Regularity of curves with a continuous tangent line

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Abstract

This note contains a proof of the fact that a Jordan curve in $\mathbb{R}^2$ with a continuous tangent line at each point admits a regular reparametrization. We extend the result both to more general curves in $\mathbb{R}^n$ and to higher orders of differentiability.

1 Introduction

An important result in the theory of the boundary regularity of the Riemann mapping, due to E. Lindelöf [Lin], asserts that a Jordan domain has a continuous tangent line at each point of the boundary if and only if the argument of the derivative of the Riemann mapping extends continuously to the boundary of the unit disk.

The traditional concept of a continuous tangent line at a point of a curve is of geometrical nature and essentially independent of the parametrization of the curve:

Definition. A Jordan curve $\gamma\colon [0,1] \to \mathbb{R}^2$ is said to have a **continuous tangent line** at each point if and only if there exists a continuous function $\beta\colon [0,1] \to \mathbb{R}$ satisfying[1] for any t_0,

$$\lim_{t\to t_0^+} \arg\{\gamma(t)-\gamma(t_0)\} = \beta(t_0)$$

and

$$\lim_{t\to t_0^-} \arg\{\gamma(t)-\gamma(t_0)\} = \beta(t_0)+\pi.$$

In the case of regular curves (having a $\mathcal{C}^1$ parametrization with nonvanishing tangent vector) the tangent line is given by the tangent direction.

The precise concept of regular curve comes from the following definitions:

Definition. A curve $\gamma\colon [0,1] \to \mathbb{R}^n$ is said to have **a regular local parametrization** if and only if:

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[1]As usual, the argument is measured with respect to the X axis in $\mathbb{R}^2$.
Clearly the referred condition is exact for $t_0 \in (0,1)$ and has a different but analogous formulation for $t_0 = 0, 1$.

A) For any $t_0 \in (0,1)$ there exists $\delta = \delta(t_0)$, $J = J_{t_0} \subset \mathbb{R}$ a bounded open interval and $\mu\colon J \to \mathbb{R}^n$, $\mathcal{C}^1$, such that $\mu(J) = \gamma(t_0 - \delta, t_0 + \delta)$ and μ' is never 0 on J.

B) There exists $\delta_0' > 0$, $J_0 \subset \mathbb{R}$ a bounded open interval and $\mu_0\colon J_0 \to \mathbb{R}^n$, $\mathcal{C}^1$ such that $\mu(J_0) = \gamma([0,\delta_0')) \cup \gamma((1-\delta_0', 1])$ and μ_0' is never 0 on J_0.

Definition. 1. A curve $\gamma\colon [0,1] \to \mathbb{R}^n$ is said to have **a regular global parametrization** if and only if there exists $\mu\colon [0,1] \to \mathbb{R}^n$ realizing the properties A) and B).

2. A curve γ is said to be **regular** if and only if it has a global regular parametrization

The assumption that the derivative is always non zero is more subtle and basic than appears on first sight. Every polygonal curve permits an infinitely differentiable parametrization γ [Dem, p. 10]. The point is that $\gamma'(t) = 0$ for t corresponding to a corner.

It is often taken for granted that definitions of having a **continuous tangent line** and being **regular** are equivalent For instance in [Pom, Sect. 3.2] the tangent definition is used to prove Lindelöf's theorem whereas in [Pom, Sect. 3.3] the other definition is used.

The proof of the fact that regular curves posses a continuous tangent line is quite elementary. In the present paper we give an accessible proof of the converse. There cannot be any doubt that the classical literature contains a proof, but the authors where unable to find a reference. For instance in [Dem, p. 11] the fact is stated as a theorem but withouth giving a proof.

However, our proof covers the case of general (not necessarily Jordan) curves in $\mathbb{R}^n$, as well as a generalization to higher orders of differentiability. We think this is the main interest of the paper.

2 Curves with continuous geometric tangent lines

Suppose, now, that $\gamma\colon (0,1) \to \mathbb{R}^n$ is a continuous arc with the natural assumption that no open interval in $(0,1)$ is applied by γ to a single point. In the remaining of the paper, the notation for the components of a curve γ will be $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, as well as $(\ ,\ )$ for the standard scalar product in $\mathbb{R}^n$.

