows from Theorem 9 for $M = \Omega$. Part (b) is a direct consequence of Theorem 9. Part (c) follows from Corollary 8. □

In fact, [27, Theorem 3, Theorem 8] are immediate consequences of Theorem 9.

Remark 11. In [2, Theorem 3.5] it is shown that if E is a Banach space and G is a closed and almost norming subspace of E', which means that E is a topological subspace of the Banach space G', then Corollary 10 (b) holds. Such as that subspace $G \subset E'$ determines boundedness in E by the Uniform Boundedness Principle. Hence Corollary 10 (b) is a proper extension of [2, Theorem 3.5]. We include an application of Corollary 10 (b) which can not be deduced from [2, Theorem 3.5]. Let X, Y be Banach spaces. For $x \in X$ and $y \in Y'$, we denote $\delta_{x,y} : L(X,Y) \to \mathbb{K}$, $T \mapsto y(T(x))$. The set $G :=$ span$\{\delta_{x,y} : x \in E, y \in F'\}$ determines boundedness in $L(X,Y)$ endowed with its norm topology. This is a consequence of the Banach-Steinhaus Theorem. Therefore, as a consequence of Corollary 10 (b), if M is a subset of a domain $\Omega \subseteq \mathbb{C}$ with an accumulation point and $f : M \to L(X,Y)$ is a function such that $z \mapsto y(f(z)(x))$ has a holomorphic extension to Ω for each $y \in Y'$ and $x \in X$, then f has an extension $\hat{f} \in \mathscr{H}(\Omega, L(X,Y))$, $L(X,Y)$ endowed with its norm topology. This implies that each $L(X,Y)$-valued holomorphic function for the Weak Operator Topology is also holomorphic for the norm topology.

It is worth remarking that [2, Theorem 1.5] shows that if E is a Banach space, $\mathbb{D}$ is the unit disc in $\mathbb{C}$ and $G \subseteq E'$ is a subspace which does not determine boundedness in E, then there exists a non continuous function $f : \mathbb{D} \to E$ such that $u \circ f \in \mathscr{H}(\mathbb{D})$ for all $u \in G$. Hence Theorem 9 is optimal if we only require M to be a set of uniqueness.

4. Extension of locally bounded functions

Let Ω be a domain in $\mathbb{C}$. A subset $M \subseteq \Omega$ is said to *fix the topology in* $\mathscr{H}(\Omega)$ (*determine local convergence* in [23]) if for all $K \subseteq \Omega$ compact there is $L \subseteq \Omega$ compact and $C \geq 1$ such that

$$\sup_{z\in K} |g(z)| \leq C \sup_{z \in M\cap L} |g(z)| \quad \text{for all } g \in \mathscr{H}(\Omega).$$

Grosse-Erdmann [23] posed the following problem (see the end of section 3 and comments below the statement of Theorem 2 in the Introduction of [23]): *let $M \subseteq \Omega$ fix the topology in $\mathscr{H}(\Omega)$, let $f : M \to E$ be a map such that $f(M \cap K)$ is bounded in E for all $K \subseteq \Omega$ compact, assume that for a separating subspace G of E', $u \circ f$ has a holomorphic extension to Ω for every $u \in G$. Does f have a holomorphic extension to Ω?* Gramsch [21] proved that this result is true if $\overline{G}^{\beta(E',E)} = E'$, which clearly includes the case that E is semireflexive (cf. [27, Theorem 6]). His result inspired Grosse-Erdmann to study this problem and he gave a positive solution for spaces E being B_r-complete. We give below a unified proof of these two cases, and show that the answer to the problem is in general negative.

The following definitions are needed to pose the problem in a more abstract form.

Definition 12. *Let Y be a Fréchet space. An increasing sequence $(B_n)_{n\in\mathbb{N}}$ of bounded subsets of Y' fixes the topology in Y if $(B_n^\circ)_{n\in\mathbb{N}}$ is a fundamental system of zero neighbourhoods of Y.*

Definition 13. *Let Ω be a domain in $\mathbb{R}^N$, let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over Ω. A subset M of $\Omega \times \mathbb{N}_0^N$ fixes the topology in $\mathscr{F}(\Omega)$ if for every compact $K \subseteq \Omega$ and every $k \in \mathbb{N}$ there is a compact $L \subseteq \Omega$, $l \in \mathbb{N}$ and $C \geq 1$ such that*

$$\sup\{|\partial^\alpha g(x)| : x \in K,\ |\alpha| \leq k\} \leq$$

$$C \sup\{|\partial^\alpha g(x)| : x \in L,\ |\alpha| \leq l,\ (x,\alpha) \in M\}$$

for all $g \in \mathscr{F}(\Omega)$.

In case of the space holomorphic functions on $\Omega \subseteq \mathbb{C}$ the subsets $M \subseteq \Omega$ fixing the topology in $\mathscr{H}(\Omega)$ can be characterized by a nice geometrical property, as we see below (cf. [25, 2.5.2,2.6.8]).

Remark 14. *(a) Let $\Omega \subseteq \mathbb{C}^N$ be a pseudo-convex domain. Then $M \subseteq \Omega$ fixes the topology in $\mathscr{H}(\Omega)$ if and only if the $\mathscr{H}(\Omega)$-hulls $\widehat{K \cap M}_\Omega$, $K \subseteq \Omega$ compact, are a fundamental system of the compact subsets of Ω.*

(b) If $\Omega \subseteq \mathbb{C}$ is a domain, then it is pseudoconvex and the $\mathscr{H}(\Omega)$-hull of a compact subset $L \subseteq \Omega$ is the union of L with the relatively compact components of $\Omega \setminus L$. Hence $M \subseteq \Omega$ fixes the topology in $\mathscr{H}(\Omega)$ (equivalently in $\hbar(\Omega)$) if and only if there is a fundamental sequence $(O_n)_n$ of Ω of relatively compact open sets O_n with $\partial O_n \subseteq \overline{O_{n+1} \cap M}$

(c) In case of $Y := \mathscr{H}(\Omega)$ and $M \subseteq \Omega$ fixing the topology in Y, the constant C in Definition 13 can be taken 1, because the powers of holomorphic functions are holomorphic.

