TOEPLITZ OPERATORS ON WEIGHTED HARDY AND BESOV SPACES

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Abstract. We obtain estimates of the norm of Toeplitz operators on weighted Hardy and Besov spaces. As an application we give characterizations of some spaces of pointwise multipliers.

1. Introduction

The main goal of this work is the study of the norms of the Toeplitz operators in a large scale of spaces of holomorphic functions on B, which includes weighted Hardy and Besov spaces. In order to introduce the problem and to state our results, we recall some results on Toeplitz operators in the classical setting of the Hardy space H^p.

Let B be the open unit ball in C^n, $\bar{B}$ its closure and S its boundary. By $d\nu$ and $d\sigma$ we denote the normalized Lebesgue measures on B and S respectively. We denote by $H(B)$ (resp. $H(\bar{B})$) the space of holomorphic functions on B (resp. on $\bar{B}$). If $p>0$, a holomorphic function f on B is in the Hardy space $H^p(B)$ if and only if

$$\|f\|_{H^p} = \sup_{r<1}\int_S |f(r\zeta)|^p d\sigma(\zeta) < +\infty.$$

If $p \geq 1$, the operator which assigns to each f its boundary values $f(\zeta) = \lim_{r\nearrow 1} f(r\zeta)$, a.e $\zeta \in S$ defines an isometry from $H^p(B)$ onto $H^p(S)$,

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the L^p-closure of the restriction to S of $H(\bar{B})$. A well known result about operators on the Hardy spaces $H^p(S)$, $1 < p < \infty$, states that for any $\psi \in L^\infty(d\sigma)$, the norm of the Toeplitz operator with symbol ψ, $T_\psi : H^p(S) \to H^p(B)$, defined by

$$T_\psi(f)(z) = \int_S \frac{\psi(\zeta)f(\zeta)}{(1 - z\bar{\zeta})^n} d\sigma(\zeta),$$

is equivalent to $\|\psi\|_\infty$.

By composing the above mentioned isometry between $H^p(B)$ and $H^p(S)$ with T_ψ we can obtain maps from $H^p(S)$ to itself, or from $H^p(B)$ to itself. All these operators will be denoted by T_ψ and we will simply write H^p to denote either $H^p(S)$ or $H^p(B)$.

We will consider the duality $(H^p)' = H^{p'}$ with respect to the pairing

$$\langle f, \bar{g} \rangle_S = \lim_{r \to 1} \int_S f_r \bar{g}_r d\sigma, \tag{1.1}$$

where $f_r(\zeta) = f(r\zeta)$, in the sense that each $\Lambda \in (H^p)'$ is given by $\Lambda(f) = \langle f, \bar{g} \rangle_S$, for some $g \in H^{p'}$ and $\|\Lambda\|_{(H^p)'} \approx \|g\|_{H^{p'}}$. Then, we have that the following equivalence between the norm of the bilinear Toeplitz form $\Gamma_\psi(f, \bar{g}) = \int_S \psi f \bar{g} d\sigma = \langle T_\psi(f), \bar{g} \rangle_S$ and the norm of the operator T_ψ holds:

$$\|\Gamma_\psi\|_{Bil(H^p \times \overline{H^{p'}} \to \mathbb{C})} \approx \|T_\psi\|_{L(H^p \to H^p)} \approx \|\psi\|_\infty. \tag{1.2}$$

There exists an extensive literature on Toeplitz operators in spaces of holomorphic functions. See for instance [11], [2] and the references therein.

We want to extend these results to other Banach spaces of holomorphic functions on B. Throughout the paper we consider Banach spaces of holomorphic functions on B, X and Y satisfying $H(\bar{B}) \subset X, Y$. We denote by X_c (resp. Y_c), the space $H(\bar{B})$ normed by $\|\cdot\|_X$ (resp. $\|\cdot\|_Y$). If $X, Y \subset H^1(B)$, then the functions f in X or Y, have boundary values $f(\zeta)$, a.e. $\zeta \in S$ in $L^1(d\sigma)$. As in the case of H^p spaces, we will identify the space X with its space of boundary values.

The Toeplitz form Γ_ψ is well-defined on $X_c \times \overline{Y}_c$ for any $\psi \in L^1(d\sigma)$, and the associated Toeplitz operator T_ψ defines an operator from X_c to $H(B)$. Moreover, notice that if $1 \le p \le \infty$, $\psi f \in L^p(d\sigma)$ and $g \in H^{p'}(B)$, then

$$\Gamma_\psi(f, \bar{g}) = \langle T_\psi(f), \bar{g}\rangle_S.$$

Therefore, if Y' is the dual space of Y with respect to the pairing (1.1), the norm of the Toeplitz operator $T_\psi : X \to Y'$ is equivalent to the norm of $\Gamma_\psi : X \times \overline{Y} \to \mathbb{C}$.

The norm of this form Γ_ψ will be computed in terms of extensions of ψ to B, given by generalized Poisson-Szegö operators. This computation will be used to obtain estimates of the norms of Toeplitz operators for different spaces of holomorphic functions on B. For $m \ge 0$, $\zeta \in S$ and $z \in B$, we consider the integral operators

$$P_m(\psi)(z) = c_{n,m} \int_S \psi(\zeta) \frac{(1-|z|^2)^{n+2m}}{|1 - z\bar{\zeta}|^{2n+2m}} d\sigma(\zeta),$$

where $c_{n,m}$ is a normalizing constant. If $m = 0$, we recover the Poisson-Szegö kernel. These operators give the solution of some generalized Dirichlet problems (see Section 2 for more details).

The fact that for any $z \in B$, $f_z(w) = (1 - w\bar{z})^{-n-m} \in H(\bar{B})$ gives that

$$|P_m(\psi)(z)| \le \|\Gamma_\psi\|_{Bil(X_c \times \overline{Y}_c)} \omega_m(z), \tag{1.3}$$

where

$$\begin{aligned} \omega_m(z) &= \omega_{m,X,Y}(z) \\ &= c_{n,m}(1-|z|^2)^{n+2m} \|(1-w\bar{z})^{-n-m}\|_X \|(1-w\bar{z})^{-n-m}\|_Y. \end{aligned} \tag{1.4}$$

Inequality (1.3) gives immediately two necessary conditions on ψ such that $\Gamma_\psi : X_c \times \overline{Y}_c \to \mathbb{C}$ is bounded.

The first condition is just that,

$$\sup_{z \in B} \frac{|P_m(\psi)(z)|}{\omega_m(z)} \le \|\Gamma_\psi\|_{Bil(X_c \times \overline{Y}_c \to \mathbb{C})}. \tag{1.5}$$

The second condition is given in terms of ψ. If $\varphi \geq 0$ is a continuous function on B and $\zeta \in S$, let $\liminf_{r\nearrow 1} \varphi(r\zeta) = \sup_r \inf_{r<t<1} \varphi(r\zeta)$.

Let

$$\tilde{\omega}_m(\zeta) = \liminf_{r\nearrow 1} \omega(r\zeta). \tag{1.6}$$

From (1.3), we obtain

$$|\psi(\zeta)| \leq \|\Gamma_\psi\|_{Bil(X_c \times \overline{Y}_c)} \tilde{\omega}_m(\zeta), \quad \text{a. e.} \quad \zeta \in S. \tag{1.7}$$

Now, if we assume that $0 < \tilde{\omega}_m(\zeta) < \infty$, a.e. $\zeta \in S$, we obtain

$$\sup_{\zeta \in S} \frac{|\psi(\zeta)|}{\tilde{\omega}_m(\zeta)} \leq \|\Gamma_\psi\|_{Bil(X_c \times \overline{Y}_c \to \mathbb{C})}. \tag{1.8}$$

Since $\|\Gamma_\psi\|_{Bil(X_c \times \overline{Y}_c \to \mathbb{C})} \leq \|\Gamma_\psi\|_{Bil(X \times \overline{Y} \to \mathbb{C})}$ it is clear that the corresponding conditions (1.5) and (1.8) are also necessary to ensure that $\Gamma_\psi : X \times \overline{Y} \to \mathbb{C}$ is bounded. Moreover, if X_c and Y_c are dense in X and Y respectively, and Γ_ψ is bounded on $X_c \times \overline{Y}_c$, then Γ_ψ extends to an unique operator on $X \times \overline{Y}$ also denoted by Γ_ψ.

In order to state conditions on X and Y such that

$$\begin{aligned} &\sup_{z \in B} \frac{|P_m(\psi)(z)|}{\omega_m(z)} \approx \|\Gamma_\psi\|_{Bil(X_c \times \overline{Y}_c \to \mathbb{C})}, \quad \text{or} \\ &\sup_{z \in B} \frac{|P_m(\psi)(z)|}{\omega_m(z)} \approx \|\Gamma_\psi\|_{Bil(X \times \overline{Y} \to \mathbb{C})}, \end{aligned} \tag{1.9}$$

we consider the functions $\tau_c, \tau : [0,1) \to \mathbb{R}$ defined by

$$\tau_c(r) = \sup \left\{ \frac{\|f_r \, \bar{g}_r \, (\omega_m)_r\|_{L^1(d\sigma)}}{\|f\|_X \|g\|_Y}; \; f, g \in H(\bar{B}), f, g \neq 0 \right\},$$

and

$$\tau(r) = \sup \left\{ \frac{\|f_r \, \bar{g}_r \, (\omega_m)_r\|_{L^1(d\sigma)}}{\|f\|_X \|g\|_Y}; \; f \in X, g \in Y, f, g \neq 0 \right\},$$

where $f_r(\zeta) = f(r\zeta)$, $0 \leq r < 1$, $\zeta \in S$.

With these notations we can now state the following result.

Theorem 1.1. *Let $m \geq 0$ and let X and Y be Banach spaces of holomorphic functions on B, such that $H(\bar{B}) \subset X, Y$.*

Then

$$\sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\} \leq \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})} \leq \|\tau_c\|_\infty \sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\}.$$

Moreover, if $X, Y \subset H^1(B)$,

$$\|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})} \leq \|\tau\|_\infty \sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\}.$$

Observe that the functions τ and τ_c depend only on the spaces X and Y and not on the function ψ. Of course the interesting case for applications is when the functions τ and τ_c are bounded.

Notice that if $X = H^p$ and $Y = H^{p'}$, choosing $m = 0$, we have $\|P_0(\psi)\|_\infty = \|\psi\|_\infty$, $\omega_0(z) \approx 1$ (by Proposition 1.4.10 in [14]) and $\|\tau\|_\infty = 1$ (by Hölder's Inequality). Therefore, (1.2) can be obtained from Theorem 1.1. We also remark that if $p = 2$, then $\omega_0 = 1$ and in consequence the equivalences in (1.2) can be replaced by identities (see for instance [8]).

Theorem 1.1 can be used to improve some well known results on Toeplitz operators in classical spaces. For instance, if B^p_s is the holomorphic Besov space on B (defined in Section 2) and $TSymb(X \to Y)$ denotes the set of symbols of all the continuous Toeplitz operators T_ψ from X to Y, then, this space coincides with $L^\infty(d\sigma)$ if X and Y satisfy one of the following conditions:

(i) $B^1_n \subset X \subset H^\infty$ and $BMOA \subset Y \subset B^\infty_0$.

(ii) $B^1_{n/p'} \subset X \subset H^p \subset Y \subset B^\infty_0$, with $1 < p < \infty$.