Definition (Continuous geometric tangent line). We will say that γ has a continuous tangent line at each point if and oly if there is a continuous map $B\colon (0,1) \to S^{n-1}$ (the euclidean unit sphere in $\mathbb{R}^n$) such that for any $t_0 \in (0,1)$, one has

$$\lim_{t \to t_0^+} \frac{\gamma(t) - \gamma(t_0)}{\|\gamma(t) - \gamma(t_0)\|} = B(t_0)$$

and

$$\lim_{t \to t_0^-} \frac{\gamma(t) - \gamma(t_0)}{\|\gamma(t) - \gamma(t_0)\|} = -B(t_0),$$

whenever $\gamma(t) \neq \gamma(t_0)$.

Remark. Observe that under the hypotheses of the definition above, **any point in the curve has finite multiplicity**. Otherwise there would be a point $t_0 \in (0,1)$ and a sequence of disjoint open intervals $I_l = (\alpha_l, \beta_l)$ whose extreme points increase (or decrease) to t_0 and satisfy $\gamma(\alpha_l) = \gamma(\beta_l) = \gamma(t_0)$, for any l. It is possible to find a sequence of points $s_l \in I_l$ such that $\frac{\gamma(s_l)-\gamma(\alpha_l)}{\|\gamma(s_l)-\gamma(\alpha_l)\|} = B(\alpha_l) + w_l$ and $\|w_l\| \to_{l\to+\infty} 0$. This means that $(\frac{\gamma(s_l)-\gamma(\alpha_l)}{\|\gamma(s_l)-\gamma(\alpha_l)\|}, B(t_0)) \to_{t\to+\infty} 1$, but $\frac{\gamma(s_l)-\gamma(\alpha_l)}{\|\gamma(s_l)-\gamma(\alpha_l)\|} = \frac{\gamma(s_l)-\gamma(t_0)}{\|\gamma(s_l)-\gamma(t_0)\|} \to -B(t_0)$, so the limit of the scalar product above should be -1. This is a contradiction.

The previous definition is the one adopted in [Gar, p. 60], for Jordan curves, in the case of $n = 2$.

Even in $\mathbb{R}^n$, the definition above imposes strong restrictions on the curve:

Proposition 1. *If γ has continuous tanget line at each point, then for every $t_0 \in (0,1)$ there exist $\delta > 0$ and $j \in \{1, \dots, n\}$ such that $\gamma_j \colon (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ is injective.*

Proof. For a fixed $t_0 \in (0,1)$, after an afine change of coordinates, we may assume that $B(t_0) = (1, 0, \dots, 0) = e_1$. Then, there exists $\delta > 0$ such that $(B(t), e_1) > 0$ for $t \in (t_0 - \delta, t_0 + \delta)$. As a consequence $\gamma_1(t)$ is injective on this interval, otherwise there would be $a, b \in (t_0 - \delta, t_0 + \delta)$ such that $\gamma_1(a) = \gamma_1(b)$, and this would imply the existence of a point $\tau \in (a, b)$ with $\gamma_1(\tau) = \gamma_1(a) = \gamma_1(b)$:

For t in a neighborhood of a and $t > a$ we have $(\gamma(t) - \gamma(a), e_1) > 0$ which implies that $\gamma_1(t) > \gamma_1(a) = \gamma_1(b)$. Analogously, for t in in a neighborhood of b and $t < b$, we have $(\gamma(t) - \gamma(b), e_1) < -\frac{1}{2}(B(b), e_1) < 0$, and therefore $\gamma_1(t) < \gamma_1(b) = \gamma_1(a)$. Then Bolzano's theorem applied to the function $f(t) = \gamma_1(t) - \gamma_1(a)$ will show the existence of τ.

Iteration of this procedure would provide points $\tau_n \to \tau_0$ with $\tau_n, \tau_0 \in (t_0 - \delta, t_0 + \delta)$ such that $\gamma_1(\tau_n) = \gamma_1(\tau_0)$. Then $(\gamma(\tau_n) - \gamma(\tau_0), e_1) = 0$, but $(B(\tau_0), e_1) > 0$. □

Corollary 1. *If $\gamma \colon [0,1] \to \mathbb{R}^n$ is a curve having continuous tangent line at every point, then for any $t_0 \in [0,1]$ there is an open neighborhood I_{t_0} (in the extended sense for the cases $t_0 = 0, 1$), such that $\gamma_{|I_{t_0}}$ is a Jordan arc.*

3 The case of Jordan arcs

Suppose now that $\gamma \colon [0,1] \to \mathbb{R}^n$ is a Jordan arc (γ continuous and injective).