Let $(L_n)_n$ be a fundamental sequence of compact (or relatively compact open) subsets of Ω. Let $M \subseteq \Omega \times \mathbb{N}_0^N$. Denote $M_n := \{(x,\alpha) \in M :\ x \in L_n,\ |\alpha| \leq n\}$ and $B_n := \{\delta_x \circ \partial^\alpha : (x,\alpha) \in M_n\} \subseteq \mathscr{F}(\Omega)'$. Observe that a set M fixes the topology in $\mathscr{F}(\Omega)$ in the sense of definition 13 if and only if the sequence $(B_n)_n$ fixes the topology in $\mathscr{F}(\Omega)$ in the sense of Definition 12. This notation will be used in the rest of the article.

Let $M \subseteq \Omega \times \mathbb{N}_0^N$ fix the topology in $\mathscr{F}(\Omega)$ and let G be a separating subspace of E'. We define

$$\mathscr{F}_G(M,E)_{lb} := \{f \in \mathscr{F}_G(M,E) : f(M_n) \text{ is bounded in } E \text{ for } n \in \mathbb{N}\}.$$

Observe that $R_{M,G}(f)$ belongs to $\mathscr{F}_G(M,E)_{lb}$ for each $f \in \mathscr{F}(\Omega,E)$. Using this terminology, our problem reads as follows: *Let $\mathscr{F}(\Omega)$ be a closed subsheaf of $\mathscr{C}^\infty$ over Ω, let $M \subseteq \Omega \times \mathbb{N}_0^N$ be a set which fixes the topology in $\mathscr{F}(\Omega)$ and let $G \subseteq E'$ be a separating subspace. Is the (injective) restriction map*

$$R_{M,G} : \mathscr{F}(\Omega,E) \to \mathscr{F}_G(M,E)_{lb},\ f \mapsto (\partial^\alpha f(x))_{(x,\alpha)\in M}$$

surjective?

Lemma 15. *Let Y be a Fréchet Schwartz space, let $(U_n)_{n\in\mathbb{N}}$ be a zero basis for Y, and let $X \subseteq Y'$ be a sequentially dense subspace. If $T : X \to E$ is a linear map into a locally convex space E then*

$$(*) \bigcap_{n\in\mathbb{N}} span(\overline{(T^t)^{-1}(U_n)}^{\sigma(E',E)}) \subseteq (T^t)^{-1}(Y).$$

Proof. It is clear that the space defined in the left side of (*) does not depend on the choice of the zero basis. Let Y be the reduced projective limit of a sequence $(G_n)_{n\in\mathbb{N}}$ of reflexive Banach spaces such that there are compact linking maps $P_{n+1,n} : G_{n+1} \to G_n$ with dense range. We denote by P_n the induced map from Y to G_n. Then $Y' = \mathrm{ind}_n G'_n$ is the injective inductive limit of an increasing sequence of reflexive Banach spaces with compact inclusions $i_{n,n+1} : G'_n \hookrightarrow G'_{n+1}$, where $i_{n,n+1} := (P_{n+1,n})^t$. Using the Grothendieck factorization theorem [32, 24.33] and the sequential density of X it is easy to see that we can assume that $X_n := G'_n \cap X$ is dense in G'_n for each $n \in \mathbb{N}$. Let B_n be the unit ball of G_n. The sets $U_n := P_n^{-1}(B_n)$, $n \in \mathbb{N}$ form a fundamental system of zero neighbourhoods in Y. We can even assume that they form a zero basis. Take $u \in \bigcap_{n\in\mathbb{N}} span(\overline{(T^t)^{-1}(U_n)}^{\sigma(E',E)})$. For each $n \in \mathbb{N}$ there is $\lambda_{n+1} \geq 1$ with $u \in \lambda_{n+1}\overline{(T^t)^{-1}(U_{n+1})}^{\sigma(E',E)}$. So, there is a net $(u^n_\alpha)_{\alpha\in I} \subseteq E'$ with $u^n_\alpha \circ T \in \lambda_{n+1}U_{n+1},\ \alpha \in I$, such that $(u^n_\alpha \circ T(x))_{\alpha\in I}$ converges to $u \circ T(x)$ for all $x \in X$. By the very definition of the U_n, for each $\alpha \in I$ there exists $v^n_\alpha \in Y$ such that $P_{n+1}(v^n_\alpha)(x) = u^n_\alpha \circ T(x)$ for each $x \in X_{n+1}$ and $P_{n+1}(v^n_\alpha) \in \lambda_{n+1}B_{n+1}$. Using now the compactness of $P_{n+1,n}$, there is a subnet $(P_n(v^n_{\sigma(\beta)}))_{\beta\in J}$ of $(P_n(v^n_\alpha))_{\alpha\in I} = (P_{n+1,n}(P_{n+1}(v^n_\alpha))_{\alpha\in I}$ such that $(P_n(v^n_{\sigma(\beta)}))_{\beta\in J}$ converges in the Banach space G_n to g_n. But for each $x \in X_n \subseteq X_{n+1}$ and for each $\alpha \in I$ we have $P_n(v^n_\alpha)(x) = P_{n+1}(v^n_\alpha)(i_{n,n+1}(x)) = u^n_\alpha \circ T(x)$. Then $g_n(x) = u \circ T(x)$ for each $x \in X_n$ and for each $n \in \mathbb{N}$. The density of X_n in G'_n yields $P_{n+1,n}(g_{n+1}) = g_n$ for each $n \in \mathbb{N}$. This means precisely $u \in (T^t)^{-1}(Y)$. □

Theorem 16. *Let Y be a Fréchet Schwartz space, let $(B_n)_{n\in\mathbb{N}}$ fix the topology in Y, and let $T : X := span(\cup\{B_n : n \in \mathbb{N}\}) \to E$ be a linear map into a locally complete space E which is bounded on each B_n. If*
a) $(T^t)^{-1}(Y)$ is strongly dense in E' or if
b) $(T^t)^{-1}(Y)$ is weak*-dense in E' and E is B_r-complete,
then T has a (unique) extension $\hat{T} \in Y\varepsilon E$.

