SCHAUDER DECOMPOSITIONS AND THE GROTHENDIECK AND DUNFORD-PETTIS PROPERTIES IN KÖTHE ECHELON SPACES OF INFINITE ORDER

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ABSTRACT. It is shown that every echelon space $\lambda_\infty(A)$, with A an arbitrary Köthe matrix, is a Grothendieck space with the Dunford-Pettis property. Since $\lambda_\infty(A)$ is Montel if and only if it coincides with $\lambda_0(A)$, this identifies an extensive class of non-normable, non-Montel Fréchet spaces having these two properties. Even though the canonical unit vectors in $\lambda_\infty(A)$ fail to form an unconditional basis whenever $\lambda_\infty(A) \neq \lambda_0(A)$, it is shown, nevertheless, that in this case $\lambda_\infty(A)$ still admits unconditional Schauder decompositions (provided it satisfies the density condition). This is in complete contrast to the Banach space setting, where Schauder decompositions never exist. Consequences for spectral measures are also given.

1. INTRODUCTION

The class of Banach spaces which are *Grothendieck spaces* with the *Dunford-Pettis property* (briefly, GDP-spaces) plays a prominent role in the theory of Banach spaces and vector measures; see Chapter VI of [15], especially the Notes and Remarks, and [14], for example. Well known examples of GDP-spaces include L^∞, $H^\infty(\mathbb{D})$, injective Banach spaces (eg. ℓ^∞) and certain $C(K)$ spaces. A sequence $\{P_n\}_{n=1}^\infty$ of continuous projections on a Banach space X is called a *(weak) Schauder decomposition* if:

(S1) $P_m P_n = P_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}$,
(S2) $\{P_n x\}$ converges (weakly) to x for each $x \in X$, and
(S3) $P_m \neq P_n$ for $m \neq n$.

Usually, a (weak) Schauder decomposition is defined in terms of closed subspaces of X, [10]. We prefer the equivalent operator theoretic definition, [20, 21]. D. Dean showed in [10] that a GDP-space does not

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have any (weak) Schauder decomposition; see also [21, Corollary 8]. As a consequence, spectral measures in GDP-spaces are of a rather special nature; they are necessarily countably additive for the operator norm topology and hence, are trivial, in the sense that they can assume only *finitely many* distinct values, [25].

Suppose now that X is a Fréchet-Montel (locally convex) space. It follows from the definitions (see Section 2) that X is necessarily a GDP-space. Hence, Schauder decompositions surely exist in many such spaces X, that is, Dean's result for Banach spaces does not extend to this class of spaces. Moreover, the Montel property ensures that all spectral measures in X are necessarily countably additive for the topology τ_b in $L(X)$ of uniform convergence on all bounded subsets of X, [28]; here $L(X)$ denotes the vector space of all continuous linear operators from X into itself. However, unlike for Banach spaces (where τ_b is the operator norm topology), it is surely not the case that all such spectral measures assume only finitely many distinct values.

Other than Montel spaces, there seem to be no other known examples of *non-normable* Fréchet spaces which are GDP-spaces. One of our aims in this note is to exhibit a class of such spaces, namely *all* Köthe echelon spaces of the kind $\lambda_\infty(A)$. Since it is known precisely when $\lambda_\infty(A)$ is Montel, namely whenever $\lambda_\infty(A) = \lambda_0(A)$ (c.f. also Proposition 2.3), this automatically produces a whole class of *non-Montel*, non-normable Fréchet spaces which are GDP-spaces. For the class of Köthe echelon spaces $\lambda_\infty(A)$, it also turns out that *every* spectral measure is τ_b-countably additive. Indeed, we establish this result for arbitrary GDP-Fréchet spaces (c.f. Proposition 4.3). Moreover, via the *density condition*, we identify a (non-empty!) subclass of the non-Montel $\lambda_\infty(A)$ spaces in which even *unconditional* Schauder decompositions always exist; see Proposition 4.4. This subclass is then of some interest; it genuinely contrasts the situation for GDP-*Banach* spaces. Moreover, in such spaces, the existence of Schauder decompositions is *not* "forced" by the rather strong Montel property; it is intrinsic to the spaces themselves. As a consequence, non-trivial spectral measures exist (even τ_b-countably additive), which then allows for a rich theory of spectral operators in such spaces, [24], [27], [29].

2. Preliminaries

For Fréchet spaces X and Y (always assumed to be locally convex) we denote the vector space of all continuous linear maps from X into Y by $L(X, Y)$; if $X = Y$, then we simply write $L(X)$. The topology of uniform convergence on all finite (resp. bounded) subsets of X is denoted by τ_s (resp. τ_b) and $L_s(X, Y)$ (resp. $L_b(X, Y)$) denotes $L(X, Y)$ equipped with the locally convex Hausdorff topology τ_s (resp.τ_b). If $Y = \mathbb{C}$, then $L_b(X, \mathbb{C})$ is also denoted by X'_β (the strong dual space of X).

A Fréchet space X is called a *Grothendieck space* if every sequence in the dual space X' which is convergent for the weak* topology $\sigma(X', X)$ is also convergent for the weak topology $\sigma(X', X'')$. A Fréchet space X is said to have the *Dunford-Pettis property* (briefly, DP-property) if every element of $L(X, Y)$, for Y an arbitrary quasicomplete locally convex Hausdorff space, which transforms bounded subsets of X into relatively weakly compact subsets of Y, also transforms weakly compact subsets of X into relatively compact subsets of Y, [16, pp.633–634]. Actually, it suffices if Y simply runs through the class of all Banach spaces. For, assume that we have the property for all Banach spaces. Let Y be a quasicomplete locally convex Hausdorff space and $T \in L(X, Y)$ map bounded sets into relatively weakly compact sets in Y. Now, Y is a topological subspace of a product Z of a family $\{Y_j\}_{j\in J}$ of Banach spaces. Denote by $p_j : Z \to Y_j$ the natural projection. For each $j \in J$, the map $p_j \circ T \in L(X, Y_j)$ maps bounded subsets of X into relatively weakly compact sets in X_j (as p_j is weakly continuous). By the property assumed of X (for Banach spaces) we have that $p_j \circ T$ maps relatively weakly compact subsets of X into relatively compact sets in Y_j. Let $C \subseteq X$ be relatively weakly compact. Then $T(C) \subseteq Y$ and $p_j(T(C))$ is relatively compact in Y_j for each $j \in J$. By Tychonoff's theorem, $T(C)$ is relatively compact in Z. Since Y is quasicomplete, the closure of $T(C)$ in Z is actually contained in Y. This implies that $T(C)$ is relatively compact in Y. So, the classical form of the DP-property as given in [16, pp.633–634] follows.