Proposition 2. *If γ has continuous tangent line at each point, then the set $\gamma((0,1))$ admits a regular local parametrization.*

Proof. We will proceed by induction on the dimension:

Fix $t_0 \in (0,1)$. After a rigid movement in $\mathbb{R}^n$ we may suppose that $\gamma(t_0) = 0$ and $B(t_0) = e_1$. By Proposition 1, there exists $\delta > 0$ such that $\gamma_1 \colon (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ is an injective map to the X_1 axis, and so it is the projection of $\gamma((t_0 - \delta, t_0 + \delta))$ onto the hyperplane $\langle e_n \rangle_{\mathbb{R}}^{\perp}$. Let $p \colon \mathbb{R}^n \to \langle e_n \rangle_{\mathbb{R}}^{\perp}$ be the orthogonal projection map.

Since p is continuous, $p \circ \gamma$ is a continuous curve, and since $t \to (p \circ \gamma(t), e_1) = \gamma_1(t)$ is injective in $I = (t_0 - \delta, t_0 + \delta)$, so is $\rho = p \circ \gamma$ in this interval.

Then ρ is a Jordan arc in $\mathbb{R}^{n-1}$. Moreover, if $t_1 \in I$, we have

$$p(B(t_1)) = p(\lim_{t\to t_1^+} \frac{\gamma(t)-\gamma(t_1)}{\|\gamma(t)-\gamma(t_1)\|}) = \lim_{t\to t_1^+} \frac{p\circ\gamma(t)-p\circ\gamma(t_1)}{\|\gamma(t)-\gamma(t_1)\|}$$

$$= \lim_{t\to t_1^+} \frac{\rho(t)-\rho(t_1)}{\|\gamma(t)-\gamma(t_1)\|},$$

and therefore

$$\|p(B(t_1))\| = \lim_{t\to t_1^+} \frac{\|\rho(t)-\rho(t_1)\|}{\|\gamma(t)-\gamma(t_1)\|}.$$

Since $(B(t_1), e_1) \neq 0$, we see that $\|p(B(t_1))\| > 0$, and

$$\lim_{t\to t_1^+} \frac{\rho(t)-\rho(t_1)}{\|\rho(t)-\rho(t_1)\|} = \lim_{t\to t_1^+} \frac{p\circ\gamma(t)-p\circ\gamma(t_1)}{\|\gamma(t)-\gamma(t_1)\|} \lim_{t\to t_1^+} \frac{\|\gamma(t)-\gamma(t_1)\|}{\|\rho(t)-\rho(t_1)\|}$$

$$= \lim_{t\to t_1^+} \frac{\|\gamma(t)-\gamma(t_1)\|}{\|\rho(t)-\rho(t_1)\|} \lim_{t\to t_1^+} p(\frac{\gamma(t)-\gamma(t_1)}{\|\gamma(t)-\gamma(t_1)\|}),$$

because p is linear, so

$$\lim_{t\to t_1^+} \frac{\rho(t)-\rho(t_1)}{\|\rho(t)-\rho(t_1)\|} = p(B(t_1)) \lim_{t\to t_1^+} \frac{\|\gamma(t)-\gamma(t_1)\|}{\|\rho(t)-\rho(t_1)\|} = p(B(t_1))\frac{1}{\|p(B(t_1))\|} \neq 0.$$

Now, if the result is true for Jordan arcs in $\mathbb{R}^{n-1}$, then ρ admits a local $\mathcal{C}^1$ parametrization. Let us call it

$$\mu\colon (\tau_0-\delta'', \tau_0+\delta'') \to \mathbb{R}^{n-1},$$

with $\mu(\tau_0) = \rho(t_0)$, for some $\delta'' > 0$.

