MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES: EQUIVALENCE OF NORMS

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ABSTRACT. For an inner function θ on the upper half-plane $\mathbb{C}_+$, we look at the star-invariant subspace $K^p_\theta := H^p \cap \theta\,\overline{H^p}$ of the Hardy space H^p. We characterize those θ for which the differentiation operator $f \mapsto f'$ provides an isomorphism between K^p_θ and a closed subspace of H^p, with $1 < p < \infty$. Namely, we show that such θ's are precisely the Blaschke products whose zero-set lies in some horizontal strip $\{a < \Im z < b\}$, with $0 < a < b < \infty$, and splits into finitely many separated sequences. We also describe the case of a single separated sequence in terms of the left inverse to the differentiation map; the description involves coanalytic Toeplitz operators. While our main result provides a criterion for the H^p-norms $\|f\|_p$ and $\|f'\|_p$ to be equivalent (written as $\|f\|_p \asymp \|f'\|_p$), where f ranges over a certain family of meromorphic functions with fixed poles, some other spaces Y that admit a similar estimate $\|f\|_Y \asymp \|f'\|_Y$ under similar conditions are also pointed out.

1. Introduction and results

Suppose E is a subset of the half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$ and $m : E \to \mathbb{N}$ is a function, to be thought of as 'multiplicity'. Further, let $\mathcal{R}_{E,m}$ be the set of rational functions whose poles (if any) are in $\bar{E} := \{\bar{\lambda} : \lambda \in E\}$ and, for each $\lambda \in E$, the pole at $\bar{\lambda}$ has multiplicity at most $m(\lambda)$. We shall address the following question: When is it true that every $f \in \mathcal{R}_{E,m} \cap L^p(\mathbb{R})$ satisfies $f' \in L^p(\mathbb{R})$ and, moreover,

$$C^{-1}\|f\|_p \le \|f'\|_p \le C\|f\|_p \tag{1.1}$$

for all such f and some fixed $C = C(p, E, m)$?

Here and below, $\|\cdot\|_p$ is the usual L^p-norm over $\mathbb{R}$, and p is restricted to the range $1 < p < \infty$ (except in Section 2). In particular, the assumption $p > 1$ guarantees that $\mathcal{R}_{E,m} \cap L^p(\mathbb{R})$ contains non-null functions whenever E is nonempty, while letting $p < \infty$ one rules out the case $f \equiv 1$, an obvious counterexample to (1.1) for $p = \infty$.

We shall often abbreviate (1.1) as $\|f\|_p \asymp \|f'\|_p$. In general, we shall write $A \asymp B$ to mean that the ratio A/B lies between two positive constants.

Now let us observe that the pair (E, m) must satisfy the *Blaschke condition*

$$\sum_{\lambda \in E} m(\lambda) \frac{\Im \lambda}{1 + |\lambda|^2} < \infty$$

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as soon as (1.1) holds true for f in $\mathcal{R}_{E,m} \cap L^p(\mathbb{R})$. Otherwise, this last set would be dense in the Hardy space $H^p = H^p(\mathbb{C}_+)$ (see [14, Chapter II]) and (1.1) would extend to all $f \in H^p$, which it does not. Consequently, we can form the Blaschke product

$$B(z) = B_{E,m}(z) := \prod_{\lambda \in E} \left\{ \alpha(\lambda) \frac{z-\lambda}{z-\bar{\lambda}} \right\}^{m(\lambda)},$$

where $\alpha(\lambda)$ are suitable 'convergence factors' of modulus 1; when $i \notin E$, one takes $\alpha(\lambda) = |\lambda^2+1|/(\lambda^2+1)$, cf. [14, p. 55]. This done, the closure of $\mathcal{R}_{E,m}$ in $L^p(\mathbb{R})$ can be identified as

$$H^p \cap B\,\overline{H^p} =: K^p_B,$$

where the Hardy space H^p is viewed as a subspace of $L^p(\mathbb{R})$ and the bar denotes complex conjugation. The estimate (1.1) is then inherited by the elements of K^p_B; these are H^p-functions on $\mathbb{C}_+$ that admit a meromorphic pseudocontinuation (see [5, 6]) into $\mathbb{C}_- := \mathbb{C} \setminus (\mathbb{C}_+ \cup \mathbb{R})$ whose poles, counted with multiplicities, are contained among those of B. Note that all functions in K^p_B will be meromorphic in $\mathbb{C}$ whenever B is.

