A COMPARISION OF TWO DIFFERENT WAYS TO DEFINE CLASSES OF ULTRADIFFERENTIABLE FUNCTIONS

JOSÉ BONET, REINHOLD MEISE, AND SERGEJ N. MELIKHOV

Dedicated to our friend Jean Schmets at the occasion of his 65th birthday.

Abstract. We characterize the weight sequences $(M_p)_p$ such that the class of ultradifferentiable functions $\mathcal{E}_{(M_p)}$ defined by imposing conditions on the derivatives of the function in terms of this sequence coincides with a class of ultradifferentiable functions $\mathcal{E}_{(\omega)}$ defined by imposing conditions on the Fourier Laplace transform. As a corollary, we characterize the weight functions ω for which there exists a weight sequence $(M_p)_p$ such that the classes $\mathcal{E}_{(\omega)}$ and $\mathcal{E}_{(M_p)}$ coincide. These characterizations also hold in the Roumieu case. Our main results are illustrated by several examples.

1. *Introduction.* Among the various ways to define ultradifferentiable functions the following two are frequently used. The older one goes back to the work of Gevrey [5] and measures the growth behaviour of such functions in terms of a weight sequence $(M_p)_{p\in\mathbb{N}_0}$, which is $((p!)^s)_{p\in\mathbb{N}_0}$, $s \geq 1$, in the Gevrey case and which satisfies certain technical conditions in the general case. Later Beurling [1], see Björck [2] for a detailed exposition, pointed out that one can also use weight functions ω to measure the smoothness of C^∞-functions with compact support by the decay properties of their Fourier transform. This method was modified by Braun, Meise, and Taylor [4] who showed that also these classes can be defined by the decay behaviour of their derivatives, if one uses the Young conjugate of the function $t \mapsto \omega(e^t)$. In Meise and Taylor [9] it was shown that under rather strong conditions both ways lead to the same class. However, in general there are classes defined in one way which cannot be defined in the other way.

The aim of the present paper is to characterize those weight sequences $(M_p)_{p\in\mathbb{N}_0}$ for which there exists a weight function ω, such that $\mathcal{E}_{(M_p)}(G) = \mathcal{E}_{(\omega)}(G)$ or $\mathcal{E}_{\{M_p\}}(G) = \mathcal{E}_{\{\omega\}}(G)$ for each/some open set G in $\mathbb{R}^n$. Our main result (Theorem 14) is based on theorems of Langenbruch [7] and shows that this happens if and only if the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies the condition (M2) of Komatsu [6] and if $\liminf_{j\to\infty} m_{Qj}/m_j > 1$ for some $Q \in \mathbb{N}$ where $m_j := M_j/M_{j-1}$, $j \in \mathbb{N}$. If these two conditions are satisfied then the associated function M of the sequence $(M_p)_{p\in\mathbb{N}_0}$ is a weight function. As a corollary we also characterize the weight functions ω for which there exists a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ such that ω-classes and the (M_p)-classes coincide. Several examples illustrate our result. Moreover, we present a class of weight sequences in Proposition 23 for which the associated function is a weight function which can be computed explicitely up to equivalence. This result is then applied to show that a quasianalytic class which came up in [3] can also be defined by weight sequences.

For undefined notation concerning locally convex spaces, in particular sequence spaces, we refer to Meise and Vogt [11].

2. **Weight functions.** A function $\omega : \mathbb{R} \to [0, \infty[$ is called a *weight function* if it is continuous, even, increasing on $[0, \infty[$, and if it satisfies $\omega(0) = 0$ and also the following conditions:

(α) $\omega(2t) = O(\omega(t))$ as t tends to infinity.
(β) $\omega(t) = O(t)$ as t tends to infinity.
(γ) $\log(t) = o(\omega(t))$ as t tends to infinity.
(δ) $\varphi : t \mapsto \omega(e^t)$ is convex on $[0, \infty[$.

If a weight function ω satisfies

(Q) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}dt = \infty$

then it is called a *quasianalytic weight*. Otherwise it is called *non-quasianalytic*.

The *radial extension* $\tilde{\omega}$ of a weight function ω is defined as

$$\tilde{\omega} : \mathbb{C}^n \to [0, \infty[, \quad \tilde{\omega}(z) := \omega(|z|).$$

It will also be denoted by ω in the sequel, by abuse of notation. The *Young conjugate* of the function $\varphi = \varphi_\omega$, which appears in (δ), is defined as

$$\varphi^*(x) := \sup\{xy - \varphi(y) : y > 0\}, \;\; x \geq 0.$$

3**. Example.** The following functions are easily seen to be weight functions:

(1) $\omega(t) := |t|(\log(1 + |t|))^{-\alpha}, \alpha > 0$.
(2) $\omega(t) := |t|^\alpha$, $0 < \alpha \leq 1$.
(3) $\omega(t) = \max(0, (\log t)^s), \; s > 1$.

4**. Weight sequences.** A sequence $(M_p)_{p\in\mathbb{N}_0}$of positive numbers is called *weight sequence*, if it satisfies the following three conditions:

(M0) $\qquad$ There exists $c > 0$ such that $(c(p + 1))^p \leq M_p, \; p \in \mathbb{N}_0$.

(M1) $\qquad M_p^2 \leq M_{p-1}M_{p+1}, \; p \in \mathbb{N}$ and $M_0 = 1$.

(M2)' $\qquad$ There are $A, H \geq 1$ such that $M_{p+1} \leq AH^pM_p, \; p \in \mathbb{N}_0$.

A weight sequence is called *non-quasianalytic*, if it satisfies

(M3)' $\qquad \displaystyle\sum_{p=1}^\infty \frac{M_{p-1}}{M_p} < \infty,$

otherwise it is called *quasianalytic*.

Later we will need the following condition which is obviously stronger than (M2)':

(M2) $\qquad$ There are $A, H > 0$ such that $M_p \leq AH^p \displaystyle\min_{0\leq q\leq p} M_qM_{p-q}, \; p \in \mathbb{N}_0$.

We will also use the condition

(M3) $\qquad$ There is $A > 0$ such that $\displaystyle\sum_{p=j+1}^\infty \frac{M_{p-1}}{M_p} \leq A\frac{M_j}{M_{j+1}}$ for each $j \in \mathbb{N}$.

5**. Example.** The following sequences $(M_p)_{p\in\mathbb{N}_0}$ are weight sequences:

(1) $M_p := (p + 1)^{ps}, \; s \geq 1$
(2) $M_p := ((p + 1)(\log(e + p))^\alpha)^p\,, \; \alpha > 0$.