This result includes the cases where $X = A$ is the ball algebra (the space of holomorphic functions on B and continuous on $\bar{B}$), and generalizes the well known result $TSymb(H^\infty \to BMOA) = TSymb(H^\infty \to B^\infty_0) = L^\infty$ in the unit disk [9].

In addition to the results on pointwise multipliers between some classical spaces which are a consequence of the fact that, if $h \in H^1$ and $f \in H(\bar{B})$, the multiplication operator $M_h(f) = hf$ corresponds to the Toeplitz operator T_h, we also prove that:

(i) If $B_n^1 \subset X \subset H^\infty \cap VMOA$, then $\mathcal{M}(X \to VMOA)$ coincides with the multiplicative algebra $H^\infty \cap VMOA$.

(ii) Let b_0^∞ be the little Bloch space, that is the closure of $H(\bar{B})$ in B_0^∞. If $B_n^1 \subset X \subset H^\infty \cap b_0^\infty$, then $\mathcal{M}(X \to b_0^\infty)$, coincides with the multiplicative algebra $H^\infty \cap b_0^\infty$.

Here, $\mathcal{M}(X \to Y)$ denotes the space of pointwise multipliers from X to Y.

The next theorem gives conditions on X and Y such that

$$\sup_{\zeta\in S} \frac{|\psi(\zeta)|}{\tilde{\omega}_m(\zeta)} \approx \|\Gamma_\psi\|_{Bil(X_c\times \overline{Y}_c\to\mathbb{C})}. \tag{1.10}$$

Theorem 1.2. *Let X and Y be Banach spaces of holomorphic functions on B containing $H(\bar{B})$.*

If $0 < \tilde{\omega}_m(\zeta) < \infty$ a.e. $\zeta \in S$, and there exists a constant $C_{X,Y}$ such that $\|fg\|_{L^1(\tilde{\omega}_m)} \le C_{X,Y}\|f\|_X\|g\|_Y$, then

$$\left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty \le \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})} \le \|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})} \le C_{X,Y}\left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty .$$

This result permit us to obtain results on Toeplitz operators in some class of holomorphic weighted Hardy spaces. In order to precise these results we consider weights in S, that for simplicity we will call **admissible weights**, satisfying:

(i) $\theta(\zeta) > 0$ a.e. $\zeta \in S$.

(ii) $\theta, \theta^{-p'/p} \in L^1(d\sigma)$,

For this class of weights, let $H^p(\theta)$ denotes the closure of the restriction of $H(\bar{B})$ on S in $L^p(\theta)$.

Observe that if $\theta^{-p'/p} \in L^1(d\sigma)$, then for each $\varphi \in L^p(\theta)$,

$$\int_S |\varphi(\zeta)|d\sigma(\zeta) \le \|\theta^{-p'/p}\|_{L^1(d\sigma)}^{1/p'}\|\varphi\|_{L^p(\theta d\sigma)}.$$

Therefore, $H^p(\theta)$ is a subspace of H^1.

With these conditions we have:

Theorem 1.3. *Let* $1 < p < \infty$*,* $\psi \in L^1(d\sigma)$ *and* θ_0, θ_1 *a pair of admissible weights. Then,*

$$\|\Gamma_\psi\|_{Bil(H^p(\theta_0)\times\overline{H^{p'}(\theta_1)})\to\mathbb{C})} \approx \left\| \frac{\psi}{\theta_0^{1/p}\theta_1^{1/p'}} \right\|_\infty .$$

From this last result we can obtain estimates on the norm of the corresponding Toeplitz operator T_ψ on weighted Hardy spaces, once we have a description of the dual of $H^{p'}(\theta_1)$ with respect to the pairing (1.1). This is the case when θ_1 is in the Muckenhoupt class $\mathcal{A}_p$ whose the definition and some properties of these weights will be stated in Section 4. Among them, we remark that we give a characterization of the fact that $\theta \in \mathcal{A}_p$ in terms of the generalized extensions $P_m(\theta)$, which extend the one obtained by [12], where it was consider the case $m = 0$ and $p = 2$.

We obtain the following theorem:

Theorem 1.4. *Let* $1 < p < \infty$*,* $\psi \in L^1(d\sigma)$*,* θ_0 *an admissible weight and* $\theta_1 \in \mathcal{A}_{p'}$*.*

Then

$$\|T_\psi\|_{L(H^p(\theta_0)\to H^p(\theta_1))} \approx \left\| \frac{\psi}{\theta_0^{1/p}\theta_1^{-1/p}} \right\|_\infty .$$

In particular, if $\theta = \theta_0 = \theta_1 \in \mathcal{A}_p$*, then* $\|T_\psi\|_{L(H^p(\theta)\to H^p(\theta))} \approx \|\psi\|_\infty$

The paper also contains some additional results on weighted Hardy spaces and Besov spaces, that may be interesting by themselves.

If $\theta \in \mathcal{A}_p$, then $H(\bar{B})$ is dense in $H^p(\theta)$, and in this case $H^p(\theta)$ consists of holomorphic functions f on B, such that

$$\|f\|^p_{H^p(\theta)} =: \sup_{0\le r<1} \int_S |f(r\zeta)|^p \theta(\zeta) d\sigma(\zeta) < \infty.$$

Let $(I + R)^k$ be the linear differential operator of order k defined by $(I + R)^k z^\alpha = (1 + |\alpha|)^k z^\alpha$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \sum_{j=1}^n |\alpha_j|$.

If $s \in R$, we denote by $H^p_s(\theta)$ the Hardy-Sobolev space of holomorphic funtions on B such that $\|(I+R)^s f\|_{H^p(\theta)} < \infty$.

It is well known (see [10] p. 334) that if $1 < p < \infty$ and $\theta \in \mathcal{A}_p$, the dual of $H^p_s(\theta)$ with the pairing given by (1.1) is $H^{p'}_{-s}(\theta^{-p'/p})$.

In order to consider weighted Besov spaces related to $H^p(\theta)$, we introduce an averaging function Θ of the weight θ in S given by $\Theta(z) = \frac{1}{|I_z|}\int_{I_z} \theta d\sigma$. If the weight θ is in $\mathcal{A}_p$, then its averaging Θ is in the class $\mathcal{B}_p$, introduced in [5] (see Section 4 for more details). The characterizations of $H^p(\theta)$ in terms of the admissible maximal function, shows that, when $\theta \in \mathcal{A}_p$ we can define an equivalent norm in $H^p(\theta)$ in terms of the averaging function Θ given by

$$\|f\|^p_{H^p(\Theta)} =: \sup_{0 \le r < 1} \int_S |f(r\zeta)|^p \Theta(r\zeta) d\sigma(\zeta).$$

In order to define weighted Besov spaces, as in the Hardy spaces cases, for $1 \le p < \infty$ we consider admisible weights on B, that is weights $\Psi > 0$ a.e. on B such that $\Psi, \Psi^{-p'/p} \in L^1(d\nu)$.

If $1 \le p < \infty$, $s \in \mathbb{R}$, a non-negative integer $k > s$ and Ψ is an admissible weight, we denote by $B^p_{s,k}(\Psi)$ the completion of the space $H(\bar{B})$ endowed with the norm

$$\|f\|^p_{B^p_{s,k}(\Psi)} =: \int_B |(I+R)^k f(z)|^p (1-|z|^2)^{(k-s)p-1} \Psi(z) d\nu(z).$$

If $1 < p < \infty$ and $\Psi \in \mathcal{B}_p$, then different values of k gives equivalent norms (see [7]), and therefore in this case the space $B^p_{s,k}(\Psi)$ will be simply denoted by $B^p_s(\Psi)$. Moreover, if $\Psi \in \mathcal{B}_p$, then the dual of $B^p_s(\Psi)$ with respect to the pairing (1.1) is $B^p_{-s}(\Psi^{-p'/p})$.

If $\theta \in \mathcal{A}_p$ the function Θ is in $\mathcal{B}_p$ and consequently, if $q < p$, the weight $\Theta^{q/p}$ is in $\mathcal{B}_{1+\frac{q}{p}(p-1)}$. Since $1 + \frac{q}{p}(p-1) > q$, we have that the weight $\Theta^{q/p}$ might not be in $\mathcal{B}_q$. If the weight θ is in a smaller class, for instance, if $\theta \in \mathcal{A}_{p_0}$, where $p_0 \le 1 + p/q'$, we have that the weight $\Theta^{q/p} \in \mathcal{B}_q$ and in particular we have that the space $B^q_{s,k}(\Theta^{q/p})$ is independent of k.

Theorem 1.5. *Let* $1 < p < \infty$*,* $1 \le q_0 < q_1 \le \min\{p, 2\}$ *and* $\theta \in \mathcal{A}_{p_0}$*, where* $p_0 = 1 + p/q_0'$*. For* $j = 0, 1$*, let* $s_j = n/q_j - n/p$ *.*

Then, $B^{q_0}_{s_0}(\Theta^{q_0/p}) \subset B^{q_1}_{s_1}(\Theta^{q_1/p}) \subset H^p(\theta)$*.*

With this result we can prove the following theorem.

Theorem 1.6. *Let* $1 < q_0 < p < q_1 < \infty$*,* $\theta_0 \in \mathcal{A}_{p_0}$*,* $\theta_1 \in \mathcal{A}_{p_1'}$ *where* $p_0 = 1 + p/q_0'$*,* $p_1' = 1 + p'/q_1$ *and* Θ_0 *and* Θ_1 *the corresponding averaging functions. Then* $\Theta_0^{q_0/p} \in \mathcal{B}_{q_0}$*,* $\Theta_1^{q_1'/p'} \in \mathcal{B}_{q_1'}$ *and*

$$\|\Gamma_\psi\|_{Bil(B^{q_0}_{n/q_0-n/p}(\Theta_0^{q_0/p})\times \overline{B^{q_1'}_{n/q_1'-n/p'}(\Theta_1^{q_1'/p'})}\to\mathbb{C})}$$

$$\approx \|T_\psi\|_{L(B^{q_0}_{n/q_0-n/p}(\Theta_0^{q_0/p})\to B^{q_1}_{n/q_1-n/p}(\Theta_1^{-q_1/p'}))} \approx \left\| \frac{\psi}{\theta_0^{1/p}\theta_1^{1/p'}} \right\|_\infty .$$

More general results of this type can be found in Section 4.

The paper is organized as follows. In Section 2, we recall some well known properties of the unweighted holomorphic Hardy-Sobolev and Besov spaces, and other technical results that will be needed in the next sections. In Section 3, we prove Theorems 1.1 and 1.2, and its application to the study of the Toepliz operators and pointwise multipliers in classical spaces and weighted Hardy spaces (Theorem 1.3). In Section 4, we start recalling the main properties of the weights $\mathcal{A}_p$ and $\mathcal{B}_p$, and we extend a result of S. Petermichl and B. Wick about characterizations of the weights in $\mathcal{A}_p$ of the sphere in terms of a generalized invariant harmonic extension to the unit ball. In the second part of this section we recall some well known properties of the weighted Hardy spaces with weights in $\mathcal{A}_p$, and of the weighted Besov spaces with weights in $\mathcal{B}_p$, and we prove Theorems 1.5 and 1.6.

Notations: Throughout the paper, the letter C may denote various non-negative numerical constants, possibly different in different places. The notation $f(z) \lesssim g(z)$ means that there exists $C > 0$, which does not depends of z, f and g, such that $f(z) \le Cg(z)$.

