# Topologizable operators on locally convex spaces

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Abstract. We give examples of non normable locally convex spaces such that every operator defined on them is topologizable, or even m-topologizable, in the sense of Żelazko. These examples are non complete metrizable spaces or complete (DF)-spaces; they solve a problem asked by Żelazko.

Motivated by the absence of a reasonable locally convex topology in the space $L(E)$ of all the continuous linear operators on a non normable locally convex space $E$, cf. [**Z1**], Żelazko introduced and studied in [**Z2**] the concept of operator algebras $A \subset L(E)$ on $E$ and the related concepts of topologizable and $m$-topologizable operators. The aim of this note is to present the examples mentioned in the abstract, thus solving a question asked by Prof. Żelazko during his lecture at the International Conference on Topological Algebras and Applications.

Our notation for locally convex spaces and functional analysis is standard; we refer the reader to [**J, K, MV, BP**]. For topological algebras we refer the reader to [**DL, MA, MI, Z0**]. We recall some terminology. For a locally convex space $E$, which we assume to be Hausdorff, $E'$ stands for its topological dual. The set of continuous seminorms on the space $E$ is denoted by $cs(E)$. We denote by $\beta(E,F)$ the strong topology and by $\sigma(E,F)$ the weak topologies on $E$ with respect to a dual pair $\langle E,F\rangle$. The strong dual $(E',\beta(E',E))$ of $E$ is also denoted by $E'_b$. If $E$ is a locally convex space, then $L(E)$ denotes the vector space of all continuous linear maps from $E$ to $E$. The composition of an operator $T$ with itself $k$ times is denoted by $T^k$. Given $T \in L(E)$ we denote by $T^t \in L(E')$ its transpose defined by $T^t(u) = u \circ T \in E'$ for each $u \in E'$. A locally convex space is called normable if it is isomorphic to a normed space, or equivalently if it has a bounded 0-neighbourhood. A complete, metrizable, locally convex space is called a Fréchet space, see [**MV, BP**], or a $B_0$-space, according to the classical terminology of Mazur and Orlicz, see [**Z2**]. A (DF)-space is a locally convex space $E$ which has a fundamental sequence of bounded sets, and such that every countable intersection of absolutely convex closed 0-neighbourhoods which absorbs the bounded sets is also a 0-neighbourhood. We refer the reader to [**K, MV, BP**]. This class was introduced by Grothendieck. The strong dual of a Fréchet space is a complete

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(DF)-space. The class of (DF)-spaces also contains the countable inductive limits of Banach spaces, called (LB)-spaces.

The following two classes of operators were defined and studied by Żelazko in [**Z2**].

Definition 1. An operator $T \in L(E)$ on a locally convex space $E$ is called *topologizable* if for every continuous seminorm $p \in cs(E)$ there is a continuous seminorm $q \in cs(E)$ such that for every $k \in \mathbb{N}$ there is $M_k > 0$ such that

$$p(T^k(x)) \leq M_k q(x)$$

for each $x \in E$.

Definition 2. An operator $T \in L(E)$ on a locally convex space $E$ is called *$m$-topologizable* if for every continuous seminorm $p \in cs(E)$ there are a continuous seminorm $q \in cs(E)$ and $C \geq 1$ such that for every $k \in \mathbb{N}$ such that

$$p(T^k(x)) \leq C^k q(x)$$

for each $x \in E$.

Observe that in the definitions above it is essential that the seminorm $q$ only depends on the seminorm $p$ and not on the iteration $k$. By Żelazko [**Z2**], Theorems 5 and 9, an operator $T \in L(E)$ is topologizable (resp. $m$-topologizable) if and only if $T$ belongs to some operator algebra $A \subset L(E)$ on $E$ (resp. $T$ belongs to an operator algebra of the form $L_\Gamma(E)$); see [**Z2**] for notation.

Clearly every $m$-topologizable operator is topologizable. If the locally convex space $E$ is normable, then every operator $T \in L(E)$ is $m$-topologizable. We give examples of non normable locally convex spaces in which every operator is topologizable or $m$-topologizable.

First of all we show that natural examples of operators on a non-normable Fréchet space are topologizable. The translation operator is mentioned in Żelazko [**Z2**], Examples 8, as a natural operator which is not topologizable in many cases.