Lemma 2.1. *Let X be a Fréchet space.*

(i) *If every $\sigma(X', X)$-null sequence in X' is also $\mu(X', X)$-null, then X has the DP-property. Here, μ denotes the Mackey topology.*
(ii) *X has the DP-property if and only if for every $\sigma(X, X')$-null sequence $\{x_n\} \subseteq X$ and every $\sigma(X', X'')$-null sequence $\{\xi_n\} \subseteq X'$, we have $\langle x_n, \xi_n\rangle \to 0$ as $n \to \infty$.*
(iii) *If X is a GDP-space, then every $\sigma(X', X)$-null sequence in X' is also $\mu(X', X)$-null.*
(iv) *If X is a GDP-space, then every complemented subspace of X is also a GDP-space.*

Proof. (i) and (iii). See [8, Proposition 11].

(ii) For Banach spaces this is the classical Brace-Grothendieck theorem; see [15, p.177] or [16, pp.635–636], for example. For general Fréchet spaces, see [8, p.397].

(iv) Let Y be a complemented subspace of X. That Y has the DP-property is known, [16, p.635].

Denote by $j : Y \to X$ the inclusion map and by $P : X \to Y$ a projection of X onto Y, in which case $P' : Y' \to X'$ denotes the *dual projection*. Let $\{\xi_n\} \subseteq Y'$ be any $\sigma(Y', Y)$-null sequence and define $v_n := P'\xi_n$, for $n \in \mathbb{N}$, in which case $\{v_n\} \subseteq X'$ is a $\sigma(X', X)$-null sequence. Since X is a Grothendieck space, we conclude that $\{v_n\}$ is a $\sigma(X', X'')$-null sequence. Fix $z \in Y''$, so that $z : Y'_\beta \to \mathbb{C}$ is continuous. The dual map $j' : X'_\beta \to Y'_\beta$ is continuous and hence, so is $z \circ j' : X'_\beta \to \mathbb{C}$. Accordingly, $\langle v_n, z \circ j' \rangle \to 0$ as $n \to \infty$. But,

$$\langle \xi_n, z \rangle = \langle \xi_n, z \circ j' \circ P' \rangle = \langle P'\xi_n, z \circ j' \rangle = \langle v_n, z \circ j' \rangle, \qquad n \in \mathbb{N},$$

and so $\langle \xi_n, z \rangle \to 0$ as $n \to \infty$, that is, $\{\xi_n\}$ is a $\sigma(Y', Y'')$-null sequence. This shows that Y is a Grothendieck space. □

Remark 2.2. Let X be a Fréchet-Montel space. Then X is reflexive, [19, p.369], and hence, is a Grothendieck space. Moreover, it follows from Lemma 2(i) that X has the DP-property, [8, p.397]. So, all Fréchet-Montel spaces are GDP-spaces. □

Let I be an index set, *always* assumed to be countable. An increasing sequence $A = (a_n)_{n=1}^\infty$ of functions $a_n : I \to (0, \infty)$ is called a *Köthe matrix* on I, where by increasing we mean

$$0 < a_n(i) \le a_{n+1}(i), \qquad i \in I,\ n \in \mathbb{N}.$$

The *Köthe echelon space* $\lambda_\infty(A)$ is defined as the vector space

$$\lambda_\infty(A) := \{x \in \mathbb{C}^I : a_n x \in \ell^\infty(I) \text{ for all } n \in \mathbb{N}\},$$

equipped with the increasing sequence of seminorms $\|\cdot\|_1 \le \|\cdot\|_2 \le \cdots$, where

$$\|x\|_k := \sup_{i \in I} a_k(i)|x_i|, \qquad x \in \lambda_\infty(A), \tag{1}$$

and elements $x \in \mathbb{C}^I$ are denoted by $x = (x_i)$. Then $\lambda_\infty(A)$ is a Fréchet space, [23, Ch.27]. We will also require its closed subspace

$$\lambda_0(A) := \{x \in \lambda_\infty(A) : a_n x \in c_0(I) \text{ for all } n \in \mathbb{N}\},$$

equipped with the seminorms (1) restricted to $\lambda_0(A)$. The *canonical unit vectors* in $\lambda_\infty(A)$, which also belong to $\lambda_0(A)$, are the vectors $\{e_j\}_{j \in I}$ given by $e_j(i) := \delta_{ij}$ for all $i \in I$. They form an unconditional

Schauder basis for $\lambda_0(A)$. In particular, the Fréchet space $\lambda_0(A)$ is always separable.

Let $\Omega_I := I$ and $\Sigma_I := 2^I$ denote the σ-algebra of all subsets of Ω_I. For each $E \in \Sigma_I$ define a projection $P_A(E) : \lambda_\infty(A) \to \lambda_\infty(A)$ by $x \mapsto \chi_E x$ (coordinatewise multiplication as functions on I), for each $x \in \lambda_\infty(A)$. Note that $\lambda_0(A)$ is an invariant subspace of each $P_A(E)$, for $E \in \Sigma_I$. Given $x \in \mathbb{C}^I$, define $|x| := (|x_i|)$. Then the seminorms (1) are *Riesz seminorms* on $\lambda_\infty(A)$, meaning that $\|x\|_k = \| \, |x| \, \|_k$ for all $x \in \lambda_\infty(A)$ and $k \in \mathbb{N}$, and

$$\|x\|_k \leq \|y\|_k, \qquad k \in \mathbb{N},$$

whenever $x, y \in \lambda_\infty(A)$ satisfy $|x| \leq |y|$ (with the order defined coordinatewise). In particular, for each $E \in \Sigma_I$,

$$\|P_A(E)x\|_k \leq \|x\|_k, \qquad x \in \lambda_\infty(A),$$

for all $k \in \mathbb{N}$. It follows that $P_A(E) \in L(\lambda_\infty(A))$ and the *Boolean algebra* of commuting projections $\{P_A(E) : E \in \Sigma_I\}$ form an equicontinuous subset of $L(\lambda_\infty(A))$. It is routine to check that the so defined set function $P_A : \Sigma_I \to L(\lambda_\infty(A))$ is multiplicative (i.e. $P_A(E \cap F) = P_A(E)P_A(F)$ for all $E, F \in \Sigma_I$), satisfies $P_A(\Omega_I) = I$ (the identity operator on $\lambda_\infty(A)$) and is finitely additive on the σ-algebra Σ_I. That is, P_A is an equicontinuous, *finitely additive* spectral measure on Σ_I, called the *canonical spectral measure* in $\lambda_\infty(A)$. Observe that $P_A(\{j\})x = x_j e_j$ for each $x \in \lambda_\infty(A)$ and $j \in \Omega_I$. For $j \in I$ define $\psi_j : \lambda_\infty(A) \to \mathbb{C}$ by $\langle x, \psi_j \rangle := x_j$, for $x \in \lambda_\infty(A)$. Since

$$|\psi_j(x)| \leq a_1(j)^{-1}\|x\|_1, \qquad x \in \lambda_\infty(A),$$

each ψ_j is continuous. Moreover, $\Gamma_\infty := \{\psi_j : j \in I\} \subseteq \lambda_\infty(A)'$ is a *total set* of functionals for $\lambda_\infty(A)$ with the property that the $\mathbb{C}$-valued set function

$$E \mapsto \langle P_A(E)x, \psi_j \rangle = \chi_E(j)x_j, \qquad E \in \Sigma_I, \tag{2}$$

is σ-additive for each $x \in \lambda_\infty(A)$ and $\psi_j \in \Gamma_\infty$.