2. Preliminaries

In this section we state some notations and some well known results, that will be needed in the forthcoming sections.

2.1. **Non-isotropic balls, tents and admissible regions.** For $\zeta \in S$ and $0 < t < 2$, let $U_{\zeta,t}$ be the non-isotropic ball on S defined by $U_{\zeta,t} = \{\eta \in S;\ |1 - \eta\bar{\zeta}| < t\}$, and let $\hat{U}_{\zeta,t}$ be the non-isotropic tent on B defined by $\hat{U}_{\zeta,t} = \{z \in B;\ |1 - z\bar{\zeta}| < t\}$.

It is well known that the Lebesgue measure on S of $U_{\zeta,t}$, denoted by $|U_{\zeta,t}|$, is of order of t^n, and the Lebesgue measure on B of $\hat{U}_{\zeta,t}$ also denoted by $|\hat{U}_{\zeta,t}|$ satisfies $|\hat{U}_{\zeta,t}| \approx t^{n+1}$.

To each $z = r\zeta$, $0 < r < 1$, $\zeta \in S$, we consider the associated non-isotropic ball $I_z = U_{\zeta,(1-|z|^2)}$ and the tent $\hat{I}_z = \hat{U}_{\zeta,(1-|z|^2)}$.

For $\psi \in L^1(d\sigma)$, let $M_{H-L}(\psi)$ denote the non-isotropic Hardy-Littlewood maximal, defined by

$$M_{H-L}(\psi)(\zeta) = \sup_{t>0} \frac{1}{|U_{\zeta,t}|} \int_{U_{\zeta,t}} \psi(\eta) d\sigma(\eta).$$

If $\zeta \in S$, let Γ_ζ be the admissible region $\Gamma_\zeta = \{z \in B;\ |1 - z\bar{\zeta}| < 1 - |z|^2\}$. Observe that if $z \in \Gamma_\zeta$, $|1 - z\bar{\zeta}| \approx 1 - |z|^2$. The admissible maximal function of a function $\psi \in L^1(d\nu)$ is $M(\psi)(\zeta) = \sup_{z \in \Gamma_\zeta} |\psi(z)|$.

If ψ is a non-negative measurable function on B, Fubini's Theorem gives that

$$\int_B \varphi(z) d\nu(z) = \int_S \int_{\Gamma_\zeta} \varphi(z) \frac{d\nu(z)}{|I_z|} d\sigma(\zeta).$$

2.2. **Integral operators.** Let P be the Cauchy integral operator given by

$$P(f)(z) = \int_S \frac{f(\zeta)}{(1 - z\bar{\zeta})^n} d\sigma(\zeta), \qquad z \in B.$$

For $m \geq 0$, $\zeta \in S$ and $z \in B$, let P_m be the non-isotropic kernel defined by

$$P_m(\zeta, z) = c_{n,m} \frac{(1 - |z|^2)^{n+2m}}{|1 - z\bar{\zeta}|^{2n+2m}}, \qquad \text{where} \quad c_{n,m} = \frac{\Gamma(n+m)^2}{(n-1)!\Gamma(n+2m)}.$$

Observe that for $m=0$, P_0 is the Poisson-Szegö kernel.

We also denote by P_m the corresponding integral operator on $L^1(d\sigma)$ defined by

$$P_m(\psi)(z)=\int_S \psi(\zeta)P_m(\zeta,z)d\sigma(\zeta), \quad z\in B.$$

Let Δ_m be the differential operator Δ_m defined by

$$\Delta_m=(1-|z|^2)\left\{\sum_{i,j=1}^n (\delta_{ij}-z_i\overline{z}_j)\,\partial_i\overline{\partial}_j+mR+m\overline{R}-m^2\mathrm{Id}\right\}.$$

This family of differential operators Δ_m generalizes the invariant Laplacian, which corresponds to $m=0$. It is shown in [1] that the generalized Dirichlet problem

$$\Delta_m u=0,\ u\in\mathcal{C}(\overline{B}),\ u=\varphi \text{ on } S,\ \varphi\in\mathcal{C}(S),$$

has a unique solution given by

$$u(z)=\int_S \varphi(\zeta)P_m(\zeta,z)\,d\sigma(\zeta)=P_m(\varphi)(z).$$

It is also shown in [1] that if $\varphi\in L^1(d\sigma)$, then $\lim_{r\to1}P_m(\varphi)(r\zeta)=\varphi(\zeta)$ a.e. $\zeta\in S$.

Proposition 1.4.10 in [14] gives in particular that if $1\le p<\infty$, $m\ge0$ and $f\in H(\bar{B})$, then

$$|f(z)|^p\approx\left|\int_S f(\zeta)\frac{(1-|z|^2)^{n+2m}}{(1-z\bar{\zeta})^n(1-\zeta\bar{z})^{n+2m}}d\sigma(\zeta)\right|^p\lesssim P_m(|f|^p)(z). \tag{2.11}$$

Finally, we also point out that $\|P_m(\psi)\|_\infty\approx\|\psi\|_\infty$, and that $\sup_r|P_m(\psi)(r\zeta)|\lesssim M_{H-L}(|\psi|)(\zeta)$.

To conclude we state the following lemma, which follows easily from Proposition 1.4.10 in [14].

Lemma 2.1. *If $k,m>0$, then for $z,w\in B$,*

$$\int_S\frac{d\sigma(\zeta)}{|1-z\bar{\zeta}|^{n+k}|1-w\bar{\zeta}|^{n+m}}\approx\frac{(1-|z|^2)^{-k}}{|1-w\bar{z}|^{n+m}}+\frac{(1-|w|^2)^{-m}}{|1-z\bar{w}|^{n+k}}.$$

2.3. **Unweighted spaces of holomorphic functions.** Let $\partial_j\psi = \frac{\partial}{\partial z_j}\psi$ and $\overline{\partial}_j\psi = \frac{\partial}{\partial \bar{z}_j}\psi$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \sum_{j=1}^n |\alpha_j|$, we denote by ∂^α the differential operator of order $|\alpha|$ defined by $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

We denote by R the radial derivative $\displaystyle\sum_{j=1}^n z_j\partial_j$. For $s \in \mathbb{R}$, we consider the invertible linear operator $(I+R)^s$ on $H(B)$, defined on the monomials $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ by $(I+R)^s z^\alpha = (1+|\alpha|)^s z^\alpha$.

For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, the holomorphic Besov space B_s^p is the set of holomorphic functions f on B such that

$$\|(1-|z|^2)^{k-s}((I+R)^k f)(z)\|_{L^p((1-|z|^2)^{-1}d\nu(z))} < +\infty,$$

for some non-negative integer $k > s$. It is well known that different values of $k > s$ give equivalent norms in B_s^p. Moreover, if we replace in the last expression $(I+R)^k f$ by $\nabla^k f =: \sum_{|\alpha|\leq k} |\partial^\alpha f|$ we also obtain an equivalent norm.

Notice that the space B_0^∞ is the Bloch space, and that if $s > 0$, then B_s^∞ coincides with the holomorphic Lipschitz space Lip_s.

It is well known that if $1 \leq p < \infty$ and $s \in \mathbb{R}$, then $H(\bar{B})$ is dense in the ball algebra A, in B_s^p and also in H_s^p. This density fails to be true for the spaces H^∞, $BMOA$ or B_s^∞. The closure of $H(\bar{B})$ in $BMOA$ is denoted by $VMOA$, and the closure of $H(\bar{B})$ in the Bloch space B_0^∞ is the little Bloch space b_0^∞.

The following two theorems summarize the inclusion and duality results on the above spaces. Their proofs can be found in [3] and [4].

Theorem 2.2. *Let $1 \leq q < p < \infty$ and $s, t \in \mathbb{R}$ satisfying $s - n/p = t - n/q$. Then,*

(i) $B_t^q \subset B_s^p$ *and* $H_t^q \subset H_s^p$.
(ii) *If $q \leq \min\{p, 2\}$, then* $B_t^q \subset H_s^p$.
(iii) $B_n^1 \subset A$, $B_{n/p}^p$, $H_{n/p}^p \subset VMOA \subset BMOA \subset B_0^\infty$.

Theorem 2.3. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. The following duality results with respect to the pairing* (2.12) *holds:*

(i) $(VMOA)' = H^1$, *and* $((B_0^\infty)_c)' = B_0^1$.
(ii) $(H^1)' = BMOA$, *and* $(B_0^1)' = B_0^\infty$.
(iii) $(B_s^p)' = B_{-s}^{p'}$, *and* $(H_s^p)' = H_{-s}^{p'}$.

We conclude this section with some remarks about the pairing $\langle f, g\rangle_S$. The fact that any function in H^p has its radial maximal function $M_r(f)(\zeta) = \sup_{0\le r<1} |f(r\zeta)|$ in $L^p(d\sigma)$ gives that

$$\langle f, g\rangle_S = \lim_{r\nearrow 1} \int_S f_r \bar{g}_r d\sigma = \int_S f\bar{g} d\sigma,$$

and hence $\int_S |f||g|d\sigma$ is finite.

However, if $f \in B_s^p$ and $g \in B_{-s}^{p'}$, then $f\overline{g}$ is not necessarily in $L^1(d\sigma)$ and we cannot, in general, interchange the limit with the integral. In these cases it is convenient to rewrite the pairing (2.12) as follows. The formula (see Section 1.4 in [14])

$$\int_S |\zeta^\alpha|^2 d\sigma = \frac{(n+|\alpha|)\cdots(n+k-1+|\alpha|)}{n\,(k-1)!} \int_B |z^\alpha|^2 (1-|z|^2)^{k-1} d\nu(z),$$

the fact that $Rz^\alpha = |\alpha| z^\alpha$, and the homogeneous expansion of the holomorphic functions f and g, give that

$$\int_S f(r\zeta)\overline{g}(r\zeta) d\sigma(\zeta) = \int_B [p(R)f](rz)[\overline{q(R)g}](rz)(1-|z|^2)^{k-1} d\nu(z),$$

where $p(R)$ and $q(R)$ are the differential operators associated to any of the one variable real polynomials $p(t)$ and $q(t)$ satisfying

$$p(t)q(t) = \frac{(n+t)\cdots(n+k-1+t)}{n\,(k-1)!}.$$

In particular, if we denote by R_j^l the differential operator of order l defined by $R_j^l = (jI+R)\cdots((j+l-1)I+R)$, then for $l \le k$

$$\begin{aligned} &\int_S f_r(\zeta)\overline{g}_r(\zeta)d\sigma \\ &\qquad = \frac{1}{n\,(k-1)!}\int_B [R_n^l f](rz)[\overline{R_{n+l}^{k-l}g(rz)}](1-|z|^2)^{k-1}d\nu(z). \end{aligned} \tag{2.12}$$

We point out that if $j>0$, then the operators $R_j^l : H \to H$ are invertible. The inverse will be denoted by R_j^{-l}.

Formula (2.12) can be used to prove that $(B_s^p)' = B_{-s}^{p'}$ with the pairing (1.1). The corresponding results on duality for the Hardy-Sobolev spaces, i.e. $(H_s^p)' = H_{-s}^{p'}$, can be deduced using instead the formula

$$\int_S f_r\bar{g}_r d\sigma = \int_S [(I+R)^{-s}f]_r\overline{[(I+R)^s g]_r}d\sigma.$$

3. Norms of Toeplitz operators

We will assume that X and Y are two Banach spaces of holomorphic functions on B, containing both of them $H(\bar{B})$. We start this section obtaining necessary conditions on a functions $\psi \in L^1(d\sigma)$, such that the bilinear Toeplitz forms $\Gamma_\psi : X_c \times \overline{Y}_c \to \mathbb{C}$ be continuous.