On the other hand, the injectivity of the projection p in $\rho(I)$ implies that for a small interval $I_0 \Subset I$, $\gamma(I_0)$ is a graph over $\rho(I_0)$, namely there exists a function $f\colon \rho(I_0) \to \mathbb{R}$, such that

$$\gamma(I_0) = \{(\mu(\tau),\ f(\mu(\tau)) :\ \tau \in (\tau_0-\delta'', \tau_0+\delta'')\}.$$

The parametrization $\tau \to (\mu(\tau),\ f(\mu(\tau))$ is $\mathcal{C}^1$ in $(\tau_0-\delta'', \tau_0+\delta'')$, because for any $t_1 \in (\tau_0-\delta'', \tau_0+\delta'')$ one has

$$\frac{f(\mu(\tau_1+h))-f(\mu(\tau_1))}{h} = \frac{f(\mu(\tau_1+h))-f(\mu(\tau_1))}{\|\mu(\tau_1+h))-\mu(\tau_1)\|}\frac{\|\mu(\tau_1+h))-\mu(\tau_1)\|}{h}.$$

The first term is

$$\frac{\gamma_n(t_1+s)-\gamma_n(t_1)}{\|p(\gamma(t_1+s))-p(\gamma(t_1))\|} = \frac{\dfrac{\gamma_n(t_1+s))-\gamma_n(t_1))}{\|\gamma(t_1+s))-\gamma(t_1)\|}}{\dfrac{\|p(\gamma(t_1+s))-p(\gamma(t_1))\|}{\|\gamma(t_1+s))-\gamma(t_1)\|}},$$

having limit

$$\frac{B_n(t_1)}{\|p(B(t_1)\|}$$

as $h \to 0^+$ (or $s \to 0^+$). The second term has limit $\|\mu'(\tau_1)\|$. Then

$$\lim_{h\to 0^+} \frac{f(\mu(\tau_1+h)) - f(\mu(\tau_1))}{h} = \frac{B_n(\gamma^{-1}(\tau_1))}{\|p(B(\gamma^{-1}(\tau_1))\|}\|\mu'(\tau_1)\|.$$

The limit when $h \to 0^-$ can be managed in a similar way.

Also, this parametrization has nonvanishing tangent vector, because $\mu'(\tau)$ is never 0.

The case $n = 1$ is trivial, and this, as first step of induction, would conclude the assertion. Nevertheless, we begin the induction by the case of $n = 2$, because it contains the basic ingredients of the general proof, and also has interest in itself, as a standard statement in the study of the boundary regularity of the Riemann conformal map. In this case, the usual presentation of the hypotheses uses the function $\beta(t) = \arctan \frac{B_2(t)}{B_1(t)}$. We will use this notation for a while.

Since γ is continuous, it follows that $J = \gamma_1(t_0 - \delta, t_0 + \delta)$ is an open interval of the X axis, and the set $\{\gamma(t) : t \in (t_0 - \delta, t_0 + \delta)\}$ is the graph of the function $f(x) = \gamma_2 \circ \gamma_1^{-1}(x)$, defined in J.

Now, f is a $\mathcal{C}^1$ function on J: If $x_0 \in J$, $x_0 = \gamma_1(t_1)$, we have

$$\lim_{h\to 0}\left\{\frac{f(x_0+h)-f(x_0)}{h} - \tan\beta(t_0)\right\} = \lim_{t\to t_0}\left\{\frac{\gamma_2(t)-\gamma_2(t_1)}{\gamma_1(t)-\gamma_1(t_1)} - \tan\beta(t_1)\right\} = 0,$$

so[2], since β is a continuous function, then $f \in \mathcal{C}^1$, and $x \to (x, f(x))$ is a $\mathcal{C}^1$ parametrization of the curve γ around the point $\gamma(t_0)$, with nonvanishing tangent vector $(1, f'(x))$. □

4 Globalization of the parametrization

Now, the curves possessing a local $\mathcal{C}^1$ parametrization have a global one in a natural way.