Proof. 1) If $V_n := T(B_n)^\circ,\ U_n := B_n^\circ$, we obtain $V_n \cap (T^t)^{-1}(Y) \subseteq (T^t)^{-1}(U_n),\ n \in \mathbb{N}$. We apply lemma 15 to get

$$(\star) \qquad \bigcap_{n\in\mathbb{N}} span(\overline{V_n \cap (T^t)^{-1}(Y)}^{\sigma(E',E)}) \subseteq (T^t)^{-1}(Y).$$

2) In view of Proposition 7 it is enough to show that $E' = (T^t)^{-1}(Y)$. Now the case a) is trivial, since the V_n are strong zero neighbourhoods and $(T^t)^{-1}(Y)$ is $\beta(E',E)$-dense. In the case of b) we have to show that $(T^t)^{-1}(Y)$ is nearly closed, i.e. the weak*-closure of the intersection of it with an equicontinuous set is contained in it. But this follows from the fact that each V_n absorbs equicontinuous sets, since they are strong zero neighbourhoods. Then, for each 0-neighbourhood U in E, $U^\circ \cap (T^t)^{-1}(Y) \subseteq$ span $(V_n \cap (T^t)^{-1}(Y))$ for each $n \in \mathbb{N}$. Taking weak star closures the conclusion is easily obtained. □

Analyzing the previous proof we see that it is enough (instead of a) or b)) to ensure that $(T^t)^{-1}(Y)$ is weak* dense and has, in addition, the following property: for each decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of strong zero neighbourhoods in E' with $(\star)$, one has $(T^t)^{-1}(Y) = E'$.

Theorem 17. *If $M \subseteq \Omega \times \mathbb{N}_0^N$ fixes the topology in $\mathscr{F}(\Omega)$ and $G \subseteq E'$ is separating, then the restriction map $R_{M,G}$ from $\mathscr{F}(\Omega, E)$ to $\mathscr{F}_G(M, E)_{lb}$ is surjective in the following two cases:*
(a) E is a B_r-complete space or

(b) E is locally complete and G is strongly dense.

Proof. Let $f \in \mathscr{F}_G(M,E)_{lb}$. There exists $f_u \in \mathscr{F}(\Omega)$ such that $u \circ f(x,\alpha) = \partial^\alpha f_u(x)$ for each $(x,\alpha) \in M$ and for each $u \in G$. Then the linear map $T : \operatorname{span} \cup_n B_n \to E$ defined by $T(\delta_x \circ \partial^\alpha) = f(x,\alpha)$ is well defined and bounded on each B_n. The conclusion follows applying Theorem 16 to T. □

From Theorem 17 we obtain a general positive solution for the problem of Wrobel, valid for the harmonic case. That is, *if $f : \Omega \to E$ is a locally bounded function such that $u \circ f$ is holomorphic or harmonic for each $u \in G \subseteq E'$ separating then f is holomorphic or harmonic.* This can be applied to the following concrete result:

Let H be a complex Hilbert space. For $x, y \in H$ we consider the continuous linear mappings $\delta_{x,y} : L(H) \to \mathbb{C}$, $T \mapsto \langle Tx, y\rangle$. The subspace $G_1 := \operatorname{span}\{\delta_{x,y} : x, y \in H\} \subseteq L(H)'$ determines boundedness in $L(H)$. This can be easily checked using the Banach-Steinhauss Theorem (see Remark 11). The subspace $G_2 = \operatorname{span}\{\delta_{x,x} : x \in H\} \subseteq L(H)'$ also determines boundedness. This can be proved observing that every $T \in L(H)$ can be decomposed as the sum of the two self-adjoint operators $\frac{1}{2}(T+T^*)$ and $\frac{1}{2i}(T-T^*)$ and that, for every $u \in G_1$ there is $v \in G_2$ such that for every $A \in L(H)$ self-adjoint $u(A) = v(A)$ (see [32, p.88],[16]). Given an orthonormal basis $(e_i)_{i\in I}$ in H, the subspace $G_3 = \operatorname{span}\{\delta_{e_i,e_j} : i,j \in I\} \subseteq L(H)'$ is $\sigma(L(H)', L(H))$ dense. Then, in view of Corollary 10 (a) and Example 29 (a), *given $\Omega \subseteq \mathbb{C}$ open and $f : \Omega \to L(H)$ the following assertions are equivalent:*

1. *$f \in \hbar(\Omega, L(H))$,*
2. *$u \circ f \in \hbar(\Omega)$ for each $u \in G_1$,*
3. *$u \circ f \in \hbar(\Omega)$ for each $u \in G_2$,*
4. *f is locally bounded and $u \circ f \in \hbar(\Omega)$ for each $u \in G_3$.*

This extends [16, Lemma 1] and hence contradicts Example 1 and the previous assertion in [16]. Grosse-Erdmann [23, Remark 1 (d)] had already observed that [16, Example 1] was not correct. We also remark that if H is a real Hilbert space then the space G_2 could be not even $\sigma(E', E)$-dense (cf.[32, Example 16.19]).

As an immediate consequence of Theorem 17, we obtain the following result, which is valid for harmonic and several variable holomorphic functions and extends [21, Satz 3.3] and [23, Theorem 2]. By a closed subsheaf $\mathscr{F}$ of $\mathscr{C}^\infty$ over Ω *satisfying the maximum principle* we mean that $\max_{z\in K} |f(z)| = \max_{z\in\partial K} |f(z)|$ for each $f \in \mathscr{F}(\Omega)$ and for each $K \subseteq \Omega$ compact.

Corollary 18. *Let $\mathscr{F}(\Omega)$ be closed in $\mathscr{C}(\Omega)$ and satisfy the maximum principle, let $(O_n)_n$ be a fundamental sequence of relatively compact subdomains of Ω, $\partial O_n \subseteq \overline{M \cap O_{n+1}}$ for each n. If*

$f : M \to E$ is a function such that $f(M \cap K)$ is bounded in E for each compact subset K of Ω and $u \circ f$ admits an extension $f_u \in \mathscr{F}(\Omega)$ for each $u \in G \subseteq E'$, then f admits an extension $\hat{f} \in \mathscr{F}(\Omega, E)$ whenever E is B_r-complete and G is separating (i.e $\sigma(E', E)$-dense) or E is locally complete and G is $\beta(E', E)$-dense.