Since

$$P_m(\psi)(z) = c_{n,m}(1-|z|^2)^{n+2m}\int_S \psi(\zeta)\,(1-\zeta\bar{z})^{-n-m}(1-z\bar{\zeta})^{-n-m}d\sigma(\zeta),$$

it is clear that for any $z \in B$,

$$\begin{aligned} &|P_m(\psi)(z)| \\ &\quad \le c_{n,m}\|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}\left\|\frac{(1-|z|^2)^{n/p'+m}}{(1-w\bar{z})^{n+m}}\right\|_X\left\|\frac{(1-|z|^2)^{n/p+m}}{(1-w\bar{z})^{n+m}}\right\|_Y \\ &\quad = \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}\omega_m(z). \end{aligned} \tag{3.13}$$

Therefore, if Γ_ψ is bounded on $X_c \times \overline{Y}_c$, we have that

$$\sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\} \le \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}.$$

This condition is given in terms of the generalized Poisson-Szëgo extension of ψ. A necessary condition in terms of the boundary values of ψ can be obtained as follows. If $z = r\eta$ and we take $\liminf_{r\nearrow 1}$ in (3.13), we have

$$|\psi(\eta)| \leq \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})} \liminf_{r\nearrow 1} \omega_m(r\eta), \quad \text{a.e. } \eta \in S. \tag{3.14}$$

Thus, if $0 < \tilde{\omega}_m(\eta) = \liminf_{r\nearrow 1} \omega_m(r\eta) < \infty$ a. e. $\eta \in S$, we obtain as a necessary condition

$$\left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty \leq \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}.$$

In order to obtain conditions on X and Y that assures that

$$\sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\} \approx \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}, \tag{3.15}$$

observe that if $f, g \in H(\bar{B})$ and $\psi \in L^1(S)$, then

$$\Gamma_\psi(f,\bar{g}) = \lim_{r\nearrow 1}\int_S \frac{P_m(\psi)(r\zeta)}{\omega_m(r\zeta)} f(r\zeta)\bar{g}(r\zeta)\omega_m(r\zeta)d\sigma(\zeta). \tag{3.16}$$

Therefore, if

$$\tau_c(r) = \sup\left\{\frac{\|f_r\,\bar{g}_r\,(\omega_m)_r\|_{L^1(d\sigma)}}{\|f\|_X\|g\|_Y};\ f,g\in H(\bar{B}), f,g\neq 0\right\},$$

is a bounded function on $[0,1)$ then (3.15) holds.

If X_c and Y_c are dense in X and Y respectively, then the bilinear form Γ_ψ can be extended to a bilinear continuous form on $X\times\overline{Y}$ preserving the norm. As usual, we also denote this extension by Γ_ψ. However, in some important examples these conditions of density are not satisfied (as it happens in the case of H^∞, $BMOA$, B_0^∞). In these cases we can modify the above argument to also obtain conditions on X and Y such that

$$\sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\} \approx \|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})}, \tag{3.17}$$

is satisfied.

Assume that $f, g \in H^1$, and $\psi \in L^1(S)$, then

$$\left|\int_S \psi(\zeta)f(\zeta)\bar{g}(\zeta)d\sigma(\zeta)\right| \leq \int_S \lim_{r\nearrow 1}|P_m(\psi)(r\zeta)||f(r\zeta)\bar{g}(r\zeta)|d\sigma(\zeta).$$

We also assume that the function

$$\tau(r) = \sup\left\{ \frac{\|f_r\,\bar{g}_r\,(\omega_m)_r\|_{L^1(d\sigma)}}{\|f\|_X\|g\|_Y};\ f\in X, g\in Y, f,g\neq 0\right\},$$

is bounded on $[0,1)$.

Then, taking a sequence $\{r_j\}\nearrow 1$ and applying Fatou's Lemma, we have

$$\begin{aligned}&\left|\int_S \psi(\zeta)f(\zeta)\bar{g}(\zeta)d\sigma(\zeta)\right|\\ &\quad\le \sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\}\sup_{0\le r<1}\int_S |f_r(\zeta)||g_r(\zeta)|(\omega_m)_r(\zeta)d\sigma(\zeta)\\ &\quad\le \|\tau\|_\infty \sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\}\|f\|_X\|g\|_Y,\end{aligned}$$

Summarizing these conditions, we have

Theorem 3.1. *Let $m\ge 0$ and let X and Y be Banach spaces of holomorphic functions on B containing $H(\bar{B})$.*

Then,

$$\sup_z\left\{\frac{P_m(\psi)(z)}{\omega_m(z)}\right\}\le \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}\le \|\tau_c\|_\infty\sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\}.$$

If in addition $X,Y\subset H^1$, then

$$\|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})}\le \|\tau\|_\infty\sup_{z\in B}\left\{\frac{|P_m(\psi)(z)|}{\omega_m(z)}\right\}.$$

Now let us state conditions on X and Y such that

$$\left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty \le \|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}. \tag{3.18}$$

is satisfied.

Assuming that $0<\tilde{\omega}_m(\zeta)<\infty$ a. e. $\zeta\in S$, we have

$$\Gamma_\psi(f,\bar{g})\lesssim \left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty\int_S |f||g|\tilde{\omega}_m d\sigma.$$

Therefore,

Theorem 3.2. *Let X and Y be Banach spaces of holomorphic functions on B containing $H(\bar{B})$.*

If $0<\tilde{\omega}_m(\zeta)<\infty$, a.e. $\zeta\in S$, and there exists a constant $C_{X,Y}$ such that $\|fg\|_{L^1(\tilde{\omega}_m)}\leq C_{X,Y}\|f\|_X\|g\|_Y$, then

$$\left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty\leq\|\Gamma_\psi\|_{Bil(X_c\times\overline{Y}_c\to\mathbb{C})}\leq\|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})}\leq C_{X,Y}\left\|\frac{\psi}{\tilde{\omega}_m}\right\|_\infty.$$

Theorem 3.2 will be used to obtain results on Toeplitz operators in weighted Hardy spaces with weights satisfying some additional conditions on S.

3.1. **Toeplitz operators in some classical spaces.** Since $\|P_m(\psi)\|_\infty\approx\|\psi\|_\infty$, we have as immediately consequence of Theorem 3.1 the following result.

Corollary 3.3. *Let X and Y be Banach spaces of holomorphic functions on B satisfying $\omega_m\approx 1$ a.e. on S, and $\|\tau\|_\infty<\infty$. Then $\|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})}\approx\|\psi\|_\infty$.*

Corollary 3.4. *Let $1\leq p\leq\infty$. Assume that X, Y satisfy $B^1_{n/p'}\subset X\subset H^p$ and $B^1_{n/p}\subset Y\subset H^{p'}$ respectively. Then, $\Gamma_\psi: X\times\overline{Y}\to\mathbb{C}$ is bounded if and only if ψ is in L^∞.*

Proof. If $l>0$, Proposition 1.4.10 in [14] gives that $\|(1-w\bar{z})^{-n-l}\|_{L^1(d\sigma)}\approx(1-|z|^2)^{-l}$. Therefore, the norms of the function $f_z(w)=(1-w\bar{z})^{-n-2m}$ both in $B^1_{n/p'}$ and in H^p, and consequently in X, are equivalent to $(1-|z|^2)^{-n/p'-m}$. Analogously, the norms of the function $f_z(w)$ in $B^1_{n/p}$ and $H^{p'}$ and consequently in Y are equivalent to $(1-|z|^2)^{-n/p-m}$. From these estimates, we obtain $\omega_m\approx 1$ a.e. on S.

Since $\|f_r\bar{g}_r(\omega_r)\|_{L^1(d\sigma)}\lesssim\|f_r\bar{g}_r\|_{L^1(d\sigma)}\lesssim\|f\|_{H^p}\|g\|_{H^{p'}}\lesssim\|f\|_X\|g\|_Y$, we have that the function τ is bounded. Consequently, the result follows from Corollary 3.3. ∎

Examples of spaces X and Y satisfying the conditions in the above corollary are the Hardy-Sobolev spaces $H^{p_0}_{s_0}$ with $1 \le p_0 < p$ and $s_0 - n/p_0 = -n/p$, or more generaly the scale of Besov spaces $B^{p_0,q_0}_{s_0}$ and the scale of Triebel-Lizorkin spaces $F^{p_0,q_0}_{s_0}$ satisfying $1 \le p_0 \le p$, $1 \le q_0 \le 2$, and $s_0 - n/p_0 = -n/p$ (see H. Triebel's book [18] and the references therein for a more complete list of embeddings between these spaces).

Corollary 3.5. *Let X and Y be Banach spaces of holomorphic functions on B. Assume that $\omega_m(r\zeta) \to 0$ as $r \nearrow 1$, a.e. $\zeta \in S$. Then the only bounded bilinear Toeplitz forms Γ_ψ are the ones corresponding to the trivial case where $\psi = 0$ a.e. $\zeta \in S$.*

Proof. By Theorem 3.1, $|P_m(\psi)(r\zeta)| \le c_{n,m}\|\Gamma_\psi\|_{L(X_c\times\overline{Y}_c\to\mathbb{C})}\omega_m(r\zeta)$, and consequently $P_m(\psi)(r\zeta) \to 0$ as $r \nearrow 1$, a.e. $\zeta \in S$. Therefore, $\psi(\zeta) = 0$ a.e. $\zeta \in S$. ∎

Corollary 3.6. *Let $1 \le p < q \le \infty$. Assume that X, Y satisfy $B^1_{n/p'} \subset X \subset H^p$ and $B^1_{n/q} \subset Y \subset H^{q'}$ respectively. Then, $\Gamma_\psi : X_c \times \overline{Y}_c \to \mathbb{C}$ is not bounded except for the trivial case where $\psi(\zeta) = 0$ a.e. $\zeta \in S$.*

Proof. The arguments used in the proof of Corollary 3.4, give that $\omega_m(r\zeta) \approx (1 - r^2)^{n/p-n/q} \to 0$ as $r \nearrow 1$. ∎

Corollary 3.4 permit us to obtain easily some well known characterizations on the symbols of Toeplitz operators. Among them we have that if $1 < p < +\infty$, then $TSymb(H^p \to H^p) = L^\infty$. It also can be obtained some improvements of the classical results that we sumarize in the following theorem.

Theorem 3.7. *Let X and Y two Banach spaces of holomorphic functions satisfying one of the following conditions:*

(i) $B^1_n \subset X \subset H^\infty$ *and* $BMOA \subset Y \subset B^\infty_0$.

(ii) $B^1_{n/p'} \subset X \subset H^p \subset Y \subset B^\infty_{-n/p}$, *for some* $1 < p < \infty$.

Then $TSymb(X \to Y) = L^\infty$.

Proof. Assume that (i) is satisfied. In this case, we only need to show that

$$TSymb(B^1_n \to B^\infty_0) \subset L^\infty \subset TSymb(H^\infty \to BMOA).$$

The second embedding is a consequence of the fact that the Cauchy projection maps L^∞ into $BMOA$.