Proposition 3. *If γ is continuous closed curve in $\mathbb{R}^n$ having a regular local parametrization, then γ admits a regular global parametrization.*

Proof. Let $t_0 \in (0,1)$ and $\varsigma_0 = \gamma(t_0)$. By the proposition 2, there are intervals $I_{t_0} \Subset (0,1)$, $J \Subset \mathbb{R}$ and $\mu\colon J \to \mathbb{R}^n$ a local $\mathcal{C}^1$ paramentrization of $\gamma(I)$ with nonvanishing derivative. We can choose $\tau_0 \in J$ such that $\mu(\tau_0) = \varsigma_0$, and since $\mu'(\tau_0) \neq 0$, there is an open interval $J'_{\tau_0} \Subset J$ where μ is injective, and $\mu(J')$ coincides with the image by γ of a corresponding interval I', like in cases A) and B) of the definition in section 1.

(WLOG we may suppose that the first component of $\mu'(\tau_0)$ is strictly positive, and so the first component of μ is an homeomorphism from an open interval J'_{τ_0} to an open interval, $K \subset \mathbb{R}$, containing the image of τ_0 in the interior. Then $\gamma^{-1}(K)$ contains t_0 in the interior, and we choose the corresponding interval.)

A similar procedure works for $t_0 = 0$ or 1.

The curve γ is rectifiable: Take a finite covering of $[0,1]$ by intervals such that the image admits a paramatrization μ in a neighborhood of the closure

[2]In fact $f'(x) = \tan\beta(\gamma_1^{-1}(x))$.

of J (μ and as J as above). Each arc $\mu(J)$ has finite length, so $\gamma([0,1])$ has it too. Let $L>0$ be the length of $\gamma([0,1])$.

Moreover, there is a finite collection of points $0<t_1<\cdots<t_p<1$ and positive numbers $\delta_1,\ldots,\delta_p,\delta'$ to which there is an associate covering of $[0,1]$ by intervals

$$I_0=[0,\delta'),\ldots,I_j=(t_j-\delta_j,t_j+\delta_j),\ldots,I_{p+1}=(1-\delta',1],$$

such that I_j only intersects I_{j-1}, I_{j+1}.

Choose points $t'_j\in I_j\cap I_{j+1}$, for $j=0,\ldots,p$, and consider the arcs $\Gamma_0=\gamma([0,t'_0]\cup[t'_p,1])$ and $\Gamma_j=\gamma([t'_j,t'_{j+1}])$. Also call $x_j=\gamma(t'_j)$.

For any j, we have for the corresponding J_j and μ_j, that $\Gamma_j\subset\mu_j(J_j)$, and since μ_j is continuous and injective in J_j, we can parametrize Γ_j by its arc length:

$$s_j(\tau)=\int_{\mu_j^{-1}(x_j)}^{\tau}\|\mu'_j(\xi)\|\,d\xi,$$

and $\lambda_j(s_j)=\mu_j(\tau(s_j))$, for $s_j\in[0,\ell(\Gamma_j))$, where ℓ means length.

Then we have a global parametrization:

In $[0,L]$ we consider the points $\sigma_j=\sum_{k=1}^{j}\ell(\Gamma_k)$.

On each interval $[L_j,L_{j+1}]$, we consider the parametrization λ_j, and define $\varrho\colon[0,L]\to\mathbb{R}^n$ as $\varrho(s)=\lambda_j(s-L_j)$, for $s\in[L_j,L_{j+1}]$.

The fact that $\|\varrho'\|\equiv1$ and that the direction of ϱ' is the same as the corresponding μ'_j, which are continuous, imply that ϱ is globally $\mathcal{C}^1$. □

Finally, we have as a corollary

Theorem 1. *If $\gamma\colon[0,1]\to\mathbb{R}^n$ is a (continuous closed) curve having a continuous tangent line at each point then γ admits a regular global parametrization.*

Remark. The case $n=2$ provides classical statement:

The Jordan curves in $\mathbb{R}^2$ having a continuous tangent line at each point admit a regular global parametrization.

5 Higher order of differentibility

In the previous sections we have seen how the geometrical condition of having a tangent line at each point implies that the curve admits a $\mathcal{C}^1$ parametrization with nonvanishing derivative, and how this geometrical property is independent of the particular parametrization γ, i. e. it can be checked from an a proiri given parametrization γ. We will study now how the existence of reparametrizations of higher order of differentiability can be checked by looking at the original parametrization too.