6**. Ultradifferentiable functions defined by weight functions.** Let ω be a given weight function. For a compact subset K of $\mathbb{R}^N$ and $m \in \mathbb{N}$ denote by $C^\infty(K)$ the space of all C^∞-Whitney jets on K and define

$$\mathcal{E}^m_{\{\omega\}}(K) := \{f \in C^\infty(K) : \|f\|_{K,m} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right) < \infty\}.$$

For an open set G in $\mathbb{R}^N$, define the space $\mathcal{E}_{\{\omega\}}(G)$ of all *ω-ultradifferentiable functions of Roumieu type* on G as

$$\mathcal{E}_{\{\omega\}}(G) := \{f \in C^\infty(G) : \text{For each } K \subset G \text{ compact there is } m \in \mathbb{N} \text{ so that } \|f\|_{K,m} < \infty\}.$$

It is endowed with the topology given by the representation

$$\mathcal{E}_{\{\omega\}}(G) = \operatorname{proj}_{\leftarrow K} \operatorname{ind}_{m\rightarrow} \mathcal{E}^m_{\{\omega\}}(K),$$

where K runs over all compact subsets of G.

Note that $\mathcal{E}_{\{\omega\}}(G)$ is a countable projective limit of (DFN)-spaces, which is ultrabornological, reflexive and complete. This follows from Rösner [14], Satz 3.25 and Vogt [16], Theorem 3.4.

The space $\mathcal{E}_{(\omega)}(G)$ of all *ω-ultradifferentiable functions of Beurling type* on G is defined as

$$\mathcal{E}_{(\omega)}(G) := \{f \in C^\infty(G) : \text{for each } K \subset G \text{ compact and each } m \in \mathbb{N}$$
$$p_{K,m}(f) := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-m\varphi^*(\frac{|\alpha|}{m})\right) < \infty\}.$$

It is easy to check that $\mathcal{E}_{(\omega)}(G)$ is a Fréchet space if we endow it with the locally convex topology given by the semi-norms $p_{K,m}$.

7. **Ultradifferentiable functions defined by weight sequences.** For a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ and an open set G in $\mathbb{R}^N$ the Carleman class $\mathcal{E}_{\{M_p\}}$ of *Roumieu type* on G is defined as

$$\mathcal{E}_{\{M_p\}}(G) := \left\{f \in C^\infty(G) : \text{ For each } K \subset G \text{ compact there is } h > 0 : \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^N}} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty\right\}$$

and the Carleman class $\mathcal{E}_{(M_p)}$ of *Beurling type* on G is defined as

$$\mathcal{E}_{(M_p)}(G) := \left\{f \in C^\infty(G) : \text{ For each } K \subset G \text{ compact and each } h > 0 : \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^N}} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty\right\}$$

8. **Notation.** Let ω be a weight function and let $(M_p)_{p\in\mathbb{N}_0}$ be a weight sequence. We will write $\mathcal{E}_{[\omega]}(G)$ (resp. $\mathcal{E}_{[M_p]}(G)$) if a statement holds in the Beurling case as well as in the Roumieu case.

Remark. For each $s > 1$ denote by $(M_p(s))_{p\in\mathbb{N}_0}$ the Gevrey sequence of exponent s, defined in 5.(1) and define the weight function $\omega_s(t) := |t|^{1/s}$, $t \in \mathbb{R}$. It is well-known that then for each open set G in $\mathbb{R}^n$ the identities

$$\mathcal{E}_{[M_p(s)]}(G) = \mathcal{E}_{[\omega_s]}(G)$$

hold as topological vector spaces. More generally, it was shown in Meise and Taylor [9], Remark 3.11, that for each weight sequence $(M_p)_{p\in\mathbb{N}_0}$ which satisfies (M2) and (M3), there is a weight function κ which is subadditive on $[0, \infty[$ such that $\mathcal{E}_{(M_p)}(G) = \mathcal{E}_{(\kappa)}(G)$ for each open set G in $\mathbb{R}^n$.

The aim of the present paper is to show that such an identity holds under much weaker conditions on the weight sequence $(M_p)_{p\in\mathbb{N}_0}$. To do so we have to introduce more notation.

9. **Definition.** Let $(M_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers tending to infinity. Then we define:

(a) The *associated function* $M : \mathbb{R} \to \mathbb{R}$ by

$$M(t) := \sup_{p\in\mathbb{N}} \log \frac{|t|^p}{M_p} \text{ if } t \neq 0 \text{ and } M(0) := 0.$$

(b) The sequence $(m_p)_{p\in\mathbb{N}}$ by $m_p := M_p/M_{p-1}$.

(c) The function $m : [0, \infty[\to \mathbb{N}_0$ by $m(t) := \#\{p \in \mathbb{N} : m_p \leq t\}$.

10. **Definition.** Let $\mathcal{F}$ be a space of functions defined on $\mathbb{R}$. Then we denote by $\mathcal{F}^{2\pi}$ the subspace of $\mathcal{F}$ consisting of all 2π-periodic functions.

In the following lemma we recall some facts that were proved in Petzsche [12], Satz 3.8, Langenbruch [7], Lemma 1.2, and Meise [8], Corollary 3.8.

11. **Lemma.** *Let $(M_p)_{p\in\mathbb{N}}$ be a weight sequence and ω a weight function. Then the isomorphisms in the following assertions are defined by $f \mapsto (\hat{f}_j)_{j\in\mathbb{Z}}$, where $\hat{f}_j$ is the j-th Fourier coefficient.*

(a) $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R})$ *is isomorphic to the Köthe sequence space*

$$\lambda_M := \Big\{x \in \mathbb{C}^{\mathbb{Z}} : \|x\|_k := \sum_{j\in\mathbb{Z}} |x_j| e^{M(kj)} < \infty \ \forall k \in \mathbb{N}\Big\}.$$

(b) $\mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ *is isomorphic to the power series space of infinite type*

$$\lambda_\omega := \left\{ x \in \mathbb{C}^{\mathbb{Z}} : \|x\|_k := \sum_{j\in\mathbb{Z}} |x_j| e^{k\omega(j)} < \infty \ \forall k \in \mathbb{N} \right\}.$$

(c) $\mathcal{E}^{2\pi}_{\{M_p\}}(\mathbb{R})$ *is isomorphic to the dual Köthe sequence space*

$$\kappa_M := \big\{x \in \mathbb{C}^{\mathbb{Z}} : \exists\, k \in \mathbb{N} : \sum_{j\in\mathbb{Z}} |x_j| e^{M(j/k)} < \infty\big\}.$$

(d) $\mathcal{E}^{2\pi}_{\{\omega\}}(\mathbb{R})$ *is isomorphic to the dual Köthe sequence space*

$$\kappa_\omega := \big\{x \in \mathbb{C}^{\mathbb{Z}} : \exists\, k \in \mathbb{N} : \sum_{j\in\mathbb{Z}} |x_j| e^{-\omega(j)/k} < \infty\big\}$$

which is isomorphic to the strong dual of $\Lambda_0((\omega(j))_{j\in\mathbb{N}})$.

Next we concentrate on the Beurling case.

12. **Lemma.** *Let $(M_p)_{p\in\mathbb{N}_0}$ be a weight sequence. Consider the following assertions:*

(1) *There exists a weight function ω such that $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R})$ is isomorphic to $\mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$.*
(2) *There exists $Q \in \mathbb{N}$ such that $\liminf_{j\to\infty} m_{Qj}/m_j > 1$.*
(3) *There exists $C > 1$ and $A > 0$ such that $m(2t) \le Cm(t) + A$, $t \ge 0$.*
(4) *The associated function M of the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies $M(2t) = O(M(t))$ as t tends to infinity.*
(5) *The associated function M of the sequence $(M_p)_{p\in\mathbb{N}_0}$ is a weight function.*

Then we have the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Rightarrow$ (2): By Lemma 11 it follows from (1) that $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R})$ is isomorphic to a power series space of infinite type. Hence it follows from Langenbruch [7], Theorem 3.1, that there exists $C \in \mathbb{N}$ such that for all large $p \in \mathbb{N}$ we have $2m_p \le m_{Cp}$. Hence (2) holds.