In order to prove the first embedding, we need to show that $\|\psi\|_\infty \lesssim \|T_\psi\|_{L(B^1_n \to B^\infty_0)}$. And that is a consequence of the duality $(B^1_0)' = B^\infty_0$, of the Corollary 3.4 with $p = \infty$ and $\|T_\psi\|_{L(B^1_n \to B^\infty_0)} \approx \|\Gamma_\psi\|_{Bil(B^1_n \times \overline{B^1_0} \to \mathbb{C})}$.

If X and Y satisfy (ii), then the result is a consequence of the fact that

$$TSymb(B^1_{n/p'} \to B^\infty_{-n/p}) \subset L^\infty = TSymb(H^p \to H^p),$$

where the first embedding is a consequence of $(B^1_{n/p})' = B^\infty_{-n/p}$ and Corollary 3.4. ■

Remark 3.8. *In the unit disk the equalities*

$$TSymb(H^\infty \to BMOA) = TSymb(H^\infty \to B^\infty_0) = L^\infty$$

were proved in [9].

Examples of spaces X *satisfying the hypothesis of Theorem 3.7 are for instance the ball algebra* A*, the Hardy-Sobolev space* H^1_n *and the multiplicative algebras* $B^p_{n/p} \cap H^\infty$*. As* Y *we can consider all the scale of Triebel-Lizorkin spaces* $F^{\infty,q}_0$*,* $2 \le q \le \infty$*. We recall that* $BMOA = F^{\infty,2}_0$ *and* $B^\infty_0 = F^{\infty,\infty}_0$.

3.1.1. ***Pointwise multipliers***. Let X and Y be Banach spaces of holomorphic functions on B, both of them containing $H(\bar{B})$. We denote by $\mathcal{M}(X \to Y)$ the space of pointwise multipliers from X to Y, that is the space of holomorphic functions h on B such that the map $M_h(f) = hf$ is continuous from X to Y.

It is clear that if $h \in H^1(S)$ and $f \in H(\bar{B})$ then $M_h(f) = T_h(f)$. Therefore, the results on Toeplitz operators lead easily to results on pointwise multipliers.

We then have:

Theorem 3.9. *Let* $1 \le p \le \infty$ *and let* X *and* Y *Banach spaces satisfying*

$$B^1_{n/p'} \subset X \subset H^p \subset Y \subset B^\infty_{-n/p}.$$

Then $\mathcal{M}(X \to Y) = H^\infty$

Proof. Observe that

$$H^\infty = \mathcal{M}(H^p \to H^p) \subset \mathcal{M}(X \to Y) \subset \mathcal{M}(B^1_{n/p'} \to B^\infty_{-n/p}) = H^\infty,$$

where the last identity is a consequence of Theorem 3.7. ∎

It is also possible to obtain other characterizations of spaces of pointwise multipliers.

Theorem 3.10. *Let* X *and* Y *Banach spaces of holomorphic functions on* B, *and let* $1 \le p < \infty$.

(i) *If* $B^1_n \subset X \subset H^\infty \cap VMOA \subset Y \subset VMOA$, *then* $\mathcal{M}(X \to Y)$ *coincides with the multiplicative algebra* $H^\infty \cap VMOA$.

(ii) *If* $B^1_n \subset X \subset H^\infty \cap b^\infty_0 \subset Y \subset b^\infty_0$, *then* $\mathcal{M}(X \to Y)$, *coincides with the multiplicative algebra* $H^\infty \cap b^\infty_0$.

(b^∞_0 denotes the little Bloch space, that is the closure of $H(\bar{B})$ in B^∞_0.)

Proof. We recall that a holomorphic function $f \in BMOA$ is in $VMOA$, if

$$\lim_{t\to 0} \frac{1}{t^n} \int_{|\{z\in B;\, |1-z\bar{\zeta}|<t\}} |Rf(z)|^2(1-|z|^2)d\nu(z) = 0 \quad \text{a.e. } \zeta \in S.$$

Therefore, it is clear that $H^\infty \cap VMOA$ is a multiplicative algebra, and $(H^\infty \cap VMOA) \cdot X \subset H^\infty \cap VMOA \subset Y$, which proves that $H^\infty \cap VMOA \subset \mathcal{M}(X \to Y)$.

In order to prove the converse, observe that $\mathcal{M}(X \to Y) \subset Y \subset VMOA$, and that $\mathcal{M}(X \to Y) \subset \mathcal{M}(X \to VMOA) \subset \mathcal{M}(X \to BMOA) = H^\infty$, where the last equality is a consequence of Theorem 3.9 with $p = \infty$.

The same arguments and the fact that a holomorphic function $f \in B_0^\infty$ is in b_0^∞ if $\lim_{r \nearrow 1}(1-r^2)|Rf(r\zeta)| = 0$ a.e. $\zeta \in S$ gives (ii). ■

3.2. **Toeplitz operators on weighted Hardy spaces.** Let us conclude this section with an application of Theorem 3.2 to the study of Toeplitz operators on weighted Hardy spaces.

We will consider weighted Hardy spaces for the class of **admissible weights** consisting of measurable functions θ on S such that

(i) $\theta(\zeta) > 0$ a.e. $\zeta \in S$,
(ii) $\theta, \theta^{-p'/p} \in L^1(d\sigma)$.

For this class of weights, $H^p(\theta)$ will denote the closure of $H(\bar{B})_{|S}$ in $L^p(\theta)$. Observe that since $\theta^{-p'/p} \in L^1(d\sigma)$, then for each $\varphi \in L^p(\theta d\sigma)$,

$$\int_S |\varphi(\zeta)| d\sigma(\zeta) \leq \|\theta^{-p'/p}\|_{L^1(d\sigma)}^{1/p'} \|\varphi\|_{L^p(\theta)}.$$

Therefore, $L^p(\theta) \subset L^1(d\sigma)$, and if $\{f_j\} \subset H(\bar{B})$ and $\lim_j f_j = \varphi$ in $L^p(\theta)$, then $\{f_j\}$ converges in L^1 to a function $f \in H^1$, whose boundary values coincide with φ a.e. on S. Therefore, $H^p(\theta)$ is a subspace of H^1.

If $(n+m)p - 2n \geq 0$, and f_z is the test function defined by $f_z(\zeta) = \dfrac{(1-|z|^2)^{n/p'+m}}{(1-\zeta\bar{z})^{n+m}}$, we have that

$$\begin{aligned} \|f_z\|_{H^p(\theta)} &= \left(\int_S \frac{(1-|z|^2)^{(n/p'+m)p}}{|1-\zeta\bar{z}|^{(n+m)p}} \theta(\zeta) d\sigma(\zeta) \right)^{1/p} \\ &= \left(\frac{1}{c_{n,(n+m)p-2n}} P_{(n+m)p-2n}(\theta)(z) \right)^{1/p}, \end{aligned}$$

which has radial limit $\dfrac{\theta^{1/p}}{c_{n,(n+m)p-2n}^{1/p}}$ a.e. on S.

We then have:

Theorem 3.11. *Let be $1 < p < \infty$, $\psi \in L^1(d\sigma)$ and θ_0, θ_1 admissible weights. Then,*

$$\|\Gamma_\psi\|_{Bil(H^p(\theta_0)\times\overline{H^{p'}(\theta_1)}\to\mathbb{C})} \simeq \left\| \frac{\psi}{\theta_0^{1/p}\theta_1^{1/p'}} \right\|_\infty .$$

Proof. Observe that the above computations give that $\tilde{\omega}_n(\theta) \approx \theta_0^{1/p}\theta_1^{1/p'}$ a.e. on S, and that, by Hölder's Inequality,

$$\int_S |f||g|\tilde{\omega}d\sigma \lesssim \|f\|_{H^p(\theta_0)}\|g\|_{H^{p'}(\theta_1)}.$$

Therefore $H^p(\theta_0)$ and $H^{p'}(\theta_1)$ satisfy the conditions in Theorem 3.2, which proves the result. ■

The estimates on the norm of the bilinear form Γ_ψ on $H^p(\theta_0) \times \overline{H^{p'}(\theta_1)}$ give estimates on the norm of the corresponding Toeplitz operator T_ψ once we can identify the dual of $H^{p'}(\theta_1)$ with respect to the pairing (1.1). This is the case when θ_1 is in the Muckenhoupt class $\mathcal{A}_p$. The next section is devoted to give the properties we will need on this class of weights and the associated weighted Hardy spaces.

4. Weighted Hardy and Besov spaces and Toeplitz operators

The main goal of this section is to prove a weighted version of the embedding $B_t^q \subset H_s^p$, $1 \le q \le \min\{p, 2\}$, $t - n/q = s - n/p$. The proof will rely on properties of weights in the Muckenhoupt class $\mathcal{A}_p$ on S and weights in the class $\mathcal{B}_p$ on B.

4.1. $\mathcal{A}_p$ and $\mathcal{B}_p$ weights.

Given a non negative weight $\theta \in L^1(d\sigma)$ and W a measurable set in S, let $\theta(W) = \int_W \theta d\sigma$. For $z = r\zeta$, $\zeta \in S$, $0 < r < 1$, we consider the average function on B associated to θ defined by $\Theta(z) = \dfrac{\theta(I_z)}{|I_z|}$, where $I_z = U_{\zeta,1-r^2} = \{\eta \in S; |1 - \eta\bar{\zeta}| < 1 - r^2\}$.

The Muckenhoupt class $\mathcal{A}_p$ on S, $1 < p < \infty$, consists of the non-negative weights $\theta \in L^1(d\sigma)$ satisfying

$$\mathcal{A}_p(\theta) = \sup_{z\in B} \left(\Theta(z)\right)^{1/p} \left(\Theta'(z)\right)^{1/p'} < \infty \tag{4.19}$$

where $\theta' = \theta^{-p'/p}$ and $\Theta'(z) = \dfrac{\theta'(I_z)}{|I_z|}$. Observe that $\theta \in \mathcal{A}_p$, if and only if, $\theta' \in \mathcal{A}_{p'}$.

If $p = 1$, the class $\mathcal{A}_1$ on S is the set of non-negative weights $\theta \in L^1(d\sigma)$ satisfying

$$\mathcal{A}_1(\theta) = \sup_{\zeta\in S} \frac{M_{H-L}(\theta)(\zeta)}{\theta(\zeta)} < \infty. \tag{4.20}$$

An immediate consequence of Hölder's Inequality is that if $1 < q < p$, then $\mathcal{A}_1 \subset \mathcal{A}_q \subset \mathcal{A}_p$, that $\mathcal{A}_p(\theta) \geq 1$, and that if $0 \leq \lambda \leq 1$, θ^λ is in $\mathcal{A}_q$, with $q = 1 + \lambda(p-1) \leq p$ (see [16] pag. 218). In particular, for any $\theta \in \mathcal{A}_p$ and $0 < \lambda \leq 1$, the weight θ^λ is also in $\mathcal{A}_p$.

Weights in the class $\mathcal{A}_p$ appear in the study of the boundedness of singular operators on weighted spaces. In particular, we have that if $1 < p < \infty$ and $\theta \in \mathcal{A}_p$, then the Cauchy projection maps $L^p(\theta)$ in $H^p(\theta)$, and as a consequence of this fact, the dual of $H^p(\theta)$ can be identified with $H^{p'}(\theta')$ with the pairing given by $\langle f, \bar{g}\rangle_S$ (see [10] p.334).

The following characterization of the class $\mathcal{A}_p$ (see [16] p.195), will be considered in the following sections.