If $G \subseteq E'$ determines boundedness in a locally complete space E then $(E, \sigma(E, G))$ is a locally complete space. Moreover, if $S \subseteq E'$ is dense in G for the strong topology $\beta(E', E)$ then it is also dense for the $\beta(G, E)$ topology, since these two topologies coincide on G. This observation together with Corollary 10 (a) yield the next result, which is relevant when $E = X'$, X a Banach space and G a dense subspace of $X \subseteq E''$.

Remark 19. *Theorem 17 and Corollary 18 remain true if E is locally complete and $\overline{G}^{\beta(E',E)}$ determines boundedness in E.*

The next example shows that Corollary 18 is not true if E is only assumed to be locally complete and G is $\sigma(E', E)$-dense.

Example 20. (a) *Let $\Omega = \mathbb{C}$, $M := \cup_n \gamma_n$, where $\gamma_n := \{ne^{it} : t \in [0, 2\pi) \cap \mathbb{Q}\}$. Then there is a function $f : M \to \oplus_{n\in\mathbb{N}} l_1$ such that $f(\gamma_n)$ is bounded for each $n \in \mathbb{N}$ and there exists a weak*-dense subspace $G \subseteq (\oplus_{n\in\mathbb{N}} l_1)'$ such that $u \circ f$ admits holomorphic extension to Ω for each $u \in G$ but f is not continuous.*

(b) *Let $\Omega = \mathbb{C}$, $M := \cup_n \gamma_n$, where $\gamma_n := \{ne^{it} : t \in [0, 2\pi)\}$. Then there is a function $f : M \to \oplus_{n\in\mathbb{N}} l_1(\gamma_n)$ such that $f(\gamma_n)$ is bounded for each $n \in \mathbb{N}$ and there exists a weak*-dense subspace $G \subseteq (\oplus_{n\in\mathbb{N}} l_1(\gamma_n))'$ such that $u \circ f$ admits a holomorphic extension to Ω for each $u \in G$ but f is not continuous.*

Proof. We show the statement (a). The proof of part (b) is analogous. Take an enumeration $\gamma_n := \{z_i^n : i \in \mathbb{N}\}$. First we observe that the linear mapping $T : \oplus_n l_1 \to \mathscr{H}(\mathbb{C})'$, $(\alpha^n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \alpha_i^n \delta_{z_i^n}$ is continuous and injective. The continuity is easily obtained from the fact that each γ_n is relatively compact, and then T maps bounded sets in the bornological space $\oplus_n l_1$ to bounded sets in $\mathscr{H}(\mathbb{C})'$. To see that it is injective we suppose that there exists a non trivial sequence $(\alpha^n)_n$ in $\oplus_n l_1$ such that $u := \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \alpha_i^n \delta_{z_i^n} = 0$. in $\mathscr{H}(\mathbb{C})'$. Fix $n_0, i_0 \in \mathbb{N}$ such that $\alpha_{i_0}^{n_0} \neq 0$ and $\alpha_i^n = 0$ for each $n > n_0$ and for each $i \in \mathbb{N}$. We enumerate the double sequence $(\alpha_i^n \delta_{z_i^n})$ by $(\alpha_k \delta_{z_k})_k$. We assume without loss of generality that $\alpha_1 \delta_{z_1} = \alpha_{i_0}^{n_0} \delta_{z_{i_0}^{n_0}}$. Take $k_0 \in \mathbb{N}$ such that $\sum_{k \geq k_0} |\alpha_k| \leq |\alpha_1|/3$. The function $f(z) := ((z + z_1)/2z_1)$ satisfies that $f(z_1) = 1$ and $|f(z_k)| < 1$ for each $k > 1$. We get $j \in \mathbb{N}$ such that $|\sum_{k=2}^{k_0} \alpha_k f^j(z_k)| < |\alpha_1|/3$. Thereby $|u(f^j)| > |\alpha_1/3|$, a contradiction.

We denote $E := \oplus_n l_1$. We return to the first enumeration of each γ_n. Now we define $f : M = \cup_n \gamma_n \to E$, $z_i^n \mapsto (\alpha^j)_j$, each $\alpha^j \in l_1$ being the zero sequence except α^n which has the i-th coordinate 1 and zero all the others. It is clear that f is not continuous because the difference in the l_1-norm $\|f(z_i^n) - f(z_j^n)\| = 2$ for $i \neq j$ and $n \in \mathbb{N}$. It is also clear that $f(\gamma_n)$ is bounded in E for each $n \in \mathbb{N}$. Now the injectivity and continuity of T yields that $G := \{g \circ T : g \in \mathscr{H}(\mathbb{C})\}$ is a $\sigma(E', E)$-dense subspace of E'. To conclude we observe that for each $g \in \mathscr{H}(\mathbb{C})$, the function $g \circ T \circ f : M \to \mathbb{C}$ can be extended to $g \in \mathscr{H}(\mathbb{C})$. □

Remark 21. (i) It is clear that the functions in the above example can not be extended holomorphically to $\mathbb{C}$. Therefore, this example solves Grosse-Erdmann's extension problem in the negative. Further, the linear mapping T in the given proof solves problems (a) and (b) in [23] also in the negative. In fact, T can not be surjective because it is an injective continuous linear mapping between two (LB) spaces and $\mathscr{H}(\mathbb{C})'$ is Montel but $\oplus_n l_1$ is not.

(ii) Example 20 contradicts [2, Corollary 3.7]. In fact, if we set $\Gamma_n := \cup_{1\leq k\leq n}\gamma_k$, $V_n := B(0, n+1)$ the open ball in $\mathbb{C}$ of radius $n+1$ centered at zero for $n \in \mathbb{N}$, $E := \oplus_{k\in\mathbb{N}} l_1$ and we consider the Banach space $E_n := \oplus_{1\leq k\leq n} l_1$ endowed with its natural norm defined as the sum of the norms in l_1 of the components of the vectors, then the restriction

$$f|_{\Gamma_n} : \Gamma_n \to E_{n+1}$$

is a function such that $f(\Gamma_n)$ is contained in the unit ball of E_{n+1} for all $n \in \mathbb{N}$. The restriction of T to each E_n is continuous. Hence, for all $n \in \mathbb{N}$,

$$G := \{g \circ T|_{E_{n+1}} : g \in \mathscr{H}(\mathbb{C})\}$$

is a $\sigma(E'_{n+1}, E_{n+1})$ dense subspace of E'_{n+1} such that, for each $g \in \mathscr{H}(\mathbb{C})$, $g \circ T|_{E_{n+1}} \circ f|_{\Gamma_n}$ admits the holomorphic extension $g|_{V_n}$ to V_n and

$$\sup_{z\in V_n} |g(z)| \leq \sup_{z\in\gamma_{n+1}} |g(z)| \leq \sup_{e\in B_{E_{n+1}}} |g \circ T|_{E_{n+1}}(e)| = \|g \circ T|_{E_{n+1}}\|_{E'_{n+1}}.$$

[2, Corollary 3.7] would imply that $f|_{\Gamma_n}$ could be extended holomorphically to $f_n \in \mathscr{H}(V_n, E_{n+1})$, but $f|_{\Gamma_n}$ is not continuous.