First of all, we have that a curve satisfiying the condition $\lim_{t\to t_0^\pm}\frac{\gamma(t)-\gamma(t_0)}{\|\gamma(t)-\gamma(t_0)\|}=\pm B(t_0)$, with $B(t)$ continuous, admits a parametrization by the arc length ϱ that is $\mathcal{C}^1$. Then it is an easy observation that the curve admits a $\mathcal{C}^k$ parametrization if and only if ρ is $\mathcal{C}^k$.

Let us consider the case $k = 2$. The curve admits a $\mathcal{C}^2$ parametrization iff the limit

$$\lim_{s \to s_0} \frac{\varrho'(s) - \varrho'(s_0)}{s - s_0} = \varrho''(s_0)$$

is continuous.

Fix t_0 and t. Since there exists a $\mathcal{C}^1$ diffeomorphism θ, such that $s = \theta(t)$, we have

$$\frac{\varrho'(s) - \varrho'(s_0)}{s - s_0} = \frac{\varrho'(\theta(t)) - \varrho'(\theta(t_0))}{\theta(t) - \theta(t_0)} = \frac{B(t) - B(t_0)}{\theta(t) - \theta(t_0)}$$

$$= \frac{1}{\theta(t) - \theta(t_0)}\{\sigma(\tau', t)\frac{\gamma(\tau') - \gamma(t)}{\|\gamma(\tau') - \gamma(t)\|} + \sigma(\tau, t_0)\frac{\gamma(\tau) - \gamma(t_0)}{\|\gamma(\tau) - \gamma(t_0)\|} + w(\tau', \tau, t, t_0)\},$$

where $w = o(|\theta(t) - \theta(t_0)|)$ and $\sigma(t', t'') = 1$ if $t'' < t'$ and -1 if $t' < t''$.

Since the curve is rectifiable, we have

$$\theta(t) - \theta(t_0) = s - s_0 = \sup\left\{\sum_i \|\gamma(\tau_i) - \gamma(\tau_{i-1})\|;\ \{\tau_i\} \subset \mathcal{P}(J_{t,t_0})\right\},$$

where $\mathcal{P}(J_{t,t_0})$ is the set of partitions of the interval between t and t_0. Then, for a given $0 < \epsilon < (s - s_0)^2$, there is a partition $\{\tau_i\}$ such that

$$\|\gamma(t) - \gamma(t_0)\| \leq s - s_0 = \alpha + \sum_i \|\gamma(\tau_i) - \gamma(\tau_{i-1})\|,$$

where $|\alpha| < \epsilon$.

The main condition on the curve implies that

$$\gamma(\tau_i) - \gamma(\tau_{i-1}) = (B(t_0) + v_i)\|\gamma(\tau_i) - \gamma(\tau_{i-1})\|,$$

where $\|v_i\| = o(|s - s_0|)$, for $|t - t_0|$ small. So

$$\begin{aligned}
\gamma(t) - \gamma(t_0) &= \sum_i \gamma(\tau_i) - \gamma(\tau_{i-1}) \\
&= B(t_0)\sum_i \|\gamma(\tau_i) - \gamma(\tau_{i-1})\| + \sum_i \|\gamma(\tau_i) - \gamma(\tau_{i-1})\| v_i \\
&= (s - s_0)B(t_0) - \alpha B(t_0) + \sum_i \|\gamma(\tau_i) - \gamma(\tau_{i-1})\| v_i,
\end{aligned}$$

and

$$(\gamma(t) - \gamma(t_0), B(t_0)) = (s - s_0)(1 + o(|s - s_0|)).$$

This implies that

$$\left(\frac{\gamma(t) - \gamma(t_0)}{\|\gamma(t) - \gamma(t_0)\|}, B(t_0)\right) = \frac{s - s_0}{\|\gamma(t) - \gamma(t_0)\|}(1 + o(|s - s_0|))$$

and so

$$\lim_{t \to t_0} \frac{s - s_0}{\|\gamma(t) - \gamma(t_0)\|} = 1.$$

Then

$$\lim_{s\to s_0}\frac{\varrho'(s)-\varrho'(s_0)}{s-s_0}=\lim_{t\to t_0}\frac{B(t)-B(t_0)}{\|\gamma(t)-\gamma(t_0)\|}$$