$\theta \in \mathcal{A}_p$, if and only if for any measurable set $U \subset S$ and $\varphi \geq 0$ in $L^p(d\sigma)$,

$$\left(\frac{1}{|U|}\int_U \varphi\, d\sigma\right)^p \leq \frac{\mathcal{A}_p(\theta)}{\theta(U)} \int_U \varphi^p \theta d\sigma. \tag{4.21}$$

From the above characterization we deduce that if $V \subset U$ are measurable sets on S, and φ is the characteristic function of V, then

$$\left(\frac{|V|}{|U|}\right)^p \leq \mathcal{A}_p(\theta)\frac{\theta(V)}{\theta(U)}.$$

Therefore, for any $\theta \in \mathcal{A}_p$, the measure $\theta d\sigma$ is a doubling measure, in the sense that there exists a constant D_S such that

$$\theta(U_{\zeta,2t}) \leq D_S \theta(U_{\zeta,t}), \quad \text{for all } \zeta \in S,\ 0 < t < 2.$$

By Proposition 5.1.4 in [14], we have that $D_S \le 2^{np}$.

An immediate consequence of this doubling property is the following lemma.

Lemma 4.1. *Let $1 < p < \infty$ and $\theta \in \mathcal{A}_p$. If $z = r\zeta$, $\zeta \in S$, $0 < r < 1$, and $N > 0$ satisfies $D_S < 2^{n+N}$, then*

$$\int_S \frac{(1-|z|^2)^N}{|1-z\bar{\eta}|^{n+N}}\theta(\eta)d\sigma(\eta) \approx \frac{\theta(I_z)}{(1-|z|^2)^n} \approx \Theta(z).$$

Proof. Let $I_{z,j} = U_{\zeta,2^{j-1}(1-r)}$. Then,

$$\int_S \frac{\theta(\eta)}{|1-z\bar{\eta}|^{n+N}}d\sigma(\eta) \approx \frac{\theta(I_z)}{(1-|z|^2)^{n+N}} + \sum_{j=2}^{\infty} \frac{\theta(I_{z,j}\setminus I_{z,j-1})}{2^{j(n+N)}(1-|z|^2)^{n+N}}.$$

Therefore, the estimate follows from $\theta(I_{z,j}\setminus I_{z,j-1}) \le D_S^j \theta(I_z)$ and $D_S < 2^{n+N}$. ∎

Observe that the above lemma permit us to rewrite the $\mathcal{A}_p$-condition (4.19) on θ in terms of its extension $P_m(\theta)$:

$$\theta \in \mathcal{A}_p, \quad \text{if and only if,} \quad \sup_z P_m(\theta)(z)^{1/p}\, P_m(\theta')(z)^{1/p'} < \infty$$

for any m satisfying $2^{2n+2m} > D_S, D'_S$. In particular for any m such that $2n+2m > \max(np, np')$.

The next theorem shows that in fact the above characterization holds for any $m \ge 0$. This characterization extends the results of Section 5 in [12] where it was considered the invariant harmonic case $m = 0$ and $p = 2$.

Theorem 4.2. *Let $1 < p < \infty$ and $m \ge 0$ and let θ be a non-negative function on S. The following conditions are equivalent:*

(i) *θ is in $\mathcal{A}_p$.*

(ii) *$\mathcal{A}_{p,m}(\theta) = \sup_z P_m(\theta)(z)^{1/p}\, P_m(\theta')(z)^{1/p'} < \infty$.*

Moreover, if $m > 0$, $\mathcal{A}_p(\theta) \approx \mathcal{A}_{p,m}(\theta)$, and $\mathcal{A}_p(\theta) \lesssim \mathcal{A}_{p,0}(\theta) \lesssim \mathcal{A}_p(\theta)^2$.

Proof. Fix $z = r\zeta \neq 0$ and let j_z be the integer part of $|\log_2(1-|z|^2)|$. If $0 \le j < j_z$, let $I_{z,j} = U_{\zeta,2^j(1-r^2)}$ and $I_{z,j_z} = S$. For $0 \le j \le j_z$, we consider

$$\Theta_j(z) = \frac{1}{|I_{z,j}|}\int_{I_{z,j}} \theta d\sigma, \quad \text{and} \quad \Theta'_j(z) = \frac{1}{|I_{z,j}|}\int_{I_{z,j}} \theta' d\sigma.$$

Now, assume that (ii) is satisfied. Clearly

$$\Theta(z) \lesssim \int_{I_z} \frac{(1-|z|^2)^{n+2m}}{|1-z\bar{\eta}|^{2n+2m}} \theta(\eta) d\sigma(\eta) \lesssim P_m(\theta)(z),$$

and analogously $\Theta'(z) \lesssim P_m(\theta')(z)$.

Therefore,

$$\begin{aligned}\mathcal{A}_p(\theta) &= \sup_z \Theta(z)^{1/p}\Theta'(z)^{1/p'}\\ &\lesssim \sup_z P_m(\theta)(z)^{1/p}\, P_m(\theta')(z)^{1/p'} = \mathcal{A}_{p,m}(\theta).\end{aligned}$$

Let us prove that if $\theta \in \mathcal{A}_p$, then (ii) is satisfied. Assume that $m > 0$. Since $|I_{z,j}| \approx 2^{jn}(1-|z|^2)^n$ and $|1-z\bar{\zeta}| \approx 2^j(1-|z|^2)$ on $I_{z,j} \setminus I_{z,j-1}$, we have

$$\begin{aligned}P_m(\theta)(z)^{1/p} &\lesssim \left(\sum_{1\le j\le j_z} 2^{-j((n+m)p-n)}\Theta_j(z)\right)^{1/p}\\ &\lesssim \sum_{1\le j\le j_z} 2^{-j(m+n/p')}\Theta_j(z)^{1/p}.\end{aligned} \tag{4.22}$$

Analogously,

$$P_m(\theta')(z)^{1/p'} \lesssim \sum_{1\le k\le j_z} 2^{-k(m+n/p)}{\Theta'}_k(z)^{1/p'}.$$

Therefore,

$$\begin{aligned}&P_m(\theta)(z)^{1/p} P_m(\theta')(z)^{1/p'}\\ &\quad\lesssim \sum_{1\le j,k\le j_z} 2^{-j(m+n/p')-k(m+n/p)}\Theta_j(z)^{1/p}{\Theta'}_k(z)^{1/p'}.\end{aligned} \tag{4.23}$$

Next, we observe that

$$\begin{aligned}
\Theta_j(z) &\leq \frac{|I_{z,k}|}{|I_{z,j}|}\,\Theta_k(z) \lesssim 2^{(k-j)n}\Theta_k(z), &&\text{if } j \leq k, \text{ and}\\
\Theta'_k(z) &\leq \frac{|I_{z,j}|}{|I_{z,k}|}\,\Theta'_j(z) \lesssim 2^{(j-k)n}\Theta'_j(z), &&\text{if } k \leq j.
\end{aligned}$$

Therefore, the left term in (4.23) is bounded by

$$\begin{aligned}
&\sum_{1\leq j\leq k\leq j_z} 2^{-j(m+n/p')-k(m+n/p)}\Theta_j(z)^{1/p}{\Theta'}_k(z)^{1/p'}\\
&\qquad+\sum_{1\leq k\leq j\leq j_z} 2^{-j(m+n/p')-k(m+n/p)}\Theta_j(z)^{1/p}{\Theta'}_k(z)^{1/p'}\\
&\lesssim \sum_{1\leq j\leq k\leq j_z} 2^{-j(m+n/p')-k(m+n/p)+(k-j)n/p}\Theta_k(z)^{1/p}{\Theta'}_k(z)^{1/p'}\\
&\qquad+\sum_{1\leq k\leq j\leq j_z} 2^{-j(m+n/p')-k(m+n/p)+(j-k)n/p'}\Theta_j(z)^{1/p}{\Theta'}_j(z)^{1/p'}\\
&\lesssim \mathcal{A}_p(\theta)\left(\sum_{1\leq j\leq k<\infty} 2^{-j(m+n)-km}+\sum_{1\leq k\leq j<\infty} 2^{-jm-k(m+n)}\right)\\
&\lesssim \mathcal{A}_p(\theta).
\end{aligned}$$

We finally consider the case $m=0$. In order to prove that (i) implies (ii), we observe that for $1\leq j\leq j_z$ there exist constants $0<\lambda,\lambda'<1$ such that

$$\Theta_{j-1}(z)\leq \lambda\frac{|I_{z,j}|}{|I_{z,j-1}|}\,\Theta_j(z), \qquad \text{and} \qquad \Theta'_{k-1}(z)\leq \lambda'\frac{|I_{z,k}|}{|I_{z,k-1}|}\,\Theta'_k(z). \tag{4.24}$$

Indeed, the constant λ can be obtained from the inequality (4.21) applied to the function $\varphi=\chi_j-\chi_{j-1}$, where χ_j denotes the characteristic function of $I_{z,j}$.

To be precise, we have

$$\left(1-\frac{|I_{z,j-1}|}{|I_{z,j}|}\right)^p \leq \mathcal{A}_p(\theta)\left(1-\frac{\int_{I_{z,j-1}}\theta d\sigma}{\int_{I_{z,j}}\theta d\sigma}\right),$$

and since $0 < c = \sup_{z,1\le j\le j_z} \frac{|I_{z,j-1}|}{|I_{z,j}|} < 1$, we obtain the first inequality in (4.24) choosing $\lambda = 1-\left(\frac{(1-c)^p}{\mathcal{A}_p(\theta)}\right) < 1$. Analogously, we obtain the second inequality choosing $\lambda' = 1 - \left(\frac{(1-c)^{p'}}{\mathcal{A}_p(\theta)}\right) < 1$.

These estimates and an argument like the one used in the proof of the case $m > 0$, give

$$\begin{aligned}\omega_0(z) &\lesssim \mathcal{A}_p(\theta)\left(\sum_{1\le j\le k<\infty} 2^{-jn}\lambda^{(k-j)/p} + \sum_{1\le k<j<\infty} 2^{-kn}\lambda'^{(j-k)/p'}\right)\\ &\lesssim \frac{\mathcal{A}_p(\theta)}{1-\lambda^{1/p}} + \frac{\mathcal{A}_p(\theta)}{1-\lambda'^{1/p'}} \lesssim \mathcal{A}_p(\theta)^2 < \infty.\end{aligned}$$

■

We have associated in a natural way to any weight θ on S its average weight Θ on B. It was proved in [6] that if θ in $\mathcal{A}_p$, then the weight Θ is in the class $\mathcal{B}_p$. We recall that this class of weights have been introduced in [5], where it has been proved that the weights Θ for which the Bergman projection is bounded in $L^p(\Theta)$ are the ones in $\mathcal{B}_p$.