We point out that both $\lambda_\infty(A)$ and $\lambda_0(A)$ are complex *Fréchet lattices*, being the complexification of the corresponding real Fréchet lattices obtained when $\mathbb{C}^I$ above is replaced with $\mathbb{R}^I$. Both $\lambda_0(A)$ and $\lambda_\infty(A)$ are *discrete* Fréchet lattices with the canonical unit vectors $\{e_j\}_{j\in I}$ forming a maximal disjoint system consisting of positive discrete elements, [17, Section 2]. It is routine to verify that $\lambda_0(A)$ has a *Lebesgue topology*, that is, whenever a sequence $\{x^{(n)}\} \subseteq \lambda_0(A)$ satisfies $|x^{(n)}| \downarrow 0$ in the order of $\lambda_0(A)$, then $x^{(n)} \to 0$ (as $n \to \infty$) in the topology of $\lambda_0(A)$. As a general reference for Fréchet (and locally convex) lattices we refer to [2].

The following result illustrates the precise connection between $\lambda_0(A)$ and $\lambda_\infty(A)$, of special interest to us because of its relationship to the Montel property of $\lambda_\infty(A)$. Some equivalences are known and several others are new.

Proposition 2.3. *Let I be a countable index set and $A = (a_n)_{n=1}^\infty$ be a Köthe matrix on I. The following assertions are equivalent.*

(i) $\lambda_\infty(A) = \lambda_0(A)$.
(ii) $\lambda_\infty(A)$ *is Montel.*
(iii) $\lambda_0(A)$ *is Montel.*
(iv) $\lambda_\infty(A)$ *is reflexive.*
(v) $\lambda_0(A)$ *is reflexive.*
(vi) $\lambda_\infty(A)$ *has a Lebesgue topology.*
(vii) $\lambda_\infty(A)$ *does not contain a complemented copy of the Banach space* ℓ^∞.
(viii) $\lambda_\infty(A)$ *is separable.*
(ix) *The canonical unit vectors* $\{e_j\}_{j\in I}$ *are an unconditional Schauder basis for* $\lambda_\infty(A)$.
(x) *The finitely additive canonical spectral measure* $P_A : \Sigma_I \to L(\lambda_\infty(A))$ *is* τ_s*-countably additive.*

Proof. The first five equivalences are known; see Theorems 27.9 and 27.15 of [23], for example.

(i) $\Leftrightarrow$ (vi). It is clear that the Riesz seminorms (1) have the *AM-property*, that is,

$$\| \, |x| \vee |y| \, \|_k = \max\{\|x\|_k, \|y\|_k\}, \qquad x, y \in \lambda_\infty(A),$$

for all $k \in \mathbb{N}$. Recall that $\{e_j\}_{j\in I}$ is a maximal disjoint system of positive, discrete elements in $\lambda_\infty(A)$. It then follows from Lemma 2.4 (and its proof) in [17], after an examination of the proof of Lemma 2.3 in [17], that the topology of $\lambda_\infty(A)$ is Lebesgue if and only if $\lambda_\infty(A) = \lambda_0(A)$.

(ii) $\Leftrightarrow$ (vii). If $\lambda_\infty(A)$ contains a complemented copy of ℓ^∞, then it is surely not Montel.

On the other hand, if $\lambda_\infty(A)$ is not Montel, then it follows from Theorem 27.9 (6) of [23] that $\lambda_\infty(A)$ contains a sectional (hence, complemented) subspace which is isomorphic to ℓ^∞.

(ii) $\Rightarrow$ (viii) is known, [19, p.370], and (viii) $\Rightarrow$ (vii) is clear.

(ix) $\Rightarrow$ (viii) is clear.

(viii) $\Rightarrow$ (x). By (2) we know that $E \mapsto \langle P_A(E)x, \psi\rangle$ is σ-additive on Σ_I, for each $x \in \lambda_\infty(A)$ and $\psi \in \Gamma_\infty$, with $\Gamma_\infty \subseteq \lambda_\infty(A)'$ a total set of functionals. Then the separability assumption on $\lambda_\infty(A)$ and Proposition 1(iii) of [26] imply that P_A is τ_s-countably additive.

(ix) $\Leftrightarrow$ (x). Suppose that P_A is τ_s-countably additive, that is, $E \mapsto P_A(E)x$ is σ-additive on Σ_I as a $\lambda_\infty(A)$-valued vector measure, for each $x \in \lambda_\infty(A)$. Since $\Omega_I = \bigcup_{j\in I}\{j\}$ is a pairwise disjoint, countable union it follows, for every $x \in \lambda_\infty(A)$, that

$$x = P_A(\Omega_I)x = \sum_{j\in I} P_A(\{j\})x = \sum_{j\in I} x_j e_j$$

with the series unconditionally convergent in $\lambda_\infty(A)$. Hence, $\{e_j\}_{j\in I}$ is an unconditional Schauder basis in $\lambda_\infty(A)$.

Conversely, suppose that $\{e_j\}_{j\in I}$ is an unconditional Schauder basis for $\lambda_\infty(A)$. Then $\lambda_\infty(A)$ is separable and by (viii) $\Rightarrow$ (i) we conclude that $\lambda_\infty(A) = \lambda_0(A)$. Then P_A is precisely the finitely additive canonical spectral measure in $\lambda_0(A)$. By Lemma 27.11 of [23], each $\xi \in \lambda_0(A)' \subseteq \mathbb{C}^I$ satisfies $\sum_{j\in I} |x_j| \cdot |\xi_j| < \infty$ for all $x \in \lambda_0(A)$ with the duality given by

$$\langle x, \xi\rangle = \sum_{j\in I} x_j\xi_j, \qquad x \in \lambda_0(A).$$

Accordingly,

$$E \mapsto \langle P_A(E)x, \xi\rangle = \sum_{j\in I} \chi_E(j)x_j\xi_j, \qquad E \in \Sigma_I,$$

is σ-additive for each $x \in \lambda_0(A)$ and $\xi \in \lambda_0(A)'$ and hence, by the Orlicz-Pettis Theorem, P_A is τ_s-countably additive in $\lambda_0(A) = \lambda_\infty(A)$. □

In relation to Proposition 2.3 we point out that if $\lambda_0(A)$ is nuclear (hence, also Montel), then *every* basis (being absolute) is necessarily unconditional, not just the canonical basis. This is surely not true for its Banach space analog c_0, where the canonical basis is unconditional but other bases, such as $\{v_n := \sum_{j=1}^n e_j\}_{n=1}^\infty$, for example, are not unconditional, [22, Theorem III.7.2].