$$=\lim_{t\to t_0}\frac{\sigma(t,t_0)}{\|\gamma(t)-\gamma(t_0)\|}\left\{\sigma(\tau',t)\frac{\gamma(\tau')-\gamma(t)}{\|\gamma(\tau')-\gamma(t)\|}+\sigma(\tau,t_0)\frac{\gamma(\tau)-\gamma(t_0)}{\|\gamma(\tau)-\gamma(t_0)\|}\right\},$$

and we have the result

Theorem 2. *If $\gamma\colon[0,1]\to\mathbb{R}^n$ is a continuous closed curve, then γ admits a $\mathcal{C}^2$ paramentrization with nonvanishing first derivative if and only if there are two vector-valued continuous functions*

$$B^{(j)}\colon[0,1]\to\mathbb{R}^n,\quad j=1,2,$$

such that

$$B^{(1)}(t_0)=\lim_{t\to t_0}\frac{1}{\|\gamma(t)-\gamma(t_0)\|}\{\sigma(t,t_0)(\gamma(t)-\gamma(t_0))\}\neq 0$$

and

$$B^{(2)}(t_0)=\lim_{t\to t_0;\ |\tau'-t|,|\tau-t_0|=o(|t-t_0|)}\frac{\sigma(t,t_0)}{\|\gamma(t)-\gamma(t_0)\|}\left\{\sigma(\tau',t)\frac{\gamma(\tau')-\gamma(t)}{\|\gamma(\tau')-\gamma(t)\|}\right.$$
$$\left.+\sigma(\tau,t_0)\frac{\gamma(\tau)-\gamma(t_0)}{\|\gamma(\tau)-\gamma(t_0)\|}\right\},$$

where the precedence indicator $\sigma(\alpha,\beta)$ is 1 if $\beta<\alpha$ and -1 if $\alpha<\beta$.

Remark. In this case, if γ is parametrized by the arc length, the term $B^{(2)}$ corresponds to the curvature parameters, so is the curvature radius and the normal vector.

Then, giving a priori these parameters and requireing their continuity, aside to the tangent direction, implies that γ admits a $\mathcal{C}^2$ parametrization.

The corresponding statement for the $\mathcal{C}^k$ case is as follows:

Theorem 3. *If $\gamma\colon[0,1]\to\mathbb{R}^n$ is a continuous closed curve, then γ admits a $\mathcal{C}^k$ paramentrization with nonvanishing first derivative if and only if there are k vector-valued continuous functions*

$$B^{(N)}\colon[0,1]\to\mathbb{R}^n,\quad N=1,\dots,k,$$

such that

$$B^{(N)}(t_0)=\lim_{\substack{t_1,\dots,t_{2^N-1}\to t_0\\ |t_p-t_q|=o(|t_{2^N-1}-t_0|),\ \forall p,q}}\frac{\sigma(t_{2^N-1},t_0)}{\|\gamma(t_{2^N-1})-\gamma(t_0)\|}\sum_{i=0}^{2^{N-1}-1}\sigma(t_{2i+1},t_{2i})\frac{\gamma(t_{2i+1})-\gamma(t_{2i})}{\|\gamma(t_{2i+1})-\gamma(t_{2i})}$$

$$\prod_{s=2}^{N-1}\sigma(t_{2^s(E[\frac{2i+1}{2^s}]+1)-1},t_{2^sE[\frac{2i+1}{2^s}]})\frac{(-1)^{E[\frac{2i+1}{2^s}]+1}}{\|\gamma(t_{2^s(E[\frac{2i+1}{2^s}]+1)-1})-\gamma(t_{2^sE[\frac{2i+1}{2^s}]})\|},$$

where the precedence indicator $\sigma(\alpha,\beta)$ is 1 if $\beta<\alpha$ and -1 if $\alpha<\beta$.

Proof. The proof results from the recurrent use of arguments completely analogous to the ones guiving the $\mathcal{C}^2$ case. The characterization is given by a similar but more complicated formula that involves 2^k points and iterated quotients of differences of values of γ at these points, with corresponding precedence signs. The denominators are always of the form $\|\gamma(t') - \gamma(t'')\|$ for t', t'' some of these points. The long formula in the statement is a compressed version of the natural formula, in the $\mathcal{C}^k$ case.

□

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