(2) $\Rightarrow$ (3): From (2) it follows that there exist $\varepsilon > 0$ and $p_0 \in \mathbb{N}$ such that $m_{Qj} \ge (1+\varepsilon)m_j$ for all $j \ge p_0$. Hence we have $m_{Q^2j} \ge (1+\varepsilon)^2 m_j$ for all $j \ge p_0$. Consequently, there exists $C = Q^\nu$ for some $\nu \in \mathbb{N}$ such that $2m_p \le m_{Cp}$ for all $p \ge p_0$. Find $t_0 > 0$ such that $m(t_0) > C(p_0+1)$. Fix $t \ge t_0$ and find the largest integer $p_1 \in \mathbb{N}$ such that $m_{p_1} \le 2t$. Find $q \in \mathbb{N}$ with $qC \le p_1 < (q+1)C$. Then we get

$$m_{C(p_0+1)} \le m_{m(t_0)} \le t_0 \le 2t$$

and hence $C(p_0+1) \le p_1$. By the choice of q this implies $q \ge p_0$ and consequently

$$2m_q \le m_{qC} \le m_{p_1} \le 2t,$$

which implies $m_q \le t$ and hence $q \le m(t)$. By the choice of q, this implies (3), since

$$m(2t) = p_1 < (q+1)C \le Cm(t) + C,\ t \ge t_0.$$

(3) $\Rightarrow$ (4): Note first that the sequence $(m_j)_{j\in\mathbb{N}}$ is increasing with $m_1 = M_1$ since $(M_p)_{p\in\mathbb{N}_0}$ satisfies condition (M1). By Komatsu [6], formula (3.11), and the definition of the function m we have for $t > 0$:

$$M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda = \int_{m_1}^t \frac{m(\lambda)}{\lambda} d\lambda.$$

From the hypothesis (3) we get the existence of $C > 1$ and $A > 0$ such that

$$m(2t) \le Cm(t) + A,\ t \ge 0.$$

The definition of the associated function M implies

$$M(t) \ge \log \frac{t}{M_1} = \log t - \log M_1,\ t > 0.$$

Hence we have

$$\begin{aligned} M(2t) &= \int_{m_1}^{2t} \frac{m(\lambda)}{\lambda} d\lambda = \int_{m_1/2}^t \frac{m(2s)}{s} ds \le \int_{m_1/2}^t \frac{Cm(s) + A}{s} ds \\ &= C\int_{m_1}^t \frac{m(s)}{s} ds + A(\log(t) - \log(m_1/2)) \\ &\le CM(t) + A(M(t) + \log M_1 + \log(2/M_1) \le (C+A)M(t) + A\log 2. \end{aligned}$$

This proves (4).

(4) $\Rightarrow$ (5): To show that M is a weight function, note first that $M_0 = 1$ implies the existence of $\delta > 0$ such that $M(t) = 0$ for $|t| \le \delta$. Next note that for each $m \in \mathbb{N}$ the supremum in the definition of $M(t)$, $t \in [-m, m]$, is in fact a maximum. Hence M is continuous on $\mathbb{R}$. From the definition of M it follows that M is increasing on $[0, \infty[$.

To check that M satisfies the conditions 2.(α)–(δ), note first that condition 2.(α) holds by (4). Then note that for each $t \in \mathbb{R}$ we have

$$M(e^t) = \sup_{p\in\mathbb{N}_0} \log(e^{pt}/M_p) = \sup_{p\in\mathbb{N}_0} (pt - \log M_p).$$

Hence $\varphi_M : t \mapsto M(e^t)$ is convex, i.e., condition 2.(δ) is satisfied. To prove condition 2.(β), define

$$\sigma(t) := \sup_{p\in\mathbb{N}_0} p \log\left(\frac{|t|}{p+1}\right),\ t \in \mathbb{R}.$$

It is well-known that there exists $D \ge 1$ such that $\sigma(t) \le Dt + D$. Therefore, condition (M0) implies for $t > 0$

$$M(t) = \sup_{p\in\mathbb{N}_0} \log(t^p/M_p) \le \sup_{p\in\mathbb{N}_0} \log(t^p/(c(p+1))^p) = \sigma\left(\frac{t}{c}\right) \le \frac{D}{c}t + D.$$

Hence condition 2.(β) holds. Since condition 2.(γ) is an obvious consequence of the definition of the associated function, we proved (5). □

13. **Proposition.** *Let $(M_p)_{p\in\mathbb{N}_0}$ be a weight sequence. Then the following assertions are equivalent:*

(1) *There is a weight function ω such that $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ as vector spaces.*
(2) *There is a weight function ω such that $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ as locally convex spaces.*
(3) *$(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2) and there exists $Q \in \mathbb{N}$ with $\liminf_{j\to\infty} m_{Qj}/m_j > 1$.*
(4) *$(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2), its associated function M is a weight function, and (1) holds with $\omega = M$.*

Moreover, whenever a weight function ω satisfies condition (1) then there exist $A \geq 1$ and $B > 0$ such that

$$(*) \qquad \frac{1}{A}M(t) - B \leq \omega(t) \leq AM(t) + B, \ t \geq 0.$$

Proof. (1) $\Rightarrow$ (2): It is easy to check that the identity map $id : \mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R}) \to \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ has closed graph, since a 2π-periodic C^∞-function is zero if all its Fourier coefficients vanish. Hence (2) follows from the open mapping theorem.

(2) $\Rightarrow$ (3): By Lemma 11, (2) implies $\lambda_M = \lambda_\omega$. Hence there exist $D > 0$ and $k \in \mathbb{N}$ as well as $E > 0$ and $\nu \in \mathbb{N}$ such that

$$\exp(M(j)) \leq D \exp(k\omega(j)) \text{ and } \exp(2k\omega(j)) \leq E \exp(M(\nu j)), \ j \in \mathbb{N}_0.$$

Consequently, we have

$$2M(j) \leq M(\nu j) + \log(ED^2) \text{ for all } j \in \mathbb{N}_0.$$

From this we get for $t > 0$ and $j \in \mathbb{N}_0$ such that $t \in [j, j+1]$

$$\begin{aligned} 2M(t) &\leq 2M(j+1) \leq M(\nu(j+1)) + \log(ED^2) \\ &\leq M(2\nu j) + M(\nu) + \log(ED^2) \leq M(2\nu t) + M(\nu) + \log(ED^2). \end{aligned}$$

By Komatsu [6], Proposition 3.6, this implies that $(M_p)_{p\in\mathbb{N}_0}$ satisfies the condition (M2). The second condition follows from Lemma 12.