The following observation will be needed later. It can be found on pp. 308–309 in the proof of Theorem 3.1(a) in [12]; see also Theorems 3 and 5 in [13].

Lemma 2.4. *Let $A = (a_n)_{n=1}^\infty$ be a Köthe matrix on I and B be a bounded subset of $\lambda_\infty(A)$. Then there exists an increasing sequence $(I_m)_{m=1}^\infty$ of subsets of the base set I such that*

(i) *each sectional subspace $\lambda_\infty(I_m, A) := \{x|_{I_m} : x \in \lambda_\infty(A)\}$ is normable, hence isomorphic to ℓ^∞, and*

(ii) *for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that*

$$\chi_{I\setminus I_m} B := \{x\chi_{I\setminus I_m} : x \in B\} \subseteq \varepsilon W_k,$$

where

$$W_k := \{x \in \lambda_\infty(A) : \|x\|_k \leq 1\}. \tag{3}$$

Let $I = \mathbb{N} \times \mathbb{N}$. A Köthe matrix $A = (a_n)_{n=1}^\infty$ on $\mathbb{N} \times \mathbb{N}$ is called a *Köthe-Grothendieck* (briefly, KG) matrix if

(KG-1) $a_n(i,j) = 1$ for all $i > n$ and all $n, j \in \mathbb{N}$,
(KG-2) $\sup_{j\in\mathbb{N}} a_n(n,j) = \infty$ for all $n \in \mathbb{N}$, and
(KG-3) $a_p(i,j) = a_q(i,j)$, for all $i, j \in \mathbb{N}$ and all $p, q \geq i$.

The original KG-matrix corresponds to

$$a_n(i,j) := \begin{cases} j \text{ for } i \leq n \text{ and } j \in \mathbb{N} \\ 1 \text{ for } i > n \text{ and } j \in \mathbb{N}, \end{cases}$$

for each $n \in \mathbb{N}$. For a KG-matrix $A = (a_n)_{n=1}^\infty$ it is known that the Köthe echelon space

$$\lambda_1(A) := \{x \in \mathbb{C}^I : q_n^{(1)}(x) := \sum_{i\in I} a_n(i)|x_i| < \infty \quad \text{for all } n \in \mathbb{N}\}$$

is *not* distinquished, [6, Thoerem 18(2)], and hence, $\lambda_\infty(A)$ cannot be Montel; see Corollary 2 and Theorem 18 of [6]. Actually, more is true.

Proposition 2.5. *Let $A = (a_n)_{n=1}^\infty$ be a KG-matrix on $\mathbb{N} \times \mathbb{N}$ and F be any Montel subspace of $\lambda_\infty(A)$. Then there exists $i_0 \in \mathbb{N}$ that*

$$F \cap \{x \in \lambda_\infty(A) : x(i,j) = 0 \ \textit{for all} \ i \leq i_0 \ \textit{and} \ j \in \mathbb{N}\} = \{0\}.$$

Proof. The statement of the result follows if we establish the (stronger) Claim: *There exists $n(0) \in \mathbb{N}$ and $d > 0$ such that, for all $x \in F$,*

$$\|x\|_1 = \sup_{\mathbb{N}\times\mathbb{N}} a_1(i,j) \cdot |x(i,j)| \leq d \sup_{1\leq i\leq n(0)} \sup_{j\in\mathbb{N}} a_{n(0)}(i,j) \cdot |x(i,j)|.$$

Suppose that the Claim is false. Then, for each $n \in \mathbb{N}$, there exists $z_n \in F$ such that

$$\|z_n\|_1 > 2^n \sup_{1\leq i\leq n} \sup_{j\in\mathbb{N}} a_n(i,j) \cdot |z_n(i,j)|.$$

Setting $x_n := z_n/\|z_n\|_1$ we have, for every $n \in \mathbb{N}$, an element $x_n \in F$ with $\|x_n\|_1 = 1$ such that

$$2^n \sup_{1\leq i\leq n} \sup_{j\in\mathbb{N}} a_n(i,j) \cdot |x_n(i,j)| < 1.$$

In particular, for each $n \in \mathbb{N}$ we have

$$a_n(i,j) \cdot |x_n(i,j)| \leq \frac{1}{2^n}, \qquad j \in \mathbb{N}, \quad 1 \leq i \leq n. \tag{4}$$

We now show that $\{x_n\}_{n=1}^{\infty}$ is bounded in F. So, fix $m \in \mathbb{N}$. Let $n > m$. Then, for all $j \in \mathbb{N}$ and $1 \leq i \leq m$, we have

$$a_m(i,j) \cdot |x_n(i,j)| \leq a_n(i,j) \cdot |x_n(i,j)| \leq \frac{1}{2^n}$$

and, for all $j \in \mathbb{N}$ and $i > m$, we have (since $a_m(i,j) = 1$ by (KG1)) that

$$a_m(i,j) \cdot |x_n(i,j)| = |x_n(i,j)| \leq \|x_n\|_1 = 1.$$

By definition of the seminorm $\|\cdot\|_m$ we have $\|x_n\|_m \leq 1$. Hence,

$$\sup_{n\in\mathbb{N}} \|x_n\|_m \leq \max\{1, \|x_1\|_m, \dots, \|x_m\|_m\} < \infty.$$

So, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in the *Montel* space F, from which it follows that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x \in F$ with $x_{n_k} \to x$ as $k \to \infty$ (in $\lambda_\infty(A)$). But, for each $i \in \mathbb{N}$ and $j \in \mathbb{N}$ (recalling that $a_n \geq 1$ on $\mathbb{N} \times \mathbb{N}$) it follows from (4), that

$$|x_n(i,j)| \leq \frac{1}{2^n}, \qquad n > i.$$

Since, $x_{n_k} \to x$ pointwise on $\mathbb{N} \times \mathbb{N}$ we conclude that $x \equiv 0$ on $\mathbb{N} \times \mathbb{N}$. This contradicts $\|x\|_1 = 1$ and hence, the Claim holds. □

Remark 2.6. (a) A.A. Albanese, [1], studied the structure of Montel subspaces of certain Fréchet spaces (of Moscatelli type). For $A = (a_n)_{n=1}^{\infty}$ a KG-matrix, her results cover the case of $\lambda_p(A)$ spaces for $p \in \{0\} \cup [1, \infty)$ but, not the case of $\lambda_\infty(A)$ (which is covered by Proposition 2.5 above).