(iii) Corollary 18 implies that [2, Corollary 3.8] is true, even for several variable holomorphic functions.

5. Wolff type results

The main tool in the proof of Grosse-Erdmann theorem [23, Theorem 1] is Wolff's theorem [40], which we state now in a more functional analytic way, as it is done in the preliminaries of [23]: If $\Omega \subseteq \mathbb{C}$ is a domain, for each $u \in \mathscr{H}(\Omega)'$ there exists a sequence $(z_i)_i$ which is relatively compact in Ω and a sequence $(\alpha_i)_i \in l_1$ such that $u = \sum_{i=1}^{\infty} \alpha_i \delta_{z_i}$. Our goal in this section is to obtain similar representations for dual spaces $\mathscr{F}'(\Omega)$ of closed subsheaves of $\mathscr{C}^\infty(\Omega)$ and to derive extension results from these representations. For further information about extensions of Wolff original result we refer to [33, sections 5.7.8 and 5.8], [39] and the references quoted there.

Let Y be a Fréchet Schwartz space and let $(B_n)_n$ be an increasing sequence of bounded subsets of Y'. We introduce a notation which will be useful in the rest of the article. We denote, for $n \in \mathbb{N}$, by $l_1(B_n)$ the Banach space of all summable families with index set B_n. The linear map

$$j_n : l_1(B_n) \to Y'_\beta, j_n((\alpha(b))_{b\in B_n}) := \sum_{b\in B_n} \alpha(b)b,$$

is well defined and continuous. We denote by $Y'(B_n)$ the image of the map j_n endowed with the quotient norm. Clearly $Y'(B_n)$ is a Banach space which is continuously embedded in Y'_β. Finally we set $Y'((B_n)_{n\in\mathbb{N}}) := \text{ind}_n Y'(B_n)$, which is an (LB)-space continuously included in Y'_β.

Remark 22. *Let Y be a Fréchet Schwartz space. If $(B_n)_n$ is an increasing sequence of bounded sets of Y', E is a locally complete locally convex space and $T:\ span(\cup_n B_n) \to E$ is a linear map such that T is bounded on each B_n and $(T^t)^{-1}(Y)$ is $\sigma(E', E)$-dense, then there exists a unique continuous linear extension $T_i : Y'((B_n)_n) \to E$. Compare with Theorem 16.*

In view of this remark, our extension problem has a positive solution whenever $(B_n)_n$ is an increasing sequence of bounded sets of Y' such that $Y'((B_n)_{n\in\mathbb{N}}) = Y'_\beta$ topologically. We characterize now these sequences of bounded sets.

Proposition 23. *Let Y be a nuclear Fréchet space with an increasing fundamental system $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms, and let $(B_n)_{n\in\mathbb{N}}$ an increasing sequence of bounded sets of Y'. Then the following assertions are equivalent:*

(i) $Y'((B_n)_{n\in\mathbb{N}}) = Y'_\beta$.

(ii) *For every $\mu \in Y'$ there is $n \in \mathbb{N}$, $(\mu_\nu)_{\nu\in\mathbb{N}} \in B_n^{\mathbb{N}}$, and $(\lambda_\nu)_{\nu\in\mathbb{N}} \in l_1$ such that*

$$\mu = \sum_{\nu=1}^{\infty} \lambda_\nu \mu_\nu .$$

(iii) *For every $k \in \mathbb{N}$ there is $n \in \mathbb{N}$, $(\mu_\nu)_{\nu\in\mathbb{N}} \in B_n^{\mathbb{N}}$, and a decreasing zero sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that*

$$\|f\|_k \leq \sup_{\nu\in\mathbb{N}} \varepsilon_\nu |\mu_\nu(f)|$$

for every $f \in Y$.

Proof. (i) and (ii) are equivalent by the open mapping theorem.
For $k \in \mathbb{N}$, let C_k denote the polar of the unit ball of the k-th seminorm. We denote by E'_k the Banach space spanned by C_k. Assume (i). By Grothendieck's factorization theorem the inductive spectra of $Y'((B_n)_{n\in\mathbb{N}})$ and that of Y'_β are equivalent. Since Y is nuclear there is $n \in \mathbb{N}$ such that the inclusion $i_{k,n} : E'_k \to Y'(B_n)$ is nuclear. Now we observe the following two facts, which can be easily checked:

a) Let E be a Banach space and let I be an index set. If $S : E \to l_1(I)$ is a nuclear linear map and B is the unit ball of E then there exists $\beta = (\beta(i))_{i\in I} \in l_1(I)$ such that

$$S(B) \subseteq \{(\lambda(i))_{i\in I} : |\lambda(i)| \leq |\beta(i)|, i \in I\}.$$

b) If E is a Banach space, F is a Hausdorff quotient of a Banach space G and $T : E \to F$ is a nuclear linear map then there exists a nuclear linear map $S : E \to G$ such that $T = p \circ S$, where p is the quotient map.