(3) $\Rightarrow$ (4): The first two assertions follow from Lemma 12. To prove also the last one, we argue as follows. Since $(M_p)_{p\in\mathbb{N}}$ satisfies the condition (M2), it follows from Komatsu [6], Proposition 3.6, that

$$2M(t) \leq M(Ht) + \log(A), \ t > 0,$$

for the constants H and A which appear in (M2). Obviously, it is no restriction to assume $H \in \mathbb{N}$. Then we get for each $\nu \in \mathbb{N}$

$$(1) \qquad 2^\nu M(j) \leq M(H^\nu j) + 2^\nu \log(A), \ j \in \mathbb{N}.$$

Since M is a weight function, condition 2.(α) implies the existence of $K \in \mathbb{N}$ such that

$$M(2t) \leq KM(t) + K, \ t > 0.$$

This implies for each $\nu \in \mathbb{N}$

$$(2) \qquad M(2^\nu j) \leq K^\nu M(j) + \nu K^\nu, \ j \in \mathbb{N}.$$

From (1) and (2) it follows that the Köthe matrix $A := (\exp(M(kj)))_{j\in\mathbb{Z},k\in\mathbb{N}}$ defines the same sequence space $\lambda^1(A)$ as the matrix $B := (\exp(kM(j)))_{j\in\mathbb{Z},k\in\mathbb{N}}$. By Lemma 11, this implies $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(M)}(\mathbb{R})$, which proves (4).

(4) $\Rightarrow$ (1): This holds obviously.

To prove the additional statement, note that by the arguments in the implication (2) $\Rightarrow$ (3) and Lemma 12.(4), there exist $A_1 \geq 1$ and $A_2 \geq 1$ such that for each $j \in \mathbb{N}_0$ we have

$$M(j) \leq A_1\omega(j) + A_1 \text{ and } \omega(j) \leq A_2 M(j) + A_2.$$

Since M is a weight function by Lemma 12, it follows easily from this that $(*)$ holds. □

14. **Theorem.** *For each weight sequence $(M_p)_{p\in\mathbb{N}_0}$ the following assertions are equivalent:*

(1) *There exists a weight function ω such that for each $n \in \mathbb{N}$ and each open set G in $\mathbb{R}^n$ the spaces $\mathcal{E}_{[M_p]}(G)$ and $\mathcal{E}_{[\omega]}(G)$ are equal as vector spaces and/or as locally convex spaces.*

(2) *There exist a weight function ω, $n \in \mathbb{N}$, and an open set G in $\mathbb{R}^n$ such that the vector spaces $\mathcal{E}_{[M_p]}(G)$ and $\mathcal{E}_{[\omega]}(G)$ are equal.*

(3) *$(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2) and there exists $Q \in \mathbb{N}$ with $\liminf_{j\to\infty} m_{Qj}/m_j > 1$.*

(4) $(M_p)_{p\in\mathbb{N}_0}$ *satisfies (M2), its associated function* M *is a weight function, and (1) holds with* $\omega = M$.

Proof. We first prove the Theorem in the Beurling case. To do so, note that the implications (1) ⇒ (2) and (4) ⇒ (1) hold trivially.

(2) ⇒ (3): By the definition of the corresponding spaces, the present hypothesis implies $\mathcal{E}_{(M_p)}(G+x) = \mathcal{E}_{(\omega)}(G+x)$ for each $x \in \mathbb{R}^n$ and hence $\prod_{x\in\mathbb{R}^n} \mathcal{E}_{(M_p)}(G+x) = \prod_{x\in\mathbb{R}^n} \mathcal{E}_{(\omega)}(G+x)$. Since $\mathcal{E}_{(M_p)}$ and $\mathcal{E}_{(\omega)}$ are sheafs on $\mathbb{R}^n$, this implies that $\mathcal{E}_{(M_p)}(\mathbb{R}^n) = \mathcal{E}_{(\omega)}(\mathbb{R}^n)$. Since $\mathcal{E}_{(M_p)}(\mathbb{R})$ and $\mathcal{E}_{(\omega)}(\mathbb{R})$ can be identified with the subspace of this space consisting of all functions on $\mathbb{R}^n$ which depend only on the first variable, the hypothesis implies that $\mathcal{E}_{(M_p)}(\mathbb{R})$ and $\mathcal{E}_{(\omega)}(\mathbb{R})$ are equal as vector spaces. Obviously, this shows that $\mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$. Hence (3) holds by Proposition 13.

(3) ⇒ (4): The first two assertions in (4) hold by Proposition 13. To prove the last one note first that by Komatsu [6], Proposition 3.2, condition (M1) implies

$$M_p = \sup_{t>0} t^p \exp(-M(t)), \quad p \in \mathbb{N}_0. \tag{3}$$

Note further that $(M_p)_{p\in\mathbb{N}}$ satisfies the condition (M2) by hypothesis. Therefore it follows from Komatsu [6], Proposition 3.6, that for $t > 0$ we have

$$2M(t) \le M(Ht) + \log A, \tag{4}$$

where H and A are the constants from (M2). Of course it is no restriction to assume $H \in \mathbb{N}$. Next we claim that the following assertions hold:

$$\begin{gathered}\text{For each } 0 < h < 1 \text{ there exist } k \in \mathbb{N} \text{ and } C > 0 \text{ such that}\\ \exp(k\varphi^*_M(p/k)) \le Ch^p M_p,\ p \in \mathbb{N}_0.\end{gathered} \tag{5}$$

$$\begin{gathered}\text{For each } m \in \mathbb{N} \text{ there exist } 0 < h < 1 \text{ and } D > 0 \text{ such that}\\ h^p M_p \le D\exp(m\varphi^*_M(p/m)),\ p \in \mathbb{N}_0.\end{gathered} \tag{6}$$

Obviously, (5) and (6) imply condition (1) for $\omega = M$.

To prove (5), fix $0 < h < 1$ and choose $m \in \mathbb{N}$ such that $1/2^m \le h$. Since M is a weight function, there exists $K \in \mathbb{N}$ such that by formula (2)

$$M(2^m t) \le K^m M(t) + mK^m$$

and consequently

$$-M(2^m t) \ge -K^m M(t) - mK^m.$$

Using this and (3), we get

$$\begin{aligned}\log\Big(\frac{1}{2^{mp}}M_p\Big) &= \sup_{t>0}\Big(p\log(\frac{t}{2^m}) - M(t)\Big) = \sup_{\tau>0}(p\log\tau - M(2^m\tau))\\ &\ge \sup_{\tau>0}(p\log\tau - K^m M(t) - mK^m) \ge \sup_{x>0}(px - K^m\varphi_M(x)) - mK^m\\ &= K^m\varphi^*_M(\frac{p}{K^m}) - mK^m\end{aligned}$$

and hence

$$\exp\Big(K^m\varphi^*_M(\frac{p}{K^m})\Big) \le e^{mK^m}\frac{1}{2^{mp}}M_p \le e^{mK^m}h^p M_p,\ p \in \mathbb{N}_0.$$

This proves (5) with $k := K^m$ and $C := e^{mK^m}$.