(b) By a classical result of C. Bessaga, A.A. Pelczyński and S. Rolewicz, [4], every Fréchet space X not isomorphic to $Y \times \mathbb{C}^{\mathbb{N}}$, with Y a Banach space, contains a closed infinite dimensional subspace F which is a nuclear Fréchet space with basis. It follows that every (non-normable) space $\lambda_\infty(A)$, with $A = (a_n)_{n=1}^{\infty}$ a KG-matrix, contains infinite dimensional nuclear Fréchet spaces F with a basis. In particular, such a space F is Montel.

3. $\lambda_\infty(A)$ is always a GDP-space

Let $A = (a_n)_{n=1}^{\infty}$ be a Köthe matrix on the countable set I. It is known that $\lambda_\infty(A)$ is always a Grothendieck space, [11, Proposition 5]. Actually, more is true.

Proposition 3.1. *Let $A = (a_n)_{n=1}^{\infty}$ be any Köthe matrix on I. Then the Fréchet space $\lambda_\infty(A)$ is a GDP-space.*

Proof. For brevity, write $X = \lambda_\infty(A)$. It was just noted above that X is a Grothendieck space.

To establish that X has the DP-property, let $\{x_k\}_{k=1}^\infty \subseteq X$ be an arbitrary $\sigma(X, X')$-null sequence and $\{u_k\}_{k=1}^\infty \subseteq X'$ be an arbitrary $\sigma(X', X'')$-null sequence. By Lemma 2.1(ii) it suffices to show that $\lim_{k\to\infty}\langle x_k, u_k\rangle = 0$. Define $\{W_n\}_{n=1}^\infty$ according to (3). There is no loss of generality in assuming that $\{W_n\}_{n=1}^\infty$ is a neighbourhood base of zero. Since X is barrelled, the $\sigma(X', X)$-bounded subset $\{u_k\}_{k=1}^\infty$ of X' is equicontinuous and hence, there exists $s \in \mathbb{N}$ such that

$$\{u_k\}_{k=1}^\infty \subseteq W_s^\circ := \{\xi \in X' : |\langle x, \xi\rangle| \leq \|x\|_s \text{ for all } x \in X\}. \tag{5}$$

The set $B := \{x_k\}_{k=1}^\infty$ is bounded in X and so, according to Lemma 2.4, there exists an increasing sequence $\{I_m\}_{m=1}^\infty$ of subsets of the index set I such that

(a) $X_m := \lambda_\infty(I_m, A)$ is isomorphic to ℓ^∞, for each $m \in \mathbb{N}$, and

(b) for every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ with $\chi_{I\setminus I_m} B \subseteq \varepsilon W_k$.

Applying (b) to the choice $k := s$ we have

$$\forall \varepsilon > 0,\ \exists m(\varepsilon) \in \mathbb{N} \text{ such that } \chi_{I\setminus I_{m(\varepsilon)}} B \subseteq \varepsilon W_s. \tag{6}$$

Fix an arbitrary $m \in \mathbb{N}$. Since the linear map $x \mapsto \chi_{I_m} x$ is continuous (hence, also weakly continuous) from X into X_m and $\{x_k\}_{k=1}^\infty$ is a $\sigma(X, X')$-null sequence, it follows that $\{\chi_{I_m} x_k\}_{k=1}^\infty$ is $\sigma(X_m, X'_m)$-null. Moreover, as the closed subspace X_m is complemented in X and $\{u_k\}_{k=1}^\infty$ is $\sigma(X', X'')$-null, it follows that the sequence of restrictions $\{u_k|_{X_m}\}_{k=1}^\infty$ is $\sigma(X'_m, X''_m)$-null. But, $X_m \simeq \ell^\infty$ (see (a)) has the DP-property and so, by Lemma 2.1(ii), $\lim_{k\to\infty}\langle \chi_{I_m} x_k, u_k|_{X_m}\rangle = 0$. That is

$$\forall m \in \mathbb{N},\ \forall \varepsilon > 0,\ \exists k_0 \in \mathbb{N} \text{ such that } |\langle \chi_{I_m} x_k, u_k\rangle| \leq \varepsilon,\ \forall k \geq k_0. \tag{7}$$

Now, fix $\varepsilon > 0$. Apply (6) to find $m(\varepsilon) \in \mathbb{N}$ such that $\chi_{I\setminus I_{m(\varepsilon)}} B \subseteq \varepsilon W_s$. In view of (5) we conclude that

$$\left|\langle \chi_{I\setminus I_{m(\varepsilon)}} x_k, u_k\rangle\right| \leq \varepsilon, \qquad k \in \mathbb{N}.$$

According to (7) there exists $k_0 \in \mathbb{N}$ such that

$$\left|\langle \chi_{I_{m(\varepsilon)}} x_k, u_k\rangle\right| \leq \varepsilon, \qquad k \geq k_0.$$

Combining these two inequalities yields $|\langle x_k, u_k\rangle| \leq 2\varepsilon$ for all $k \geq k_0$. That is, $\lim_{k\to\infty}\langle x_k, u_k\rangle = 0$ and the proof is complete. □

For every KG-matrix $A = (a_n)_{n=1}^\infty$ we know that $\lambda_\infty(A)$ is *not* Montel yet, by Proposition 3.1, it is still a GDP-space. For $\lambda_0(A)$ spaces, even with A arbitrary, this is impossible.

Corollary 3.2. *Let $A = (a_n)_{n=1}^{\infty}$ be any Köthe matrix on I. The following assertions are equivalent.*

(i) *$\lambda_0(A)$ is Montel.*
(ii) *$\lambda_0(A)$ is a GDP-space.*
(iii) *$\lambda_0(A)$ is a complemented subspace of $\lambda_\infty(A)$.*

Proof. (i) $\Rightarrow$ (ii) is precisely Remark 2.2.

(i) $\Rightarrow$ (iii) is immediate from [23, Proposition 27.15] which states that (i) is equivalent with $\lambda_0(A) = \lambda_\infty(A)$.

(iii) $\Rightarrow$ (ii) follows from Lemma 2.1(iv) and Proposition 3.1.

(ii) $\Rightarrow$ (i) is established via a contrapositive argument which uses Theorem 27.9 and Proposition 27.15 of [23], together with the fact that the Banach (sequence) space c_0 is not a GDP-space. □

Observe that (ii) and (iii) of Corollary 3.2 provide two further equivalences with those in (i)–(x) of Proposition 2.3.