Example 3. Let $G := \{z \in \mathbb{C} \mid |z| < t\}$, $t > 0$ or $t = \infty$, and let $E = H(G)$ be the Fréchet space of all holomorphic functions on $G$, endowed with the compact open topology. The topology of $H(G)$ is defined by the seminorms $p_r(f) := \sup_{|z| \leq r} |f(z)|, 0 < r < t$. We show that *the operator of differentiation $T \in L(E), T(f) := f'$, is topologizable in $L(E)$.*

Given $0 < r < t$, select $s$ such that $r < s < t$. For each $k \in \mathbb{N}$, we can apply Cauchy integral formula to conclude

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_{|u|=s} \frac{f(u)}{(u-z)^{k+1}} du, \quad |z| \leq r$$

therefore

$$p_r(T^k(f)) = p_r(f^{(k)}) \leq \frac{k!\ s}{(s-r)^{k+1}} p_s(f).$$

Theorem 4. *If $E$ is a (DF)-space, then every operator $T \in L(E)$ is topologizable. In particular, the strong dual $E = F'_b$ of a Fréchet space $F$ is a complete (DF)-space such that every operator $T \in L(E)$ is topologizable.*

PROOF. Let $T \in L(E)$ be an operator on a (DF)-space $E$. We fix a continuous seminorm $p \in cs(E)$. For each $k \in \mathbb{N}$, the continuity of $T^k$ yields a continuous seminorm $p_k \in cs(E)$ such that $p(T^k(x)) \leq p_k(x)$ for each $x \in E$. Every (DF)-space satisfies the countable neighbourhood property [**BP**, 8.3.5], hence there is a continuous seminorm $q \in cs(E)$ and there is a sequence $(M_k)_k$ of positive numbers such that $p_k(x) \leq M_k q(x)$ for each $x \in E$. this implies $p(T^k(x)) \leq M_k q(x)$ for each $k \in \mathbb{N}$ and each $x \in E$, and the operator $T$ is topologizable. □

Not every operator defined on a (DF)-space is $m$-topologizable. To see this we need the notation about Köthe echelon and co-echelon spaces. A Köthe matrix $A = (a_n)_{n\in\mathbb{N}}$ is an increasing sequence of strictly positive functions on $\mathbb{N}$. Corresponding to each Köthe matrix $A = (a_n)_n$ we associate the Fréchet space

$$\lambda_1(A) = \{x = (x(i))_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \text{ (or } \mathbb{R}^{\mathbb{N}}) \mid \forall n \in \mathbb{N} : \ p_n(x) := \textstyle\sum_i (a_n(i)|x(i)|) < \infty\},$$

The $i$-th unit vector is denoted by $e_i$, $i \in \mathbb{N}$.

The space $\lambda_1(A)$ is called a *Köthe echelon space* of order 1, it is a Fréchet space with the sequence of norms $p_n$, $n = 1, 2, ....$ Let $\alpha = (\alpha_i)_i$ be a monotonically increasing sequence of strictly positive numbers tending to $\infty$. A *power series space of infinite type* $\Lambda_\infty(\alpha)$ is a Köthe echelon space $\lambda_1(A)$ for the Köthe matrix $a_n(i) = \exp(n\alpha_i), n, i \in \mathbb{N}$. The space of entire functions on $\mathbb{C}$ and the space $S$ of rapidly decreasing smooth functions of Schwartz are isomorphic to power series spaces of infinite type; see Meise, Vogt [**MV**], Chapters 27 and 29.