To prove (6), note first that by Braun, Meise, and Taylor [4], Lemma 1.5, for $k \le l$ and each $p \in \mathbb{N}_0$ we have $k\varphi^*_M(p/k) \le l\varphi^*_M(p/l)$. Hence it suffices to prove (6) for $m = 2^k$, $k \in \mathbb{N}$. To do so, fix $k \in \mathbb{N}$ and let $h := 1/H^k$, where H is the constant from (4). Then (4) implies

$$2^k M(t) \le M(H^k t) + 2^k \log A$$

and hence

$$-M(H^k t) \le -2^k M(t) + 2^k \log A.$$

From this and (3) we get

$$\begin{aligned}\log\Big(\frac{M_p}{H^{kp}}\Big) &= \sup_{t>0}\Big(p\log(\frac{t}{H^k}) - M(t)\Big) = \sup_{\tau>0}(p\log\tau - M(H^k\tau))\\ &\le \sup_{\tau>0}(p\log\tau - 2^k M(\tau)) + 2^k\log A \le \sup_{x>0}(px - 2^k\varphi_M(x)) + 2^k\log A\\ &= 2^k\varphi_M^*(\frac{p}{2^k}) + 2^k\log A\end{aligned}$$

and consequently

$$h^p M_p = \Big(\frac{1}{H^{kp}}\Big)M_p \le A^{2^k}\exp\Big(2^k\varphi_M^*(\frac{p}{2^k})\Big),\ p \in \mathbb{N}_0.$$

This proves (6) for $m = 2^k$.

Next we prove the Theorem in the Roumieu case. As in the Beurling case, the implications (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) hold trivially.

(2) $\Rightarrow$ (3): The arguments that we used for the same implication in the Beurling case apply also in the Roumieu case and show that $\mathcal{E}^{2\pi}_{\{M_p\}}(\mathbb{R}) = \mathcal{E}^{2\pi}_{\{\omega\}}(\mathbb{R})$ as vector spaces. By the closed graph theorem, these spaces are equal even as locally convex spaces. By Lemma 11.(d), this shows that $\mathcal{E}^{2\pi}_{\{M_p\}}(\mathbb{R})$ is isomorphic to the strong dual of a power series space of finite type. Hence Langenbruch [7], Theorem 4.3, implies that $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2) and condition 12.(3). By Proposition 13, this implies (3).

(3) $\Rightarrow$ (4): This implication follows from a suitable modification of the arguments that we used in the same implication in the Beurling case. □

15. *Remark.* Theorem 14 extends a result of Petzsche, which was announced in Meise [8], 2.6 (2), but for which a proof was never published. Petzsche claimed that for a sequence $(M_p)_{p\in\mathbb{N}_0}$ which satisfies (M1) and (M2), the condition $M(2t) = O(M(t))$ for t tending to infinity is equivalent to the existence of $Q \in \mathbb{N}$ with $\liminf_{j\to\infty} m_{Qj}/m_j > 1$.
Note that Example 21 below shows that there exist weight sequences which do not satisfy (M2) but which satisfy condition 12.(2) and hence $M(2t) = O(M(t))$ for t tending to infinity.

16. **Corollary.** *For each weight function ω the following assertions are equivalent:*

(1) *There exists a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}$ and each open set G in $\mathbb{R}^n$ the spaces $\mathcal{E}_{[\omega]}(G)$ and $\mathcal{E}_{[M_p]}(G)$ are equal as vector spaces and/or locally convex spaces.*
(2) *There exists a weight sequence $(M_p)_{p\in\mathbb{N}_0}$, $n \in \mathbb{N}$, and an open set G in $\mathbb{R}^n$ such that the vector spaces $\mathcal{E}_{[\omega]}(G)$ and $\mathcal{E}_{[M_p]}(G)$ are equal.*
(3) *There exists $H \ge 1$ such that, for all $t \ge 0$,*

$$2\omega(t) \le \omega(Ht) + H,$$

and the sequence $(M_p)_{p\in\mathbb{N}_0}$, $M_p := \varphi_\omega(p)$, is a weight sequence for which (1) holds.

Proof. We first prove the corollary in the Beurling case. To do so, note first that the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold trivially.

(2) $\Rightarrow$ (3): Obviously, condition (2) implies that 14.(2) holds for the existing weight sequence $(M_p)_{p\in\mathbb{N}_0}$. By Theorem 14, also the condition 14.(4) holds. Hence $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2) and by the Proposition 13, also condition 13.($*$) holds. Consequently, it follows from Komatsu [6], Proposition 3.6, that there exist $H \ge 1$, $C > 0$, and $D > 0$ such that in the notation of 13.($*$) we have

$$2\omega(t) \le 2AM(t) + 2B \le \frac{1}{A}M(Ht) + D + 2B \le \omega(Ht) + C + DB,\ t \ge 0,$$

which implies the first condition in (3).
To show that also the second holds, note that condition 13.($*$) implies for $y \geq 0$:

$$\varphi^*_\omega(y) \leq \frac{1}{A}\varphi^*_M(Ay) + B \text{ and } \varphi^*_\omega(y) \geq A\varphi^*_M(\frac{y}{A}) - B.$$

From these estimates it follows that $\mathcal{E}_{(\omega)}(G) = \mathcal{E}_{(M)}(G)$ holds for each open set G in $\mathbb{R}^n$. Since $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2) and since M is a weight function, it follows from Theorem 14 that $\mathcal{E}_{(M)}(G) = \mathcal{E}_{(M_p)}(G)$ holds for each open set G in $\mathbb{R}^n$. Hence we proved (3).

The proof of the Corollary in the Roumieu case is the same as in the Beurling case, since Theorem 14 holds for both cases. □

17. **Example.** There exists a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ which satisfies the conditions (M2) and (M3)' such that for each weight function ω, each $n \in \mathbb{N}$, and each open set G in $\mathbb{R}^n$ we have $\mathcal{E}_{(M_p)}(G) \neq \mathcal{E}_{(\omega)}(G)$.

To show this, let $(M_p)_{p\in\mathbb{N}_0}$ be the sequence which is constructed in Langenbruch [7], Example 3.3. It satisfies (M1), (M2), and (M3)'. By its definition it satisfies $m_p \geq p^2$ and hence (M0). Hence it is a weight sequence. If we assume that for some weight function ω, some $n \in \mathbb{N}$, and some open set G in $\mathbb{R}^n$ we have $\mathcal{E}_{(M_p)}(G) = \mathcal{E}_{(\omega)}(G)$, then it follows from Corollary 14 that there exists $Q \in \mathbb{N}$ such that $\liminf_{j\to\infty} m_{Qj}/m_j > 1$. From this it follows as in the proof of Lemma 12 that there exists $C \in \mathbb{N}$ such that

$$2m_p \leq m_{Cp} \text{ for all large } p \in \mathbb{N}. \tag{7}$$

However, it was shown in [7], Example 3.3, that (7) does not hold for this sequence $(M_p)_{p\in\mathbb{N}_0}$. Hence it has the property claimed above.

To show that there are a number of weight functions ω for which $\mathcal{E}_{(\omega)}(G) \neq \mathcal{E}_{(M_p)}(G)$ for each weight sequence $(M_p)_{p\in\mathbb{N}_0}$ and each open set G in $\mathbb{R}^n$, we have to recall some definitions.