We apply a) and b) to $i_{k,n}$ to obtain $(\mu_\nu)_{\nu\in\mathbb{N}} \in B_n^{\mathbb{N}}$ and $(\beta_\nu)_{\nu\in\mathbb{N}} \in l_1$ such that

$$C_k \subseteq \{\sum_{\nu=1}^{\infty} \lambda_\nu \mu_\nu : |\lambda_\nu| \leq \beta_\nu,\ \nu \in \mathbb{N}\}.$$

If we choose a decreasing zero sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that $C := \sum_{\nu=1}^\infty |\frac{\beta_\nu}{\varepsilon_\nu}| < \infty$, we obtain

$$\begin{aligned} \|f\|_k &\leq \sup\{\sum_{\nu=1}^\infty \lambda_\nu\mu_\nu(f) : |\lambda_\nu| \leq \beta_\nu,\ \nu\in\mathbb{N}\} \\ &\leq C\sup_{\nu\in\mathbb{N}} \varepsilon_\nu|\mu_\nu(f)| \end{aligned}$$

for every $f \in Y$.
Assume now (iii) and fix $k \in \mathbb{N}$. Then C_k is contained in the closure D of the absolutely convex hull of $\{\varepsilon_\nu\mu_\nu : \nu\in\mathbb{N}\}$ and from [35, 3.2.13] we get that $D = \{\sum_{\nu=1}^\infty \lambda_\nu\varepsilon_\nu\mu_\nu : \sum_{\nu=1}^\infty |\lambda_\nu| \leq 1\}$. This shows (ii). □

Remark 24. (a) Nuclearity plays an important role in the proof of the equivalence between (ii) and (iii), but (i) and (ii) are equivalent for distinguished Fréchet spaces. In our setting, $\mathscr{F}(\Omega)$ is a nuclear space because it is a subspace of $\mathscr{C}^\infty(\Omega)$.

(b) Notice that if $Y'((B_n)_{n\in\mathbb{N}}) = Y'_\beta$ holds then $(B_n)_n$ fixes the topology in the Fréchet Schwartz space Y, because the polar of the unit ball of $Y'(B_n)$ in Y coincides with the polar of B_n for each $n \in \mathbb{N}$. Example 20 together with Remark 21 (i) show that $(B_n)_n$ fixing the topology in Y is not enough to have the equality $Y'((B_n)_{n\in\mathbb{N}}) = Y'$.

To obtain concrete examples about Wolff descriptions we introduce the following notation. Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over an open set $\Omega\subseteq\mathbb{R}^N$ and let $U\subseteq V\subseteq\Omega$ open sets. If $B\subseteq\mathscr{F}(U)'$ we denote $B|_{\mathscr{F}(V)} := \{u\circ\rho_{V,U} : u\in B\} = (\rho_{V,U})^t(B)\subseteq\mathscr{F}(V)'$.

Lemma 25. *Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over an open domain $\Omega\subseteq\mathbb{R}^N$, let $(U_n)_n$ be an increasing covering of Ω by open subsets, and let $(B_j^n)_j\subseteq\mathscr{F}(U_n)'$ be a sequence of bounded sets which fixes the topology in $\mathscr{F}(U_n)$ for $n\in\mathbb{N}$. Assume also that $(B_j^n)|_{\mathscr{F}(U_{n+1})}\subseteq B_j^{n+1}$ and that $B_n := \cup_j(B_j^n)|_{\mathscr{F}(\Omega)}$ is bounded in $\mathscr{F}(\Omega)'$ for each $n\in\mathbb{N}$. Suppose that $T:$ span$(\cup_n B_n)\to E$ is a linear mapping into a locally complete space E such that $(T^t)^{-1}(\mathscr{F}(\Omega))$ is separating and that, for all $n\in\mathbb{N}$, there exists a Banach space $E_n\hookrightarrow E$ continuously embedded such that $T(B_n)$ is bounded in E_n. Then there exists an extension $\widehat{T}\in\mathscr{F}(\Omega)\varepsilon E$ of T.*

Proof. First we observe that span$(\cup_n B_n)$ is dense in $\mathscr{F}(\Omega)'$ and then $(T^t)^{-1}(\mathscr{F}(\Omega))$ is meaningful. This can be easily deduced from the description $\mathscr{F}(\Omega)$ as a projective limit of the spaces $\mathscr{F}(U_n)$ with respect to the restrictions, using the density of span $\cup_j (B_j^n)$ in $\mathscr{F}(U_n)'$. For $n\in\mathbb{N}$, we define $T_n :$ span$(\cup_j B_j^n)\to E_n$, $\mu\mapsto T(\mu|_{F(\Omega)})$. By the construction, we have that $T_n(B_j^n)$ is bounded in E_n for each $j\in\mathbb{N}$ and the subspace $H_n := (T_n^t)^{-1}(\mathscr{F}(U_n))$ of E_n' contains $(T^t)^{-1}(\mathscr{F}(\Omega))|_{E_n}$. Hence H_n is $\sigma(E_n', E_n)$ dense. Theorem 16 b) implies that there exists a continuous linear extension $\widehat{T}_n : \mathscr{F}(U_n)'\to E_n\hookrightarrow E$ of T_n. We observe that $\widehat{T}_{n+1}\circ(\rho_{U_{n+1},U_n})^t|\text{span}_{\cup_j B_j^n} = T_n$ and that span$\cup_j B_j^n$ is dense in $\mathscr{F}(U_n)'$ for each $n\in\mathbb{N}$. Therefore, we apply Remark 2 to define $\widehat{T} : \mathscr{F}(\Omega)' =$ ind$_n\mathscr{F}(U_n)'\to E$ by $\widehat{T}(b) = \widehat{T}_n(b)$ whenever $b\in\mathscr{F}(U_n)'$. $\widehat{T}$ is the desired extension of T. □

The following abstract Wolff type result is now a consequence of the previous extension lemma.

Theorem 26. *Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over an open set $\Omega \subseteq \mathbb{R}^N$, let $(U_n)_n$ be an increasing covering of Ω by open subsets and let $(B_j^n)_j \subseteq \mathscr{F}(U_n)'$ be a sequence of bounded sets which fixes the topology in $\mathscr{F}(U_n)$ for $n \in \mathbb{N}$. Assume also that $(B_j^n)|_{\mathscr{F}(U_{n+1})} \subseteq B_j^{n+1}$ and that $B_n := \cup_j (B_j^n)|_{\mathscr{F}(\Omega)}$ is bounded in $\mathscr{F}(\Omega)'$. Then $\mathscr{F}(\Omega)' = \mathscr{F}(\Omega)'((B_n)_{n\in\mathbb{N}})$ (topologically).*

Proof. Set $E := Y'((B_n)_{n\in\mathbb{N}})$ and $E_n := Y'(B_n)$. We consider the inclusion $T :\ \mathrm{span}(\cup_n B_n) \to E$. Since $E \hookrightarrow \mathscr{F}(\Omega)'$ continuously it follows that $\mathscr{F}(\Omega)$ is $\sigma(E', E)$ dense. We apply Lemma 25 to obtain a continuous linear map $\widehat{T} : \mathscr{F}(\Omega)' \to E$ which extends T. But $\mathrm{span}(\cup_n B_n)$ is dense in $\mathscr{F}(\Omega)'$. Hence the continuous inclusion $E \hookrightarrow \mathscr{F}(\Omega)'$ is surjective. The topological equality follows from Proposition 23. □

We see below that we can obtain Wolff type results for not necessarily increasing coverings by open sets.