For a Köthe matrix $A = (a_n)_n$, denote by $V = (v_n)_n$ the associated decreasing sequence of functions $v_n = 1/a_n$, $n \in \mathbb{N}$. For a given decreasing sequence $V = (v_n)_n$ of strictly positive functions on $\mathbb{N}$ or for the corresponding Köthe matrix $A = (a_n)_n$, we associate as in [**BMS**] the system

$$\overline{V} := \{\overline{v} = (\overline{v}(i))_i \in \mathbb{R}_+^{\mathbb{N}};\ \forall n \in \mathbb{N} : \ \sup_i \frac{\overline{v}(i)}{v_n(i)} = \sup_i a_n(i)\overline{v}(i) < \infty\}.$$

The strong dual $\lambda_1(A)'_b$ coincides algebraically and topologically with the space

$$K_\infty(\overline{V}) := \{u = (u(i))_i \mid p_{\overline{v}}(u) := \sup_i \overline{v}(i)|u(i)| < \infty \ \text{ for all } \ \overline{v} \in \overline{V}\},$$

endowed with the topology defined by the seminorms $p_{\overline{v}}, \overline{v} \in \overline{V}$. An element $u = (u(i))_i$ belongs to $K_\infty(\overline{V})$ if and only if there is $n \in \mathbb{N}$ such that $\sup_i v_n(i)|u(i)| < \infty$. A fundamental sequence of bounded sets in $K_\infty(\overline{V})$ is given by the sets $B_n := \{u = (u(i))_i \mid \sup_i v_n(i)|u(i)| \leq n\}, n \in \mathbb{N}$. We refer the reader to [**BB, BMS**] for more information and details.

EXAMPLE 5. We exhibit *a (DF)-space $E$ and an operator $T \in L(E)$ which is not $m$-topologizable*. In fact, the space $E$ is even an (LB)-space. Recall that, by Theorem 4, every operator $T \in L(E)$ is topologizable. The space $s$ of rapidly decreasing sequences is the Köthe echelon space $\lambda_1(A)$ defined by the Köthe matrix $a_n = i^n$, $i, n \in \mathbb{N}$. Its strong dual $E = s'_b$ is a complete (DF)-space. Define the operator $T \in L(E)$ by $T((u(i))_i) := (0, u(1), 2u(2), 3u(3), ...), u \in E$. Observe that, if we write $T(x) = ((Tx)_i)_i$, we have $(Tx)_i = (i-1)u(i-1), i \in \mathbb{N}$, with $u(0) := 0$, $T(e_i) = ie_{i+1}$, $i \in \mathbb{N}$, and $T^k(e_1) = k!\ e_{k+1}$ for each $k \in \mathbb{N}$. Since $s$ is a Fréchet Schwartz space, $E$ is bornological. Therefore, to see that $T$ is linear and continuous, it is enough to show that it maps bounded sets into bounded sets: it can be easily verified that $T(B_n) \subset B_{n+1}$, for $B_n := \{u = (u(i))_i \mid \sup_i i^{-n}|u(i)| \leq n\}$. Now suppose that $T$ is $m$-topologizable. Consider the weight $w(i) = 2^{-i}, i \in \mathbb{N}$. Clearly

$w$ belongs to the set $\overline{V}$, since $\sup_i i^n 2^{-i} < \infty$ for each $n \in \mathbb{N}$. Given the seminorm $p_w$ on $E$, there is a seminorm $p_{\overline{v}}, \overline{v} \in \overline{V}$, and there is $C \geq 1$ such that for each $k \in \mathbb{N}$

$$p_w(T^k(u)) \leq C^k p_{\overline{v}}(u) = C^k \sup_i \overline{v}(i)|u(i)|,$$

for each $u = (u(i))_i \in E$. Evaluating this inequality at $x = e_1$, we get, for each $k \in \mathbb{N}$,

$$k!\ 2^{-(k+1)} = p_w(k!\ e_{k+1}) = p_w(T^k(e_1)) \leq C^k p_{\overline{v}}(e_1) \leq C^k \overline{v}(1).$$

This implies that the sequence $\left(\frac{k!}{2^{k+1}C^k}\right)_k$ is bounded, a contradiction.