18. **Definition.** Let ω be a weight function.

(a) We say that ω is a (DN)-weight function, if for each $C > 1$ there exist $\delta > 0$ and $R_0 > 0$ such that

$$\omega^{-1}(CR)\omega^{-1}(\delta R) \leq (\omega^{-1}(R))^2 \text{ for all } R \geq R_0.$$

(b) We say that ω has the poperty (ε) if it satisfies:

(ε) There exists $C > 0$ such that $\displaystyle\int_1^\infty \frac{\omega(yt)}{t^2}dt \leq C\omega(y) + C$ for each $y > 0$.

19. **Lemma.** *Let ω be a weight function which satisfies (ε) and the condition 16.(3). Then ω is a (DN)-weight function.*

Proof. By Corollary 16, there is a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ such that $\mathcal{E}_{(\omega)}(\mathbb{R}) = \mathcal{E}_{(M_p)}(\mathbb{R})$. Then it follows from Theorem 14 that its associated function M is a weight function and that $\mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(M)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R})$. Since ω satisfies (ε), it follows from Meise and Taylor [9], Theorem 3.10, that the Borel map

$$B : \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R}) \to \mathbb{C}^{\mathbb{N}_0},\ B(f) := (f^{(j)}(0))_{j\in\mathbb{N}_0}.$$

has the image

$$\Lambda_\omega := \Big\{x = (x_j)_{j\in\mathbb{N}_0} : \|x\|_m := \sup_{j\in\mathbb{N}} |x_j| \exp(-m\varphi^*(j/m)) < \infty \text{ for each } m \in \mathbb{N}\Big\}.$$

Next note that by Proposition 13 the equality $\mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(M_p)}(\mathbb{R})$ implies condition 13.($*$). From it we get that the Köthe matrices $(k\varphi^*_M(j/k))_{j\in\mathbb{N},k\in\mathbb{N}}$ and $(k\varphi^*_\omega(j/k))_{j\in\mathbb{N},k\in\mathbb{N}}$ define the same Köthe sequence space. Since the present hypothesis also implies that the assertions (5) and (6) hold, we get that

$$\Lambda_\omega = \lambda^\infty\Big(\big(k\varphi^*_\omega(j/k)\big)_{j,k\in\mathbb{N}}\Big) = \lambda^\infty\Big(\big(k\varphi^*_M(j/k)\big)_{j,k\in\mathbb{N}}\Big).$$

On the other hand, the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies condition (γ_1) of Petzsche [13], since ω satisfies (ϵ). By Petzsche [13], Theorem 2.1, this implies that the image of B is equal to the Fréchet space

$$\left\{x = (x_j)_{j\in\mathbb{N}_0} : \ |x|_m := \sup_{j\in\mathbb{N}_0} \frac{m^j|x_j|}{M_j} < \infty \text{ for each } m \in \mathbb{N}\right\}.$$

Hence Λ_ω is isomorphic to the power series space $\Lambda_\infty((j)_{j\in\mathbb{N}_0})$ of infinite type. Since $\Lambda_\omega = A_\omega(\mathbb{C})'_b$ in the notation of Meise and Taylor [10], it follows from [10], Theorem 3.4, that ω is a (DN)-weight function. □

20**. Example.** For each $s > 1$ the function $\omega(t) := \max(0, (\log|t|)^s)$ is a weight function which satisfies (ε) by Meise and Taylor [9], Example 1.8 (a), and which is not a (DN)-weight function by Meise and Taylor [10], Example 3.5 (5). Hence Lemma 19 together with Corollary 16 implies

$$\mathcal{E}_{(\omega)}(G) \neq \mathcal{E}_{(M_p)}(G)$$

for each weight sequence $(M_p)_{p\in\mathbb{N}_0}$, each $n \in \mathbb{N}$, and each open set G in $\mathbb{R}^n$.

Of course, this fact follows also from Corollary 16, since ω does not satisfy the condition in 16.(3).

The next example shows that there are weight sequences $(M_p)_{p\in\mathbb{N}_0}$ for which the associated function M is a weight function but for which $\mathcal{E}_{(M_p)}(G) \neq \mathcal{E}_{(M)}(G)$ for each open set G in $\mathbb{R}^n$.

21**. Example.** For $s > 1$ and $p \in \mathbb{N}_0$ define $M_p := \exp(p^s)$. Then $(M_p)_{p\in\mathbb{N}_0}$ is a weight sequence having the following properties:

(1) The associated function M of the sequence $(M_p)_{p\in\mathbb{N}_0}$ is a weight function. There exist $A \geq 1$ and $t_0 > 0$ such that

$$\frac{1}{A}(\log(1+|t|))^{s/(s-1)} \leq M(t) \leq A(\log(1+|t|))^{s/(s-1)}, \ |t| > t_0.$$

(2) $(M_p)_{p\in\mathbb{N}_0}$ does not satisfy (M2).

It is easy to check that $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M0), (M1), and (M2)'. From $m_p = e^{2p-1}$ it follows that $m_{2p}/m_p = e^{2p}$. Hence condition 12.(2) is satisfied and by Lemma 12, M is a weight function. The upper and lower estimate for M can be checked directly and were stated already in Meise [8], 2.6 (4). Thus (1) holds.

(2) follows by direct computation. It also follows from part (1), Example 20, and Proposition 13.

To show that some quasianalytic classes that provided interesting examples in [3] can be defined also by weight sequences, we need some preparation.

22**. Lemma.** *Let $g : [0,\infty[\to [1,\infty[$ be a continuous increasing function which satisfies $g(0) = 1$ and has the following additional properties*

(1) $g(2x) = O(g(x))$ *as x tends to infinity,*

(2) $\limsup_{x\to\infty} \left(\dfrac{g(x+1)}{g(x)}\right)^x < \infty$,

(3) $h : [0,\infty[\to [0,\infty[$, $h(x) := x\log((x+1)g(x))$ *is convex.*

Then the associated function M of the sequence $(M_p)_{p\in\mathbb{N}_0}$ defined by $M_p := ((p+1)g(p))^p$ is a weight function which satisfies (M2) and condition 12.(2). Moreover, if $\sum_{p=1}^\infty 1/pg(p) = \infty$ then $(M_p)_{p\in\mathbb{N}_0}$ is quasianalytic.

Proof. We have to show that $(M_p)_{p\in\mathbb{N}_0}$ satisfies the conditions (M0), (M1), (M2), and 12.(2). Note first that (M0) holds since $g(p)\geq 1$ for all $p\in\mathbb{N}_0$ by hypothesis. Next note that

$$\log M_p = p\log((p+1)g(p)) = h(p),\ p\in\mathbb{N}_0. \tag{8}$$

Since h is convex by hypothesis, it follows easily that $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M1).