$\mathcal{B}_p$ is the class of the non-negative weights $\Psi \in L^1(d\nu)$ satisfying

$$\mathcal{B}_p(\Psi) = \sup_{z\in B}\left(\frac{1}{|\hat{I}_z|}\int_{\hat{I}_z}\Psi d\nu\right)^{1/p}\left(\frac{1}{|\hat{I}z|}\int_{\hat{I}_z}\Psi^{-p'/p}d\nu\right)^{1/p'} < \infty. \tag{4.25}$$

The class $\mathcal{B}_p$ appears in the study of the boundedness of the Bergman projection on weighted L^p spaces on B (see [5]). From now on, for a measurable set V on B, we will write $\Psi(V) = \int_V \psi d\nu$. This class $\mathcal{B}_p$ has similar properties to the class $\mathcal{A}_p$, replacing the non-isotropic balls I_z on S by the non-isotropic tents $\hat{I}_z$ on B. In particular, Hölder's Inequality gives that $\mathcal{B}_p(\Psi) \ge 1$, $\mathcal{B}_q \subset \mathcal{B}_p$, $q < p$, and that if $0 \le \lambda \le 1$, then Ψ^λ is also in the class $\mathcal{B}_q$, $q = 1 + \lambda(p-1)$. Moreover, if $\psi \in \mathcal{B}_p$, $\Psi d\nu$ is a doubling measure on tents, that is there is a constant D_B such that

$$\Psi(\hat{U}_{\zeta,2t}) \le D_B\,\Psi(\hat{U}_{\zeta,t}), \quad \text{for all } \zeta \in S,\ 0 < t < 2.$$

As an immediate consequence we have that:

Lemma 4.3. *Let* $1 < p < \infty$ *and* $\Psi \in \mathcal{B}_p$. *If* $z = r\zeta$, $\zeta \in S$, $0 \le r < 1$. *If* $D_B < 2^N$, *then*

$$\int_B \frac{\Psi(w)}{|1 - z\bar{w}|^N} d\nu(w) \approx \frac{\Psi(\hat{I}_z)}{(1 - |z|^2)^N}.$$

4.2. **Holomorphic weighted Hardy-Sobolev and Besov spaces.** Let us start recalling some well known facts on the weighted Hardy-Sobolev spaces $H^p_s(\theta)$ with weights θ in $\mathcal{A}_p$.

If $1 \le p < \infty$ and $s \in R$, the space $H^p_s(\theta)$ consists of the holomorphic functions on B such that

$$\|f\|_{H^p_s(\theta)} = \|(I + R)^s f\|_{H^p(\theta)} = \left(\sup_r \int_S |(I + R)^s f|^p \theta d\sigma\right)^{1/p} < \infty.$$

As in the unweighted case, these spaces can be characterized in terms of maximal radial functions, maximal admissible functions [10] and area functions [6].

In particular, if $\Gamma_\zeta = \{z \in B; |1 - \zeta\bar{z}| < 1 - |z|^2\}$ is an admissible region and $M(f)(\zeta)$ is the maximal admissible function $M(f)(\zeta) = \sup\{|f(z)|; z \in \Gamma_\zeta\}$, we have

$$\|f\|_{H^p(\theta)} \approx \|M(f)\|_{L^p(\theta)} \approx \|M_{H-L}(f)\|_{L^p(\theta)}.$$

Our next observation is that the space $H^p(\theta)$ can also be described in terms of the averaging function Θ, with an equivalent norm. Indeed, since $\lim_{r\nearrow 1} |f(r\zeta)|^p \Theta(r\zeta) = |f(\zeta)|^p \theta(\zeta)$, a.e. $\zeta \in S$, we have

$$\|f\|_{H^p(\theta)} \le \|f\|^p_{H^p(\Theta)} =: \sup_r \int_S |f(r\zeta)|^p \Theta(r\zeta) d\sigma(\zeta),$$

and by Fubini's Theorem,

$$\|f\|^p_{H^p(\Theta)} \lesssim \int_S M(f)^p(\zeta)\theta(\zeta) d\sigma(\zeta) \lesssim \|f\|^p_{H^p(\theta)}. \tag{4.26}$$

As in the unweighted case, it is possible to obtain pointwise estimates for the derivatives of the functions in $H^p_s(\theta)$ (see Lemma 2.12 in [6]).

Lemma 4.4. *If $\theta \in \mathcal{A}_p$ and $f \in H^p_s(\theta)$, then for any positive integer $k > s$*

$$(1-r^2)^{k-s}|\nabla^k f(r\zeta)| \lesssim \|f\|_{H^p_s(\theta)}(\theta(I_z))^{-1/p} \approx \|f\|_{H^p(\theta)} \frac{(\theta'(I_z))^{1/p'}}{|I_z|}.$$

(the last inequality is an immediate consequence of the fact that $\theta \in \mathcal{A}_p$.)

We next introduce weighted Besov spaces. For $1 \leq p < \infty$, $s \in \mathbb{R}$, a non-negative integer $k > s$ and a weight $0 < \Psi \in L^1(B)$, the weighted holomorphic Besov space $B^p_{s,k}(\Psi)$ is the completion of the space $H(\bar{B})$ endowed with the norm

$$\|f\|_{B^p_{s,k}(\Psi)} = \left(\int_B |\nabla^k f(z)|^p (1-|z|^2)^{(k-s)p-1}\Psi(z)d\nu(z)\right)^{1/p}.$$

In [7] it is proved that if $\psi \in \mathcal{B}_p$, provided $1 \leq p < \infty$, different values of k give equivalent norms. From now on we will simply write $B^p_s(\Psi)$. Moreover, as it happens in the unweighted case, we can replace $|\nabla^k f(z)|$ by $|(I+R)^k f(z)|$ in the definition of $\|f\|_{B^p_{s,k}(\Psi)}$.

For $s = -1/p$ the space $B^p_s(\Psi)$ is the space of holomorphic function on B intersection $L^p(\Psi d\nu)$. Several properties of these space can be found in [10].

In particular, Theorem 2.1 in [10] with $\beta = (k-s)p-1$, $\gamma = (k+s)p'+1$ together with (1.1), gives the following result on duality.

Theorem 4.5. *Let $1 < p < \infty$, $s \in \mathbb{R}$, $\Psi \in \mathcal{B}_p$ and $\Psi' = \Psi^{-p'/p}$. Then $(B^p_s(\Psi))' = B^{p'}_{-s}(\Psi')$ with the pairing given by $\langle f, g\rangle_B = \int_B f\bar{g}d\nu$.*

The next theorem extends the embedding $B^q_{n/q-n/p} \subset H^p$, $1 \leq q \leq \min\{p, 2\}$ to the weighted case.

Theorem 4.6. *Let $1 < p < \infty$, $1 \leq q_0 < q_1 \leq \min\{p, 2\}$ and $\theta \in \mathcal{A}_p$. For $j = 0, 1$, let $s_j = n/q_j - n/p$ and let $k > s_0$ be a non-negative integer.*

Then, $B^{q_0}_{s_0,k}(\Theta^{q_0/p}) \subset B^{q_1}_{s_1,k}(\Theta^{q_1/p}) \subset H^p(\theta)$.

Proof. In order to prove the theorem, we will show that:

(i) If $1 \leq q \leq \min\{p, 2\}$ and $k > n/q - n/p$ a non-negative integer, then $B^q_{n/q-n/p,k}(\Theta^{q/p}) \subset H^p(\theta)$.

(ii) For $f \in H^p(\theta)$, $\|f\|^{q_1}_{B^{q_1}_{s_1,k}(\Theta^{q_1/p})} \lesssim \|f\|^{q_1-q_0}_{H^p(\theta)} \|f\|^{q_0}_{B^{q_0}_{s_0,k}(\Theta^{q_0/p})}$.

In order to show (i), we use Theorem 2.10 in [6], which gives an embedding between weighted Triebel-Lizorkin spaces. To be precise, if $1 \leq q \leq \min\{p, 2\}$, then $F^{p,1}_0(\theta) \subset F^{p,q}_0(\theta) \subset F^{p,2}_0(\theta) = H^p(\theta)$. Consequently,

$$
\begin{aligned}
&\|f\|_{H^p(\theta)} \approx \|f\|_{F^{p,2}_0(\theta)} \\
&\approx \left(\int_S \left(\int_{\Gamma_\zeta} |\nabla^k f(z)|^2 (1-|z|^2)^{2k-1-n} d\nu(z) \right)^{p/2} \theta(\zeta) d\nu(\zeta) \right)^{1/p} \\
&\lesssim \left(\int_S \left(\int_{\Gamma_\zeta} |\nabla^k f(z)|^q (1-|z|^2)^{kq-1-n} d\nu(z) \right)^{p/q} \theta(\zeta) d\nu(\zeta) \right)^{1/p} \\
&\approx \|f\|_{F^{p,q}_0(\theta)}.
\end{aligned} \tag{4.27}
$$

By duality, we can replace the last term in (4.27) by

$$
\sup_{\|\psi\|_{L^{(p/q)'}(\theta)}=1} \left(\int_S \int_{\Gamma_\zeta} |\nabla^k f(z)|^q (1-|z|^2)^{kq-1-n} d\nu(z) \psi(\zeta) \theta(\zeta) d\nu(\zeta) \right)^{1/q}.
$$

Fubini's Theorem gives that this last expression is equivalent to

$$
\left(\int_B |\nabla^k f(z)|^q (1-|z|^2)^{kq-1-n} \int_{I_z} \psi(\zeta)\theta(\zeta) d\sigma(\zeta) d\nu(z) \right)^{1/q}, \tag{4.28}
$$

and by Hölder's Inequality

$$
\int_{I_z} \psi(\zeta)\theta(\zeta) d\sigma(\zeta) \leq \left(\int_{I_z} \theta(\zeta) d\sigma(\zeta) \right)^{q/p} \lesssim (1-|z|^2)^{nq/p} \Theta(z)^{q/p}.
$$

Therefore (4.28) is bounded by

$$
\left(\int_B |\nabla^k f(z)|^q (1-|z|^2)^{(k-(n/q-n/p))q-1} \Theta(z)^{q/p} d\nu(z) \right)^{1/q},
$$

which proves (i).

In order to prove (ii), observe that if $f \in H^p(\theta)$, then by Lemma 4.4

$$|\nabla^k f(z)|^{q_1} \lesssim |\nabla^k f(z)|^{q_0} \|f\|_{H^p(\theta)}^{q_1-q_0} \left(\frac{(1-|z|^2)^{-k}}{\theta(I_z)^{1/p}} \right)^{q_1-q_0}$$

$$\lesssim \|f\|_{H^p(\theta)}^{q_1-q_0} |\nabla^k f(z)|^{q_0} \left(\frac{(1-|z|^2)^{-k-n/p}}{\Theta(z)^{1/p}} \right)^{q_1-q_0}.$$

Therefore,

$$|\nabla^k f(z)|^{q_1} (1-|z|^2)^{(k-n/q_1+n/p)q_1-1} \Theta(z)^{q_1/p}$$

$$\lesssim \|f\|_{H^p(\theta)}^{q_1-q_0} |\nabla^k f(z)|^{q_0} (1-|z|^2)^{(k-n/q_0+n/p)q_0-1} \Theta(z)^{q_0/p}.$$

Integrating both terms on B, we conclude the proof. ∎

Proposition 4.7. *Let $1 < p < \infty$, $k > n+1$ and θ in $\mathcal{A}_p$. If $D_S < 2^{Np}$, and f_z are the test functions defined by $f_z(w) = (1 - w\bar{z})^{-N}$, then*

$$\|f_z\|_{B^1_{n/p',k}(\Theta^{1/p})} \approx \|f_z\|_{H^p(\theta)} \approx \frac{\theta(I_z)^{1/p}}{(1-|z|^2)^N}.$$

Proof. The fact that the norms in $B^1_{n/p',k}(\Theta^{1/p})$ and $H^p(\theta)$ of the test functions are equivalent, will be deduced from Theorem 4.6 and the estimate

$$\|(1-w\bar{z})^{-N}\|_{B^1_{n/p',k}(\Theta^{1/p})} \lesssim \|(1-w\bar{z})^{-N}\|_{H^p(\theta)}.$$