Example 6. *There are complete (DF)-spaces which are not normable in which every operator is $m$-topologizable.* Indeed, let $I$ be an uncountable index set, and let $\ell_2(I)$ be the vector space of all square summable families $x = (x(i))_{i \in I}$. We denote by $||.||$ the usual Hilbert norm on $\ell_2(I)$ and define $E$ as the space $\ell_2(I)$ endowed with the topology $\tau$ of uniform convergence on the separable bounded subset of the dual $\ell_2(I)'$ of $\ell_2(I)$. Clearly $\tau$ is a topology of the dual pair $(\ell_2(I), \ell_2(I)')$, hence it has the same bounded sets as the norm topology of $\ell_2(I)$. Moreover $E = (\ell_2(I), \tau)$ is a complete (DF)-space which is not normable. Spaces of this type have been used often as counter-examples in the theory of (DF)-spaces; see e.g. Köthe [**K**], page 401 of volume 1. The topology $\tau$ of $E$ can be described by the following family of seminorms: for each countable subset $J$ of $I$, we set

$$p_J(x) := \left(\sum_{i \in J} |x(i)|^2\right)^{1/2},\ x = (x(i)) \in \ell_2(I).$$

We show that every continuous linear operator $T \in L(E)$ is $m$-topologizable. Fix a continuous seminorm $p_{J(0)}$ on $E$, $J(0)$ a countable subset of $I$. Since each $T^k, k \in \mathbb{N}$ is continuous, we find, for each $k$, a countable subset $J(k)$ of $I$ and a constant $C_k > 0$ such that $p_{J(0)}(T^k(x)) \leq C_k p_{J(k)}(x)$ for each $x \in E$. Set $J$ for the union of the sets $J(0), J(1), J(2), ....$ If $y \in E$ satisfies $y(i) = 0$ for each $i \in J$, then $p_{J(0)}(T^k(x)) = 0$ for each $k \in \mathbb{N}$. Since $T$ is continuous on $E$, it maps bounded sets into bounded sets, hence it is a continuous linear map on $\ell_2(I)$, and there is $C \geq 1$ such that, for all $k \in \mathbb{N}$, $||T^k(x)|| \leq C^k||x||$ for every $x \in \ell_2(I)$. We claim that $p_{J(0)}(T^k(x)) \leq C^k p_J(x)$ for each $k \in \mathbb{N}, x \in E$, which implies that $T$ is $m$-topologizable in $L(E)$. Fix $k \in \mathbb{N}$ and $x \in E$, and write $x = x_1 + x_2$, with $x_1(i) = 0$ if $i \in I \setminus J$ and $x_2(i) = 0$ if $i \in J$. we have

$$p_{J(0)}(T^k(x)) \leq p_{J(0)}(T^k(x_1)) + p_{J(0)}(T^k(x_2)) = p_{J(0)}(T^k(x_1)) \leq$$
$$\leq ||T^k(x_1)|| \leq C^k||x_1|| = C^k p_J(x).$$

There is a relevant class of operators which are $m$-topologizable on every locally convex space. An operator $T \in L(E)$ is called *bounded* if there is 0-neighbourhood $U$ in $E$ such that $T(U)$ is bounded in $E$. Bounded operators play an important role in the theory of Fréchet spaces, see e.g. Vogt [**V**].

Proposition 7. Every bounded operator $T \in L(E)$ on a locally convex space $E$ is $m$-topologizable.

Proof. If $T$ is bounded, there is a continuous seminorm $p_0 \in cs(E)$ such that for every $p \in cs(E)$ there is $C(p) \geq 1$ such that, for every $x \in E$, we have $p(T(x)) \leq$

$C(p)p_0(x)$. In particular, we can find $C(0) \geq 1$ such that $p_0(T(x)) \leq C(0)p_0(x)$. Fix $p \in cs(E)$ and $x \in E$. We have, for $k \in \mathbb{N}, k \geq 2$,

$$p(T^k(x)) \leq C(p)p_0(T^{k-1}(x)) \leq C(p)C(0)^{k-1}p_0(x).$$

Taking $q(x) := C(p)p_0(x), x \in E$, we get $p(T^k(x)) \leq C(0)^k q(x)$ for each $k \in \mathbb{N}, x \in E$. □

Many non normable Fréchet spaces $E$ admit an operator $T \in L(E)$ which is not topologizable. Our next result covers the spaces given by Żelazko in [**Z2**], Examples 8 (a), (b), (c). The operator in the examples (a) and (b) is the translation operator.