To prove (M2), fix $p\in\mathbb{N}$. Then (8) implies

$$\mu_p := \min_{x\in[0,p]}(h(x)+h(p-x)) \leq \min_{0\leq q\leq p}(\log M_q + \log M_{p-q}).$$

Since h is convex by (3), the function

$$h_p : [0,p]\to\mathbb{R},\ h_p(x) := h(x)+h(p-x)$$

is convex and symmetric with respect to $\frac{p}{2}$, i.e., $h_p(\frac{p}{2}+y) = h_p(\frac{p}{2}-y)$ for $y\in[0,\frac{p}{2}]$. Consequently,

$$\mu_p = h_p(\frac{p}{2}) = 2h(\frac{p}{2}) = p\log\left((\frac{p}{2}+1)g(\frac{p}{2})\right),$$

and hence

$$\min_{0\leq q\leq p} M_q M_{p-q} \geq \left((\frac{p}{2}+1)g(\frac{p}{2})\right)^p.$$

Now note that because of (1) we can choose $p_1\in\mathbb{N}$ and $H>1$ such that

$$g(p)\leq Hg(\frac{p}{2}) \text{ for } p\geq p_1.$$

Hence we get for $p\geq p_1$

$$M_p = ((p+1)g(p))^p \leq (2H)^p\left(\frac{p+1}{2}g(\frac{p}{2})\right)^p \leq (2H)^p \min_{0\leq q\leq p} M_q M_{p-q}.$$

Obviously, this implies that $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M2).

To show that $(M_p)_{p\in\mathbb{N}}$ satisfies the condition 12.(2), note that by (12) we have

$$m_p = pg(p)(1+\frac{1}{p})^p\left(\frac{g(p)}{g(p-1)}\right)^{p-1},\ p\in\mathbb{N}. \tag{9}$$

Then we use condition (2) to choose $\nu\in\mathbb{N}$, $\nu\geq 6$, such that

$$\left(\frac{g(p)}{g(p-1)}\right)^{p-1} \leq \frac{\nu}{2e} \text{ for } p\in\mathbb{N}. \tag{10}$$

For $p\in\mathbb{N}$ this implies

$$m_p \leq pg(p)e\frac{\nu}{2e} = \frac{\nu}{2}pg(p).$$

Since g is increasing, we get

$$m_{\nu p} = \nu pg(\nu p)(1+\frac{1}{\nu p})^{\nu p}\left(\frac{g(\nu p)}{g(\nu p-1)}\right)^{\nu p-1} \geq \nu pg(p)\cdot 2 \geq 4m_p$$

for $p\in\mathbb{N}$ and consequently

$$\liminf_{p\to\infty}\frac{m_{\nu p}}{m_p} > 2. \tag{11}$$

Thus, the condition in 12.(2) holds. By Lemma 12, this proves that M is a weight function.

To show that $(M_p)_{p\in\mathbb{N}_0}$ is quasianalytic if $\sum_{p=1}^\infty 1/pg(p) = \infty$, note first that

$$\frac{M_{p-1}}{M_p} = \frac{1}{pg(p)}\frac{1}{(1+\frac{1}{p})^p}\left(\frac{g(p-1)}{g(p)}\right)^{p-1}. \tag{12}$$

Then let $\mu = \limsup_{x\to\infty} \left(\frac{g(x+1)}{g(x)}\right)^x$. Obviously, $\mu \geq 1$ and without restriction we can assume that p_2 is so large that

$$\left(\frac{g(p)}{g(p-1)}\right)^{p-1} \leq 2\mu \text{ for } p \geq p_2.$$

Hence we get from (12) that

$$\frac{M_{p-1}}{M_p} \geq \frac{1}{e2\mu} \cdot \frac{1}{pg(p)} \text{ for } p \geq p_2.$$

Therefore, the present hypothesis shows that $(M_p)_{p\in\mathbb{N}_0}$ is quasianalytic. □

From Proposition 13 and Lemma 22 it is clear that the associated function M of the sequence $(M_p)_{p\in\mathbb{N}_0}$ defined in Lemma 22 is a weight function. Next we show that it can be computed in terms of the function g, if g satisfies two additional conditions.

23. Proposition. *Let $g : [0,\infty[\to [1,\infty[$ be a continuous increasing function which satisfies the assumptions of Lemma 22 and has the following two properties:*

(a) *there is $D > 0$ such that $g(xg(x)) \leq Dg(x)$ for all $x \geq 0$,*
(b) *the function $\varphi : y \mapsto y/g(y)$ is increasing for $y \geq y_0 > 0$.*

Let $M_p := ((p+1)g(p))^p$, $p \in \mathbb{N}_0$. Then there is $B \geq 1$ such that the associated function M satisfies

$$\frac{1}{B}M(t) \leq \frac{t}{g(t)} \leq BM(t),\ t \geq B.$$

Proof. We keep the notations of the proof of Lemma 22 and note that (11) holds. Since $(M_p)_{p\in\mathbb{N}_0}$ satisfies (M1), the sequence $(m_p)_{p\in\mathbb{N}}$ is increasing. It is easy to check that

$$M(t) = \log \frac{t^{p-1}}{M_{p-1}},\ m_{p-1} \leq t \leq m_p,\ p \geq 2,$$

where m_p is given by (9). Hence, for $p \geq 2$

$$M(m_p) = (p-1)\log\left[\frac{g(p)}{g(p-1)}\left(1+\frac{1}{p}\right)^p\left(\frac{g(p)}{g(p-1)}\right)^{p-1}\right].$$

By (10), there is $A_1 \geq 1$ with

$$\frac{p-1}{A_1} \leq M(m_p) \leq A_1(p-1),\ p \geq 2. \tag{13}$$

By (9) and (10) we get $C \geq 1$ with

$$pg(p) \leq m_p \leq Cpg(p),\ p \in \mathbb{N}. \tag{14}$$

In particular $p \leq m_p$ for $p \in \mathbb{N}$. Moreover, since g is increasing we have for $p \geq 2$:

$$m_p \leq Cpg(p) \leq Cpg(m_p) \leq 2C(p-1)g(m_p).$$

Therefore

$$\varphi(m_p) = m_p/g(m_p) \leq 2C(p-1),\ p \geq 2.$$

For $p \in \mathbb{N}$ we have, by (14),

$$(p-1)g(m_p) \leq (p-1)g(Cpg(p)) \leq pg(Cpg(p)).$$

Next we apply condition (a) together with condition 22.(1) to find $D_1 \geq D$ such that

$$pg(Cp(g(p))) \leq pg(Cpg(Cp)) \leq pDg(Cp) \leq D_1pg(p) \leq D_1m_p$$

and hence

$$p-1 \leq D_1\varphi(m_p),\ p \in \mathbb{N}.$$

If $A_2 := \max(2C, D_1)$, we have

$$\frac{1}{A_2}(p-1) \leq \varphi(m_p) \leq A_2(p-1),\ p \geq 2. \tag{15}$$

Now, by (10), there is $\nu \in \mathbb{N}$ such that, for $p \geq 2$,

$$m_p \leq \frac{\nu}{2} p g(p) \leq \nu(p-1)g(p).$$

Since $\lim_{p\to\infty}(\frac{\nu}{2e})^{1/p-1} = 1$, we get $p_4 \geq 2$ such that for $p \geq p_4$ it follows from (10) and (14) that