In order to prove the above estimate, it is enough to show that if $k > n+1$, then

$$\int_B \frac{(1-|z|^2)^{k-n/p'-1}\Theta^{1/p}(z)}{|1-w\bar{z}|^{N+k}} d\nu(w) \lesssim \left(\int_S \frac{\theta(\zeta)}{|1-\zeta\bar{z}|^{Np}} d\sigma(\zeta) \right)^{1/p}.$$

Hölder's Inequality give that if $\varepsilon > 0$ the left term in the above inequality is bounded, up a constant depending of ε, by

$$\left(\int_B \frac{(1-|z|^2)^{kp-np/p'-1-\varepsilon p}\Theta(z)}{|1-w\bar{z}|^{Np+kp-np/p'-2\varepsilon p}} d\nu(w) \right)^{1/p} \left(\int_B \frac{(1-|w|^2)^{\varepsilon p'-1}}{|1-w\bar{z}|^{n+2\varepsilon p'}} d\nu(w) \right)^{1/p'}$$

$$\lesssim (1-|z|^2)^{-\varepsilon} \left(\int_B \frac{(1-|z|^2)^{kp-np/p'-1-\varepsilon p}\Theta(z)}{|1-w\bar{z}|^{Np+kp-np/p'-2\varepsilon p}} d\nu(w) \right)^{1/p}.$$

Since $k > n+1$, let $\varepsilon > 0$ be small enough such that $kp-np/p'-1-\varepsilon p-n > 0$. By Fubini's Theorem, the last term is bounded by

$$(1-|z|^2)^{-\varepsilon}\left(\int_S\int_{\Gamma_\zeta}\frac{(1-|z|^2)^{kp-np/p'-1-\varepsilon p-n}}{|1-w\bar z|^{Np+kp-np/p'-2\varepsilon p}}d\nu(w)\theta(\zeta)d\sigma(\zeta)\right)^{1/p}$$
$$\lesssim (1-|z|^2)^{-\varepsilon}\left(\int_S\int_{\Gamma_\zeta}\frac{d\nu(w)}{|1-w\bar z|^{n+1+Np-\varepsilon p}}\theta(\zeta)d\sigma(\zeta)\right)^{1/p}$$
$$\lesssim (1-|z|^2)^{-\varepsilon}\left(\int_S\frac{\theta(\zeta)d\sigma(\zeta)}{|1-\zeta\bar z|^{Np-\varepsilon p}}\right)^{1/p}.$$

Since $D_S < 2^{Np}$, Lemma 4.1 gives that, provided $\varepsilon > 0$ is small enough, we have

$$(1-|z|^2)^{-\varepsilon}\left(\int_S\frac{\theta(\zeta)d\sigma(\zeta)}{|1-\zeta\bar z|^{Np-\varepsilon p}}\right)^{1/p}\approx\frac{\theta(I_z)}{(1-|z|^2)^N}\approx\left(\int_S\frac{\theta(\zeta)d\sigma(\zeta)}{|1-\zeta\bar z|^{Np}}\right)^{1/p}$$

which concludes the proof. ∎

4.3. **Toeplitz operators.** The results of the above section together Theorem 3.2 gives the following result, which generalizes Corollary 3.4.

Theorem 4.8. *Let $1 < p < \infty$, $k > n+1$ and let $\theta_0 \in \mathcal{A}_p$ and $\theta_1 \in \mathcal{A}_{p'}$. Let Θ_0 and Θ_1 be the corresponding averaging functions.*

If X and Y are two Banach spaces satisfying,

(i) $B^1_{n/p',k}(\Theta_0^{1/p}) \subset X \subset H^p(\theta_0)$, *and*
(ii) $B^1_{n/p,k}({\Theta_1}^{1/p'}) \subset Y \subset H^{p'}(\theta_1)$,

then $\|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})}\approx\left\|\dfrac{\psi}{\theta_0^{1/p}\theta_1^{1/p'}}\right\|_\infty.$

Proof. We want to prove that X and Y satisfy the conditions in Theorem 3.2.

By Proposition 4.7 it is clear that, if $m > 0$, then we have $\omega_{m,X,Y}(z)\approx\omega_{m,H^p(\theta_0),H^{p'}(\theta_1)}(z)$. Therefore, as in the proof of Theorem 3.11, in both cases the corresponding functions $\tilde\omega_m$ are equivalent to $\theta_0^{1/p}\theta_1^{1/p'}$ a.e. on S.

Hence, if $f \in X$ and $g \in Y$, then

$$\int_S |f||g|\tilde{\omega}_{m,X,Y} d\sigma \approx \int_S |f||g|\theta_0^{1/p}\theta_1^{1/p'} d\sigma$$
$$\lesssim \|f\|_{H^p(\theta_0)}\|g\|_{H^{p'}(\theta_1)} \lesssim \|f\|_X\|g\|_Y.$$

and as a consequence of Theorem 3.2

$$\|\Gamma_\psi\|_{Bil(X\times\overline{Y}\to\mathbb{C})} \approx \left\|\frac{\psi}{\theta_0^{1/p}\theta_1^{1/p'}}\right\|_\infty.$$

■

If the weights θ_0 and θ_1 are in smaller classes, we can drop the k in the last theorem and we have the following corollary.

Corollary 4.9. *Let $1 < q_0 < p < q_1 < \infty$, $\theta_0 \in \mathcal{A}_{p_0}$, $\theta_1 \in \mathcal{A}_{p_1'}$ where $p_0 = 1 + p/q_0'$, $p_1' = 1 + p'/q_1$ and Θ_0 and Θ_1 the corresponding averaging functions. Then, $\Theta_0^{q_0/p} \in \mathcal{B}_{q_0}$, $\Theta_1^{q_1'/p'} \in \mathcal{B}_{q_1'}$ and*

$$\|\Gamma_\psi\|_{Bil(B^{q_0}_{n/q_0-n/p}(\Theta_0^{q_0/p})\times\overline{B^{q_1'}_{n/q_1'-n/p'}(\Theta_1^{q_1'/p'})}\to\mathbb{C})} \approx \left\|\frac{\psi}{\theta_0^{1/p}\theta_1^{1/p'}}\right\|_\infty.$$

As we stated in the introduction, the results on Toeplitz forms together duality results lead to results on Toeplitz operators. For instance, we have the following theorem.

Theorem 4.10. *Let $1 < p < \infty$, $\psi \in L^1(d\sigma)$, θ_0 an admissible weight and $\theta_1 \in \mathcal{A}_{p'}$.*

Then

$$\|T_\psi\|_{L(H^p(\theta_0)\to H^p(\theta_1))} \approx \left\|\frac{\psi}{\theta_0^{1/p}\theta_1^{-1/p}}\right\|_\infty.$$

In particular, if $\theta = \theta_0 = \theta_1$, then $\|T_\psi\|_{L(H^p(\theta)\to H^p(\theta))} \approx \|\psi\|_\infty$.

Proof. This result is a consequence of

$$\|T_\psi\|_{L(H^p(\theta_0)\to H^p(\theta_1))} \approx \|\Gamma_\psi\|_{Bil(H^p(\theta_0)\times\overline{H^{p'}(\theta_1^{-p'/p})})} \approx \left\|\frac{\psi}{\theta_0^{1/p}\theta_1^{-1/p}}\right\|_\infty.$$

The last equivalence is a consequence of Theorem 4.8. ■

In the case of Besov spaces, we can state simmilar results once we have a description of the dual of the Besov spaces involved. In general, if $\theta \in \mathcal{A}_{p'}$, the weight $\Theta^{q_1'/p'}$ is not necessarily in $\mathcal{B}_{q_1'}$. In order to ensure that $\Theta^{q_1'/p'} \in \mathcal{B}_{q_1'}$, we can take for instance a weight $\theta \in \mathcal{A}_{p_1'}$ with p_1' satisfying that $1+(p_1'-1)q_1'/p' \le q_1'$.

We then have:

Theorem 4.11. *Let* $1<q_0<p<q_1$, $k>n+1$, $\theta_0 \in \mathcal{A}_p$ *and* $\theta_1 \in \mathcal{A}_{p_1'}$, $p_1' \le 1+p'/q_1$. *Then,* $\Theta_1^{q_1'/p'} \in \mathcal{B}_{q_1'}$ *and*

$$\|T_\psi\|_{L(B^{q_0}_{n/q_0-n/p,k}(\Theta_0^{q_0/p})\to B^{q_1}_{n/q_1-n/p}(\Theta_1^{-q_1/p'}))} \approx \left\|\frac{\psi}{\theta_0^{1/p}\theta_1^{-1/p}}\right\|_\infty .$$

Proof. By hypothesis, we have $p_1' < p'$, and by Theorem 4.6 the spaces $X = B^{q_0}_{n/q_0-n/p,k}(\Theta_0^{q_0/p})$ and $Y = B^{q_1'}_{n/q_1'-n/p'}(\Theta_1^{q_1'/p'})$ satisfy the conditions in Theorem 4.8. Since $\theta_1 \in \mathcal{A}_{p_1'}$, $\Theta_1 \in \mathcal{B}_{p_1'}$, and $\Theta_1^{q_1'/p'} \in \mathcal{B}_{1+(p_1'-1)q_1'/p'}$. Since $p_1' \le 1+p'/q_1$, this class is included in $\mathcal{B}_{q_1'}$.

Therefore, $\left(B^{q_1'}_{n/q_1'-n/p'}(\Theta_1^{q_1'/p'})\right)' = B^{q_1}_{n/q_1-n/p}(\Theta_1^{-q_1/p'})$. Consequently we have

$$\begin{aligned}&\|T_\psi\|_{L(B^{q_0}_{n/q_0-n/p,k}(\Theta_0^{q_0/p})\to B^{q_1}_{n/q_1-n/p}(\Theta_1^{-q_1/p'}))}\\ &\qquad \approx \|\Gamma_\psi\|_{Bil\left(B^{q_0}_{n/q_0-n/p,k}(\Theta_0^{q_0/p})\times \overline{B^{q_1'}_{n/q_1'-n/p'}(\Theta_1^{q_1'/p'})\to\mathbb{C}}\right)} .\end{aligned}$$

Theorem 4.8 ends the proof. ■

If in addition the weight θ_0 is in the smaller class $\mathcal{A}_{p_0}$ with $p_0 \le 1+p/q_0'$, the Besov space involved in the last theorem does not depend on k, and we deduce that the following corollary holds.

Corollary 4.12. *Let* $1<q_0<p<q_1$, $\theta_0 \in \mathcal{A}_{p_0}$, $p_0 \le 1+p/q_0'$ *and* $\theta_1 \in \mathcal{A}_{p_1'}$, $p_1' \le 1+p'/q_1$. *Then,* $\Theta_0^{q_0/p} \in \mathcal{B}_{q_0}$, $\Theta_1^{q_1'/p'} \in \mathcal{B}_{q_1'}$ *and*

$$\|T_\psi\|_{L(B^{q_0}_{n/q_0-n/p}(\Theta_0^{q_0/p})\to B^{q_1}_{n/q_1-n/p}(\Theta_1^{-q_1/p'}))} \approx \left\|\frac{\psi}{\theta_0^{1/p}\theta_1^{-1/p}}\right\|_\infty .$$

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