4. Families of commuting projections in GDP-spaces

The definition of a (weak) Schauder decomposition $\{P_n\}_{n=1}^{\infty}$ in Banach spaces, as given in Section 1 via (S1)-(S3), is purely algebraic and topological and so applies equally well in Fréchet spaces X; see [18] and the references therein, for example. Note, by (S2), that $\{P_n\}_{n=1}^{\infty}$ is an equicontinuous subset of $L(X)$. Recall, for a Fréchet space X, that the weakly and strongly bounded subsets of X' coincide. For Banach spaces, the next two results are due to H.P. Lotz, [20], [21].

Proposition 4.1. *Let X be a Fréchet space which is a Grothendieck space and let $\{P_n\}_{n=1}^{\infty} \subseteq L(X)$ be any Schauder decomposition.*

(i) *If $\{\xi_n\} \subseteq X'$ is any bounded sequence, then $\{(I - P_n)'\xi_n\}$ is a $\sigma(X', X'')$-null sequence in X'.*
(ii) *If $\{x_n\} \subseteq X$ is any bounded sequence, then $\{(I - P_n)x_n\}$ is a $\sigma(X, X')$-null sequence in X.*

Proof. For simplicity of notation set $Q_j := (I - P_j)$ for $j \in \mathbb{N}$.

(i) According to (S2) we have $\lim_{j\to\infty} Q_j = 0$ in $L_s(X)$. Since $B := \{\xi_j\}_{j=1}^{\infty} \subseteq X'$ is equicontinuous, it follows that $\lim_{j\to\infty} \sup_{\xi\in B} |\langle Q_j x, \xi\rangle| = 0$ for each $x \in X$. Accordingly, $\lim_{j\to\infty}\langle x, Q_j'\xi_j\rangle = 0$ for each $x \in X$, that is, $\{Q_j'\xi_j\}_{j=1}^{\infty}$ is a $\sigma(X', X)$-null sequence. By the Grothendieck property of X, the sequence $\{Q_j'\xi_j\}_{j=1}^{\infty}$ is also $\sigma(X', X'')$-null.

(ii) By part (i) we have

$$\lim_{j\to\infty} P_j'\xi = \xi \quad \text{(relative to } \sigma(X', X'')\text{), for each } \xi \in X'. \tag{8}$$

Define a linear subspace of X' by

$$H := \{\xi \in X' : \lim_{j\to\infty} P'_j \xi = \xi \ \text{ in } \ X'_\beta\}.$$

Claim 1. *H is a closed subspace of X'_β.*

To establish the claim, let $\{\xi_\alpha\}_{\alpha\in A} \subseteq H$ be a net (so, $\lim_{j\to\infty} P'_j \xi_\alpha = \xi_\alpha$ in X'_β for each $\alpha \in A$) such that $\lim_\alpha \xi_\alpha = \xi$ (in X'_β) for some $\xi \in X'$. Fix a bounded set $D \subseteq X$. By equicontinuity of the set $\{P_j\}_{j=1}^\infty \subseteq L(X)$ we can conclude that $C := D \cup (\bigcup_{j\in\mathbb{N}} P_j(D))$ is also bounded in X. Since $\lim_\alpha \xi_\alpha = \xi$ in X'_β, there exists $\alpha_0 \in A$ such that

$$|\langle c, (\xi_\alpha - \xi)\rangle| \le \frac{1}{3}, \qquad \alpha \ge \alpha_0, \quad c \in C.$$

For each $\alpha \ge \alpha_0$ and $d \in D$, it follows from

$$\langle d, Q'_j \xi\rangle = \langle d, (\xi - \xi_\alpha)\rangle + \langle d, Q'_j \xi_\alpha\rangle + \langle d, P'_j (\xi_\alpha - \xi)\rangle,$$

valid for each $j \in \mathbb{N}$, and the triangle inequality (after noting that $P_j d \in C$ for all $j \in \mathbb{N}$), that

$$\left|\langle d, Q'_j \xi\rangle\right| \le \frac{2}{3} + \left|\langle d, Q'_j \xi_\alpha\rangle\right|, \qquad j \in \mathbb{N}.$$

In particular, for each $j \in \mathbb{N}$ and $d \in D$, we have

$$\left|\langle d, Q'_j \xi\rangle\right| \le \frac{2}{3} + \left|\langle d, Q'_j \xi_{\alpha_0}\rangle\right|. \tag{9}$$

Since $\lim_{j\to\infty} Q'_j \xi_{\alpha_0} = 0$ in X'_β, there exists $j_0 \in \mathbb{N}$ such that

$$\sup_{d\in D} \left|\langle d, Q'_j \xi_{\alpha_0}\rangle\right| \le \frac{1}{3}, \qquad j \ge j_0.$$

It follows that $\left|\langle d, Q'_j \xi\rangle\right| \le 1$ for all $d \in D$ and $j \ge j_0$, that is, $Q'_j \xi \in D^\circ$ for all $j \ge j_0$. Equivalently, since, D is an arbitrary bounded subset of X, we conclude from (9) that $\lim_{j\to\infty} P'_j \xi = \xi$ in X'_β and hence, $\xi \in H$.

Claim 2. $\left(\bigcup_{k\in\mathbb{N}} P'_k\right) H \subseteq H$.

To see this, fix $\xi \in H$, in which case $\lim_{j\to\infty} P'_j \xi = \xi$ in X'_β. For each (fixed) $k \in \mathbb{N}$ it follows, after noting that $\{P'_j\}_{j=1}^\infty \subseteq L(X'_\beta)$, that

$$P'_k \xi = P'_k(\lim_{j\to\infty} P'_j \xi) = \lim_{j\to\infty} P'_j(P'_k \xi)$$

in X'_β. Hence, $P'_k \xi \in H$. Since $k \in \mathbb{N}$ and $\xi \in H$ are arbitrary, Claim 2 is established.

It now follows from (8) and Claim 2 that H is $\sigma(X', X'')$-dense in X'. Since both $\sigma(X', X'')$ and $\beta(X', X)$ are topologies of the dual pairing

(X', X''), we conclude that H is also dense in X'_β. Then Claim 1 yields that $H = X'_\beta$, that is,

$$\lim_{j\to\infty} Q'_j = 0 \quad \text{in} \;\; L_s(X'_\beta). \tag{10}$$

Finally, to complete the proof of part (ii), let $\{x_k\}_{k=1}^\infty \subseteq X$ be a bounded sequence. According to (10) we have

$$\lim_{j\to\infty} \sup_{k\in\mathbb{N}} \left|\langle x_k, Q'_j\xi\rangle\right| = 0, \qquad x \in X'.$$

It follows that $\lim_{j\to\infty}\langle Q_j x_j, \xi\rangle = 0$ for each $\xi \in X'$, that is, $\{(I - P_j)x_j\}_{j=1}^\infty$ is a $\sigma(X, X')$-null sequence.

□

We have the following important application.