$$m_p \leq 2\nu(p-1)g(p-1) \leq 2\nu m_{p-1}.$$

Therefore, there is $S \in \mathbb{N}$, such that

$$m_p \leq S m_{p-1},\ p \geq p_4. \tag{16}$$

Moreover, as g is increasing we have for $t \geq 0$,

$$\varphi(2t) = 2t/g(2t) \leq 2t/g(t) = 2\varphi(t). \tag{17}$$

Now fix $p \geq p_4 + 1$ with $m_{p-1} \geq y_0$, $y_0 > 0$ given by the condition (b). If $t \in [m_{p-1}, m_p]$ we get from (15), (13), and 16

$$\begin{aligned} t/g(t) = \varphi(t) \leq \varphi(m_p) \leq A_2(p-1) \leq A_1A_2M(m_p) \\ \leq A_1A_2M(Sm_{p-1}) \leq A_1A_2B_1M(m_{p-1}) \leq A_1A_2B_1M(t), \end{aligned}$$

for some $B_1 \geq 1$ which exists because M is a weight satisfying 2.1 (α) by Proposition 13. On the other hand, it follows from ((13)) and (15) that

$$\begin{aligned} M(t) \leq M(m_p) \leq A_1(p-1) \leq A_1A_2\varphi(m_p) \leq A_1A_2\varphi(Sm_{p-1}) \\ \leq A_1A_2B_2\varphi(m_{p-1}) \leq A_1A_2B_2\varphi(t) = A_1A_2B_2 t/g(t), \end{aligned}$$

for some $B_2 \geq 1$ which exists by (17). □

In order to derive examples from Proposition 23, we will use the following lemma.

24. **Lemma.** *Let $g : [0, \infty[\to [1, \infty[$ be a continuously differentiable increasing function with $g(0) = 1$.*

(i) *If $A := \sup_{x\geq 0}(xg'(x))/g(x) < \infty$ then the function g satisfies the condition 22.(2).*

(ii) *If $B := \sup_{x\geq 0}(xg'(x)) < \infty$ then the function g satisfies the conditions (a) and (b) in Proposition 23.*

Proof. (i) We put $\varepsilon(x) := (xg'(x))/g(x)$, $x \geq 0$. Then for $x > 0$ the equality

$$g(x) = \exp\left(\int_0^x \frac{\varepsilon(t)}{t} dt\right)$$

holds. From it and condition (i) we get the following estimate

$$\left(\frac{g(x+1)}{g(x)}\right)^x = \left(\exp\left(\int_x^{x+1} \frac{\varepsilon(t)}{t} dt\right)\right)^x \leq \left(\exp\left(A\log\frac{x+1}{x}\right)\right)^x = \left(1+\frac{1}{x}\right)^{Ax} \leq e^A.$$

Hence the condition (2) of Lemma 22 is satisfied.

(ii) By the properties of g it follows from (ii) that for $x > 0$ we have the estimate

$$\frac{g(xg(x))}{g(x)} = \exp\left(\int_x^{xg(x)} \frac{tg'(t)}{tg(t)}dt\right) \le \exp\left(B\frac{1}{xg(x)}(xg(x)-x)\right) \le e^B.$$

Consequently the condition (a) of Proposition 23 holds.

The condition (b) in Proposition 23 is valid since

$$\left(\frac{y}{g(y)}\right)' = \frac{g(y)-yg'(y)}{(g(y))^2} \ge \frac{g(y)-B}{(g(y))^2} > 0$$

for large $y > 0$. □

We remark that the integral representation for the function g in the proof of Lemma 24 plays an important role in the theory of regularly varying functions (see for example [15], Ch.I, 1.2).

25. **Example.** For $k \in \mathbb{N}_0$ define recursively

$$e_0 := 1, e_k := \exp(e_{k-1}), \log_0 x := x, \log_k x := \log(\log_{k-1} x).$$

Then fix $s \in \mathbb{N}$, $\alpha \in]0,1]$, if $s = 1$ and $\alpha > 0$ if $s \ge 2$, and define the sequence $(M_p(\alpha,s))_{p\in\mathbb{N}_0}$ by

$$M_p(\alpha,s) := \left((p+1)(\log_s(e_s+p))^\alpha\right)^p.$$

Then there exists $C_{\alpha,s} \ge 1$ such that the associated function $M_{\alpha,s}$ of $(M_p(\alpha,s))_{p\in\mathbb{N}_0}$ satisfies

$$\frac{1}{C_{\alpha,s}}M_{\alpha,s}(t) \le \frac{|t|}{(\log_s(e_s+|t|))^\alpha} \le C_{\alpha,s}M_{\alpha,s}(t),\ \ |t| \ge C_{\alpha,s}.$$

To show this, we put $g_\alpha(t) := (\log_s(e_s+t))^\alpha$, $t \ge 0$ and show that the function g_α satisfies all the conditions in Lemma 22 and Proposition 23. Then the assertion follows from Proposition 23.

Note first that condition (1) in Lemma 22 holds obviously. Condition (2) of Lemma 22 holds by Lemma 24 (i). To check the condition (3) in Lemma 22 we show that for arbitrary $s \in \mathbb{N}$ the (infinitely differentiable) function $h(x) := x\log(\log_s(e_s+x))$ is convex on $[0,\infty[$.

For all $x \ge 0$ we have

$$h''(x) = (e_s+x)^{-2}\left[2(e_s+x)\left(\prod_{j=1}^s \log_j(e_s+x)\right)^{-1} - x\left(\prod_{j=1}^s \log_j(e_s+x)\right)^{-2} - \right.$$

$$\left.\sum_{m=1}^{s-1} x\left(\prod_{j=m+1}^s \log_j(e_s+x)\right)^{-1}\left(\prod_{j=1}^m(\log_j(e_s+x)\right)^{-2} - x\left(\prod_{j=1}^s \log_j(e_s+x)\right)^{-1}\right].$$

Hence for all $x \ge 0$

$$h''(x) \ge \left(\prod_{j=1}^s \log_j(e_s+x)\right)^{-1}(e_s+x)^{-1}\left(2-\sum_{m=0}^s e^{-m}\right) \ge 0.$$

Consequently the function h is convex on $[0,\infty[$. From this and the convexity of the function $x \mapsto x\log(x+1)$ it follows that the function $x \mapsto x\log((x+1)g_\alpha(x))$ is convex on $[0,\infty[$, too.

The conditions (a) and (b) in Proposition 23 hold by Lemma 24 (ii).

Acknowledgements: S. Melikhov thanks DAAD for financial support.
The research of J. Bonet and R. Meise was partially supported by FEDER and MEC, Project MTM 2004–02262.

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Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, E - 46071, Valencia, Spain

E-mail address: jbonet@mat.upv.es

Mathematisches Institut, Heinrich-Heine-Universität, Universitätsstrasse 1, D-40225 Düsseldorf, Germany

E-mail address: meise@math.uni-duesseldorf.de

Department of Mechanics and Mathematics, Rostov State University, Zorge st. 5, 344090 Rostov on Don, Russia, and Vladikavkaz institute of applied mathematics and computer science with VNTS of Russ. Acad. Sci.

E-mail address: melih@ms.math.rsu.ru