Corollary 27. *Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over an open domain $\Omega \subseteq \mathbb{R}^N$, let $(U_n)_n$ be a covering of Ω by open subsets and let $(B_j^n)_j \subseteq \mathscr{F}(U_n)'$ be a sequence of bounded sets which fixes the topology in $\mathscr{F}(U_n)$ for $n \in \mathbb{N}$. Assume that $B_n := \cup_j (B_j^n)|_{\mathscr{F}(\Omega)}$ is bounded in $\mathscr{F}(\Omega)'$ for each $n \in \mathbb{N}$. Then, for each $\mu \in \mathscr{F}(\Omega)'$ there exists $(\alpha_i)_i \in l_1$, $k \in \mathbb{N}$ and $(\mu_i)_i \subseteq \cup_{1\le j\le k} B_j$ such that $\mu = \sum_i \alpha_i \mu_i$*

Proof. We show that, if U, V are two open subsets of Ω, $(B_j^U)_j$ is a sequence of bounded subsets of $\mathscr{F}(U)'$ which fixes the topology in $\mathscr{F}(U)$, $(B_j^V)_j$ is a sequence of bounded subsets of $\mathscr{F}(V)'$ which fixes the topology in $\mathscr{F}(V)$ and we define $C_n := B_n^U|_{\mathscr{F}(U\cup V)} \cup B_n^V|_{\mathscr{F}(U\cup V)}$, then $(C_n)_n$ is a sequence of bounded subsets of $\mathscr{F}(U \cup V)'$ which fixes the topology in $\mathscr{F}(U \cup V)$. Since $\mathscr{F}$ is a sheaf, $\mathscr{F}(U \cup V)$ is the projective limit of the spaces $\mathscr{F}(U)$ and $\mathscr{F}(V)$ with respect to the restrictions $\rho_{U\cup V,U}$ and $\rho_{U\cup V,V}$. This implies that a fundamental system of 0-neighbourhoods is given by the sets

$$W_n := (\rho_{U\cup V,U})^{-1}((B_n^U)^\circ) \cap (\rho_{U\cup V,V})^{-1}((B_n^V)^\circ),$$

$n \in \mathbb{N}$. It is straightforward to show that $W_n = C_n^\circ$. Now the conclusion can be obtained by applying Theorem 26 to the covering $(V_n)_n$ of Ω defined by $V_n := \cup_{1\le j\le n} U_j$. □

From Theorem 26 we have also the following corollary.

Corollary 28. *Let $\mathscr{F}$ be a closed subsheaf of $\mathscr{C}^\infty$ over an open domain $\Omega \subseteq \mathbb{R}^N$. If $M \subseteq \Omega \times \mathbb{N}_0^N$ satisfies that there exists an increasing countable covering $(U_k)_k$ of Ω by relatively compact open sets such that the sets $M_n := \{(x, \alpha) \in M : x \in U_n, |\alpha| \le n\}$ fix the topology in $\mathscr{F}(U_n)$, $n \in \mathbb{N}$, then $\mathscr{F}(\Omega)' = \mathscr{F}(\Omega)'((B_n)_{n\in\mathbb{N}})$. Hence the restriction map $R_{M,G} : \mathscr{F}(\Omega, E) \to \mathscr{F}_G(M, E)_{lb}$ is surjective for each locally complete locally convex space E.*

We enumerate below examples satisfying the hypothesis of Corollary 28.

Example 29. (a) *$\mathscr{F}$ a sheaf of smooth functions which is closed in the sheaf of continuous functions $\mathscr{C}$ over $\Omega \subseteq \mathbb{R}^N$ and $M := \Omega$. For any covering $(U_k)_k$ of Ω by relatively compact open subsets, $M_k = U_k$ fixes the topology in $\mathscr{F}(U_k)$.*

(b) *The sheaf* $\mathscr{H}$ *of holomorphic functions over* $\Omega \subseteq \mathbb{C}^N$, *and* $M := \Omega \setminus K$, $K \subseteq \Omega$ *compact. For any increasing covering* $(U_k)_k$ *of* Ω *relatively compact open subsets such that* $K \subseteq U_1$ *the sets* $M_k := M \cap U_k$ *fix the topology of* $\mathscr{H}(U_k)$ *(cf.* [23, Corollary 1]*).*

(c) *The sheaf* $\mathscr{H}$ *of holomorphic functions over* $\Omega := \mathbb{C}^N$, $N \in \mathbb{N}$, *and* $M := \cup_{k,n\in\mathbb{N}} S(0, k - 1/n)$. *For* $U_k := B(0,k)$, $k \in \mathbb{N}$, *the sets* $M_k := M \cap U_k$ *fix the topology* $\mathscr{H}(U_k)$ *(here* $B(a,r)$ *and* $S(a,r)$ *denote the ball and the sphere centered and a with radius* r *respectively). Example 20 shows that* $M := \cup_k S(0,k)$ *does not satisfy the hypothesis of Corollary 28 in the one variable case.*

We have also the following consequences:

(i) In the three examples above, Corollary 28 can be formulated in the following way. For each $\mu \in \mathscr{F}(\Omega)'$ there exists $k \in \mathbb{N}$, a sequence $(z_\nu)_\nu$ in M_k and a sequence $(\alpha_\nu)_\nu \in l_1$ such that

$$\mu = \sum_{\nu=1}^{\infty} \alpha_\nu \delta_{z_\nu}.$$

Hence, if E is a locally complete space, G is a weak* dense subspace of E' and $f : M \to E$ is a function such that $f(M \cap K)$ is bounded in E for each compact subset K of Ω and $u \circ f$ admits extension in $\mathscr{F}(\Omega)$ for all $u \in G$ then there exists an extension $\hat{f} \in \mathscr{F}(\Omega, E)$ of f. We also remark that in (b) and (c) one can take the sheaf of harmonic functions on $\Omega \subseteq \mathbb{C}$ instead of the sheaf of holomorphic functions. Thus, we have proper extensions of [23, Theorem 1, Corollary 1].