Proposition 4.2. *Let X be a Fréchet GDP-space and $\{P_n\}_{n=1}^\infty \subseteq L(X)$ be a Schauder decomposition. Then $P_n \to I$ in $L_b(X)$ as $n \to \infty$.*

Proof. Suppose that $\{P_n\}_{n=1}^\infty$ fails to converge to I in $L_b(X)$. Then there exist a bounded set $B \subseteq X$, a continuous seminorm q in X and $\varepsilon > 0$ such that $\sup_{x\in B} q(x - P_{j_k}x) > \varepsilon$ for some increasing sequence $\{j_k\}_{k=1}^\infty \subseteq \mathbb{N}$. For each $k \in \mathbb{N}$, select $x_k \in B$ such that $q(x_k - P_{j_k}x_k) > \varepsilon$. Recall that $q(x) = \sup_{\xi\in U_q^\circ} |\langle x, \xi\rangle|$ for all $x \in X$, where

$$U_q^\circ := \{\xi \in X' : |\langle x, \xi\rangle| \le q(x) \;\text{ for all }\; x \in X\}.$$

For each $k \in \mathbb{N}$, it follows from the Hahn-Banach theorem that there exists $\xi_k \in U_q^\circ$ satisfying $|\langle (x_k - P_{j_k}x_k), \xi_k\rangle| > \varepsilon$. Using $(I - P_{j_k})^2 = (I - P_{j_k})$ we conclude that

$$\varepsilon < |\langle (I - P_{j_k})x_k, (I - P_{j_k})'\xi_k\rangle|\,, \qquad k \in \mathbb{N}. \tag{11}$$

But, $\{x_k\}_{k=1}^\infty \subseteq B$ is bounded in X and $\{\xi_k\}_{k=1}^\infty \subseteq U_q^\circ$ is bounded in X'. Then Proposition 4.1 shows that $(I - P_{j_k})x_k \to 0$ in $(X, \sigma(X, X'))$ as $k \to \infty$ and $(I - P_{j_k})'\xi_k \to 0$ in $(X', \sigma(X', X''))$ as $k \to \infty$. Since X has the DP-property, Lemma 2.1(ii) shows that the right-hand-side of (11) converges to 0 as $k \to \infty$, which is clearly impossible. Accordingly, $P_n \to I$ in $L_b(X)$ as $n \to \infty$. □

Let X be a Fréchet space and Σ be a σ-algebra of subsets of a non-empty set Ω. A map $P : \Sigma \to L(X)$ which is multiplicative, satisfies $P(\Omega) = I$ and is σ-*additive* in $L_s(X)$ is called a *spectral measure* (in X). If, in addition, P is σ-additive in $L_b(X)$, then P is called *boundedly σ-additive*. As noted in Section 1, in a Fréchet-Montel space every spectral measure is automatically boundedly σ-additive. The following result is an extension of this fact.

Proposition 4.3. *Let X be a Fréchet GDP-space and $P : \Sigma \to L(X)$ be any spectral measure. Then P is necessarily boundedly σ-additive.*

Proof. Let $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ be pairwise disjoint sets, in which case we have $P(E_n)P(E_m) = 0$ for all $m \neq n$ in $\mathbb{N}$. By the τ_s-countable additivity of P we have $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$, where the series converges unconditionally in $L_s(X)$. Let $Q_n := \sum_{i=1}^{n} P(E_i)$ for $n \in \mathbb{N}$ and $Q := \sum_{n=1}^{\infty} P(E_n)$. Then $\{Q_n\}_{n=1}^{\infty}$ is increasing (in the partial order specified by range inclusion) and satisfies $Q_n \to Q$ in $L_s(X)$ as $n \to \infty$. Of course, $Q_n \leq Q$ for all $n \in \mathbb{N}$.

Let $Y := Q(X)$ and $P_n := Q_n|_Y$ be the restriction of Q_n to the closed invariant subspace Y of X. Then Y is a GDP-space (by Lemma 2.1(iv)) and $Q|_Y$ is the identity operator I_Y in Y.

Suppose there exists $n_0 \in \mathbb{N}$ such that $P_n = P_{n_0}$ for all $n \geq n_0$. Since $P_n \to I_Y$ in $L_s(Y)$ as $n \to \infty$ we conclude that $P_n = I_Y$ for all $n \geq n_0$ and hence, $P_n \to I_Y$ in $L_b(Y)$. If no such n_0 exists, then there exists an increasing sequence $n_k \uparrow \infty$ in $\mathbb{N}$ such that $P_{n_k} \neq P_{n_{k+1}}$ for all $k \in \mathbb{N}$. Since $P_n \leq P_{n+1}$ for all $n \in \mathbb{N}$, the n_k can be chosen so that $P_j = P_{n_k}$ for all $n_k \leq j < n_{k+1}$ and all $k \in \mathbb{N}$. Then $\{P_{n_k}\}_{k=1}^{\infty}$ satisfies (S1)–(S3), that is, it is a Schauder decomposition in Y. By Proposition 4.2 we conclude that $P_{n_k} \to I_Y$ in $L_b(Y)$ as $k \to \infty$ and hence, that $P_n \to I_Y$ in $L_b(Y)$ as $n \to \infty$. So, in both cases, we have $P_n \to I_Y = Q|_Y$ in $L_b(Y)$.

Since $Q_n|_Z$, with $Z := (I - Q)(X)$, is the zero operator 0_Z in Z for *every* $n \in \mathbb{N}$, we have $Q_n|_Z \to 0_Z$ in $L_b(Z)$ as $n \to \infty$. Noting that $Q(B)$, resp. $(I - Q)(B)$, is a bounded set in Y, resp. Z, whenever $B \subseteq X$ is bounded, and that $Q_n = P_n \oplus 0_Z$ (with $Y \oplus Z = X$) for every $n \in \mathbb{N}$, it follows that

$$Q_n \to Q|_Y \oplus 0_Z = Q, \quad \text{for } n \to \infty,$$

in $L_b(X)$. This is precisely bounded σ-additivity of P. □

Given a Schauder decomposition $\{P_n\}_{n=1}^{\infty}$ in a Fréchet space X, it follows from (S1)–(S3) that $\sum_{n=0}^{\infty}(P_{n+1} - P_n) = I$ (where $P_0 := 0$) with the series convergent in $L_s(X)$. If this series is unconditionally convergent in $L_s(X)$, then $\{P_n\}_{n=1}^{\infty}$ is called an *unconditional Schauder decomposition*, [24]. It is precisely such decompositions which are associated with (non-trivial) spectral measures; see (the proof of) Proposition 4.3 above and also Lemma 5 and Theorem 6 in [24].