Theorem 8. *If $E$ is a Fréchet space without a continuous norm, then there is $T \in L(E)$ which is not topologizable.*

Proof. By a classical result of Bessaga and Pelczynski, see e.g. [**BP**, 2.6.13], there is a complemented subspace $F$ of $E$ which is isomorphic to the space $\omega$ of all sequences endowed with the pointwise topology. By Żelazko [**Z2**], Example 8 (c), the backward shift $B(x_1, x_2, x_3, ...) := (x_2, x_3, ...)$ on $\omega$ is not topologizable. Note that the notation for the space $\omega$ in [**Z2**] is $(s)$. Consider $B$ as an operator on $F$. By assumption, there are continuous linear maps $\pi : E \to F$ and $j : F \to E$ such that $\pi \circ j$ coincides with the identity on $F$. Define $T \in L(E)$ by $T(x) := (j \circ B \circ \pi)(x), x \in E$. It is easy to see that if $T$ were topologizable on $L(E)$, then $B$ would be topologizable in $L(F)$. □

Our next result covers examples of non normable Fréchet spaces with a continuous norm and, in particular, the space which is used in Example 8 (d) in [**Z2**], since the space of entire functions $H(\mathbb{C})$ is isomorphic to the power series space $\Lambda_\infty(\alpha)$ with $\alpha = (\alpha_i)_i, \alpha_i = i, i \in \mathbb{N}$. However, the operator mentioned in [**Z2**] is more natural, it is the translation operator.

Theorem 9. *If $E = \Lambda_\infty(\alpha)$ is a power series space of infinite type, then there is $T \in L(E)$ which is not topologizable.*

Proof. Define the diagonal operator $T \in L(E)$ by $T((x(i))_i) := (\exp(\alpha_i)x(i))_i$ for every $x \in E$. It is easy to see that $T$ is continuous. Suppose that $T$ is topologizable. Given the seminorm $p_1(x) = \sum_i \exp(\alpha_i)|x(i)|, x \in E$, there is $n \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$, there is $M_k > 0$ such that, for each $k \in \mathbb{N}$ and $x \in E$,

$$p_1(T^k(x)) \leq M_k p_n(x) = M_k \sum_i \exp(n\alpha_i)|x(i)|.$$

This implies, for each $k \in \mathbb{N}$, taking $x = e_i$, $\exp(\alpha_i)^{k+1} \leq M_k \exp(n\alpha_i)$ for each $i \in \mathbb{N}$. For $k = n+1$ we conclude $\alpha_i \leq \log(M_{n+1})$ for each $i \in \mathbb{N}$. This is a contradiction, since the sequence $\alpha$ tends to infinity. □

Every composition operator $T = C_\varphi, f \to f \circ \varphi$, on the space $H(\mathbb{D})$ of holomorphic functions on the unit disc, defined by a holomorphic self map $\varphi : \mathbb{D} \to \mathbb{D}$ such that the sequence of iterates $(\varphi^k(0))_k$ of $0 \in \mathbb{D}$ converges to a point in the boundary of $\mathbb{D}$, is a continuous linear operator on $T \in L(H(\mathbb{D}))$ which is not topologizable. We refer the reader to Shapiro [**S**] for details.

*We do not know if every non normable Fréchet space with a continuous norm admits an operator which is not topologizable.*

There are non complete non normable metrizable locally convex spaces on which every operator is $m$-topologizable. The following result is based on a deep result due to Valdivia about the existence of dense hyperplanes of separable Fréchet spaces with special properties, see [**BP**, 6.3.11].

THEOREM 10. *Every infinite dimensional non normable separable Fréchet space $F$ contains a dense hyperplane $E$ which is not normable and satisfies that every operator $T \in L(E)$ is $m$-topologizable.*

PROOF. We apply Bonet, Frerick, Peris, Wengenroth [**BFPW**], Lemma 3.1, to conclude that $F$ contains a dense hyperplane $E$ such that every operator $T \in L(E)$ is of the form $T = \lambda I + B$, with $B$ an operator with finite dimensional range. Since $F$ is not normable and $E$ is dense in $F$, the metrizable space $E$ is not normable. Take $T = \lambda I + B \in L(E)$, with the range of $B$ finite dimensional. Every operator with finite dimensional range is bounded, hence $m$-topologizable by Proposition 7. Since the operators $\lambda I$ and $B$ commute, we can apply Żelazko [**Z2**], Proposition 12, to conclude that $T$ is $m$-topologizable too. □

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