(ii) Example 29 (a) and Proposition 23 together imply that for each compact subset K of the open set $\Omega \subset \mathbb{C}^N$, there exists a decreasing zero sequence of positive numbers $(\varepsilon_\nu)_\nu$ and a relatively compact sequence $(z_\nu)_\nu \subseteq \Omega$ such that, for each $f \in \mathscr{H}(\Omega)$

$$\sup_{z\in K} |f(z)| \leq \sup_{\nu\in\mathbb{N}} \varepsilon_\nu |f(z_\nu)|.$$

In case $\Omega = \mathbb{C}^N$, Example 29 shows that for each K compact there exists $k_0 \in \mathbb{N}$ such that the sequence $(z_\nu)_\nu$ can be even taken in $\cup_{1\leq k\leq k_0} \cup_{n\in\mathbb{N}} S(0, k-1/n)$. Again we remark that there is no $k_0 \in \mathbb{N}$ such that the sequence above could be taken in $\cup_{k\leq k_0} S(0,k)$. This is a consequence of Example 20 and Proposition 23.

Finally, we obtain Wolff type results for closed subsheaves $\mathscr{F}(\Omega, E)$ of $\mathscr{C}^\infty(\Omega, E)$ with E Fréchet. To do this, we consider in these spaces the natural topology of uniform convergence of the derivatives on compact subsets of Ω, which makes it a Fréchet space. This topology coincides with the one endowed by the ε-product $\mathscr{F}(\Omega)\varepsilon E$. We refer to [26, 16.7] for the proof of this fact for the sheaf of one variable holomorphic functions.

Proposition 30. *Let* $\mathscr{F}$ *be a closed subsheaf of* $\mathscr{C}^\infty$ *over an open set* $\Omega \subseteq \mathbb{R}^N$. *Let* $(B_n)_n$ *be an increasing sequence of bounded subsets of* $\mathscr{F}(\Omega)'$ *such that* $\mathscr{F}(\Omega)' = \mathscr{F}(\Omega)'((B_n)_{n\in\mathbb{N}})$. *Let* E *be a Fréchet space. For each* $\mu \in \mathscr{F}(\Omega, E)'$ *there exists a sequence* $(\alpha_k)_k \in l_1$, $n_0 \in \mathbb{N}$, *a sequence* $(b_k)_k \subseteq B_{n_0}$ *and a bounded sequence* $(v_k)_k \subseteq E'$ *such that*

$$\mu(f) = \sum_{k=1}^{\infty} \alpha_k b_k (v_k \circ f).$$

for each $f \in \mathscr{F}(\Omega, E)$.

Proof. The space $\mathscr{F}(\Omega)$ is nuclear and hence $\mathscr{F}(\Omega)$ has the Approximation Property. Moreover, $\mathscr{F}(\Omega)$ is also separable. Thus, there exists a projective spectrum $(H_n)_n$ of separable Hilbert spaces such that $\mathscr{F}(\Omega)$ is its reduced projective limit. Let E be the reduced projective limit of a sequence $(E_n)_n$ of Banach spaces. We have

$$\mathscr{F}(\Omega, E) = \mathscr{F}(\Omega)\hat{\otimes}_\varepsilon E = \text{ proj}_n H_n \hat{\otimes}_\varepsilon E_n.$$

We can apply [13, 16.6] to obtain

$$\mathscr{F}(\Omega, E)' = \text{ind}_n H_n' \hat{\otimes}_\pi E_n'.$$

algebraically. Let $\mu \in \mathscr{F}(\Omega, E)'$. There exists $k_0 \in \mathbb{N}$ such that $\mu \in H_{k_0}' \hat{\otimes}_\pi E_{k_0}'$. Thereby, there exists a bounded sequence $(h_k)_k$ in $H_{k_0}' \hookrightarrow \mathscr{F}(\Omega)'$, a bounded sequence $(e_k)_k$ in $E_{k_0}' \hookrightarrow E$ and a sequence $(\lambda_k)_k$ in l_1 such that

$$\mu(f) = \sum_{k=1}^{\infty} \lambda_k (h_k \otimes v_k)(f) = \sum_{k=1}^{\infty} \lambda_k h_k (v_k \circ f).$$

Since $(h_k)_k$ is bounded in $\mathscr{F}(\Omega)' = \mathscr{F}(\Omega)'((B_n)_{n\in\mathbb{N}})$, there exists $n_0 \in \mathbb{N}$ such that $(h_k)_k$ is bounded in $\mathscr{F}(\Omega)'(B_{n_0})$. Hence we can get a sequence $(b_i)_i \subseteq B_{n_0}$ and $M > 0$ such that, for each $k \in \mathbb{N}$ there exists $(\alpha_i^k)_i \in l_1$ such that $\sum_i |\alpha_i^k| < M$ and $h_k = \sum_i \alpha_i^k b_i$. Therefore, for each $f \in \mathscr{F}(\Omega, E)$

$$\mu(f) = \sum_{k=1}^{\infty} \lambda_k (h_k \otimes v_k)(f) = \sum_{k=1}^{\infty} \lambda_k h_k (v_k \circ f) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \lambda_k \alpha_i^k b_i (v_k \circ f).$$

An enumeration of the double series gives the desired formula. □

By Example 29 (a), we have that in the sheaf $\mathscr{H}$ of holomorphic functions over $\Omega \subseteq \mathbb{C}^N$, for each $\mu \in \mathscr{H}(\Omega, E)'$, there exists $(\alpha_k)_k \in l_1$, $(v_k)_k \subseteq E'$ bounded and $(z_k)_k \subseteq \Omega$ relatively compact such that, for each $f \in \mathscr{H}(\Omega, E)$

$$\mu(f) = \sum_{k=1}^{\infty} \alpha_k v_k (f(z_k)).$$

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