As noted in Section 1, a GDP-*Banach* space admits no Schauder decompositions, a fact which *fails* in general Fréchet spaces; see Remark 2.2. In view of Remark 2.2, Proposition 3.1 and the discussion immediately after Proposition 3.1, the question arises of whether there exist spaces $\lambda_\infty(A)$, *other* than Montel ones, which also admit unconditional Schauder decompositions? Recall that a Fréchet space X satisfies the *density condition* if the bounded sets of its strong dual X'_β are metrizable. For equivalent conditions we refer to [5], [6], for example. Fréchet spaces of the kind $\lambda_\infty(A)$ satisfy the density condition if and only if they are distinguished, [3].

Proposition 4.4. *Let $A = (a_n)_{n=1}^\infty$ be a Köthe matrix on I such that $\lambda_\infty(A)$ is non-normable and satisfies the density condition. Then $\lambda_\infty(A)$ admits an unconditional Schauder decomposition.*

Proof. The density condition implies that A satisfies Condition D, namely, there exists an increasing sequence $(I_m)_{m\in\mathbb{N}}$ of subsets of I such that:

$$\text{(D1)} \qquad \forall m\ \exists n(m)\ \forall k > n(m) : \inf_{i\in I_m} a_{n(m)}(i)/a_k(i) > 0$$

and

(D2) $\forall n\ \forall I_0 \subseteq I$ with $I_0 \cap (I \setminus I_m) \neq \emptyset\ (\forall m \in \mathbb{N}), \exists n^* = n^*(n, I_0) > n$ such that $\inf_{i\in I_0} a_n(i)/a_{n^*}(i) = 0$,

[6, Theorem 18]. Since $\lambda_\infty(A)$ is non-normable, the argument in the proof of [9, Corollary 2.4] shows that the increasing sequence $(I_m)_{m\in\mathbb{N}}$ given by Condition D satisfies $\bigcup_{m=1}^\infty I_m = I$, but that no I_m coincides with I.

Suppose there exists $x \in \lambda_\infty(A)$ such that $\{x\chi_{I_m}\}_{m=1}^\infty$ fails to converge to x. Then there exists $n \in \mathbb{N}$, $\varepsilon > 0$ and a sequence $\{j(m)\}_{m=1}^\infty \subseteq I$ with $j(m) \notin I_m$ for each m such that $a_n(j(m)) \cdot |x(j(m))| > \varepsilon$ for all $m \in \mathbb{N}$. Apply (D2) to $I_0 := \{j(m) : m \in \mathbb{N}\}$ and the above n to choose an $n^* > n$ such that

$$\varepsilon < \frac{a_n(j(m))}{a_{n^*}(j(m))} \cdot |x(j(m))| \cdot a_{n^*}(j(m)) \leq \frac{a_n(j(m))}{a_{n^*}(j(m))} \cdot \|x\|_{n^*}$$

for all $m \in \mathbb{N}$. This is impossible since the right-hand-side has infimum 0 (as m varies). Accordingly, for every $x \in \lambda_\infty(A)$ we have $x\chi_{I_m} \to x$ (in $\lambda_\infty(A)$) as $m \to \infty$.

Let P_m denote the projection of $\lambda_\infty(A)$ onto the sectional subspace $\lambda_\infty(I_m, A) := \{x\chi_{I_m} : x \in \lambda_\infty(A)\}$, for each $m \in \mathbb{N}$. It was just shown that $P_m \to I$ in $L_s(\lambda_\infty(A))$ as $m \to \infty$, with the P_m, for $m \in \mathbb{N}$, pairwise distinct and increasing. So, $\{P_m\}_{m=1}^\infty$ is a Schauder decomposition in $\lambda_\infty(A)$. The claim is that $\{P_m\}_{m=1}^\infty$ is actually an unconditional Schauder decomposition, that is, $\sum_{m=0}^\infty (P_{m+1} - P_m)$ is unconditionally convergent in $L_s(\lambda_\infty(A))$, where $P_0 := 0$. To verify this it suffices to show that for each given $x \in \lambda_\infty(A)$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a finite set $F_0 \subseteq \mathbb{N}$ such that for each finite set $F \subseteq \mathbb{N}$ with $F \cap F_0 = \emptyset$ we have

$$\sup_{i \in I} a_n(i) \cdot \left| \left(\sum_{m \in F} (P_{m+1} - P_m)x \right)(i) \right| < \varepsilon.$$

Since $\lim_{m\to\infty} x\chi_{I_m} = x$ (in $\lambda_\infty(A)$), there is $m(0) \in \mathbb{N}$ such that $a_n(i)|x(i)| < \varepsilon$ for all $i \in I \setminus I_{m(0)}$. Set $F_0 := \{1, 2, \dots, m(0)\}$. Then, for any finite set $F \subseteq \mathbb{N}$ with $F \cap F_0 = \emptyset$, we have

$$\sup_{i \in I} a_n(i) \cdot \left| \left(\sum_{m \in F} (P_{m+1} - P_m)x \right)(i) \right| \le \sup_{i \in I \setminus I_{m(0)}} a_n(i) \cdot |x(i)| < \varepsilon.$$

This completes the proof. □

Proposition 4.4 applies to *all* non-normable spaces $\lambda_\infty(A)$ which satisfy the density condition. Of course, if $\lambda_\infty(A)$ is also Montel, then Proposition 2.3(ix) already implies the existence of unconditional Schauder decompositions (consisting even of rank 1 projections). The real interest in Proposition 4.4 lies in the *non-Montel* case. For the existence of such spaces $\lambda_\infty(A)$, we observe that the product of ℓ^∞ and an infinite dimensional nuclear space $\lambda_\infty(B)$ is a space of type $\lambda_\infty(A)$ which is not normable, not Montel but, satisfies the density condition. Further examples are obtained by taking a Köthe matrix A which is regularly decreasing (in the sense of [7]) and such that $\lambda_\infty(A)$ is neither normable nor Montel; concrete examples of such spaces $\lambda_\infty(A)$ are given in [7].

In conclusion we point out that the density condition in Proposition 4.4 is not necessary for the existence of unconditional Schauder decompositions. Indeed, the cartesian product of any Montel space $\lambda_0(C)$, such as the space s of rapidly decreasing sequences, with the classical space $\lambda_\infty(B)$ corresponding to any KG-matrix B (which is known to fail the density condition; combine [9, Proposition 2.6] with [6, Theorem 18], for example), is a $\lambda_\infty(A)$ space (just put each of $\lambda_0(C)$ and $\lambda_\infty(B)$ in the even-odd coordinates, respectively) which fails the density condition. However, $\lambda_\infty(A)$ admits an unconditional Schauder decomposition; just add to the usual one coming from the canonical basis of $\lambda_0(C)$ (see Proposition 2.3) the projection of $\lambda_\infty(A)$ onto $\lambda_\infty(B)$.

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