In fact, we shall deal with the following, somewhat more general, problem. Suppose θ is an *inner function* on $\mathbb{C}_+$; that is, $\theta \in H^\infty(\mathbb{C}_+)$ and $\lim_{y\to 0^+} |\theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$. For $1 < p < \infty$, consider the *star-invariant subspace*

$$K^p_\theta := H^p \cap \theta\,\overline{H^p} \tag{1.2}$$

that θ generates in H^p. When do we have (1.1) for all $f \in K^p_\theta$?

The term "star-invariant", as used above, means invariant under the semigroup of *backward shifts* $\{S^*_a : a > 0\}$, where S^*_a is the adjoint of the *forward shift* S_a given by

$$(S_a f)(x) := e^{iax} f(x) \qquad (f \in H^q, \quad p^{-1} + q^{-1} = 1).$$

It is a well-known consequence of the Beurling–Lax theorem on S_a-invariant subspaces (cf. [16, Lecture XI]) that the general form of a nontrivial star-invariant subspace in H^p is actually provided by (1.2), with θ inner; see also [5, 6]. Our star-invariant subspaces are also known as *model subspaces*, especially when $p = 2$. We refer, once again, to [16] for an explanation of this terminology.

Going back to our problem, we now remark that 'half' of it is solved by our previous result from [8]; see also [11] for an alternative proof. This result, cited as Theorem A below, describes the θ's with the property that K^p_θ-functions are all (locally) absolutely continuous on $\mathbb{R}$ and the differentiation operator $\frac{d}{dx} : f \mapsto f'$ is a continuous mapping from K^p_θ to $L^p(\mathbb{R})$.

Theorem A. *Let $1 < p < \infty$, and let θ be an inner function in $\mathbb{C}_+$. In order that $\frac{d}{dx}$ be a bounded operator from K^p_θ to $L^p(\mathbb{R})$, it is necessary and sufficient that $\theta' \in L^\infty(\mathbb{R})$. Moreover, the norm of this operator is bounded by $C_p\|\theta'\|_\infty$, where C_p is a suitable constant depending only on p.*

The condition $\theta' \in L^\infty(\mathbb{R})$, understood in any natural sense, is actually equivalent to $\theta' \in H^\infty$ and also to the requirement that

$$\inf\{|\theta(z)| : 0 < \Im z < \delta\} > 0 \tag{1.3}$$

for some $\delta > 0$ (see, e.g., [11, Lemma 2]). It follows that the singular factor of such a θ can only be of the form e^{iaz}, with $a \ge 0$, while the associated Blaschke product $B = B_{\{z_k\}}$ is meromorphic in $\mathbb{C}$ (and analytic on $\mathbb{R}$): its zeros $z_k = x_k + iy_k$ must satisfy $\inf_k y_k > 0$, so they can only cluster at ∞. The inner functions θ occurring in Theorem A are thus given by $\theta = e^{iaz}B_{\{z_k\}}$, with

$$|\theta'(x)| = a + 2\sum_k \frac{y_k}{|x - z_k|^2}, \qquad x \in \mathbb{R}, \tag{1.4}$$

being bounded on the real line. In connection with (1.4), we refer to [1] for similar (and more general) formulas involving derivatives of inner functions on the boundary; see also [10, Lemma 3] for the case of a meromorphic Blaschke product.

The inequality

$$\|f'\|_p \le C_p \|\theta'\|_\infty \|f\|_p, \qquad f \in K^p_\theta, \tag{1.5}$$

coming from Theorem A, should be viewed – and actually arose – as a generalization of the classical Bernstein inequality for entire functions. The latter reads

$$\|f'\|_p \le a\|f\|_p,$$

where f is an entire function of exponential type $\le a$, and can be essentially recovered from (1.5) by taking $\theta(z) = e^{iaz}$ (except that (1.5) does not give $C_p = 1$). This time, our plan is to couple (1.5) with the 'reverse Bernstein inequality'

$$\|f'\|_p \ge \text{const} \cdot \|f\|_p,$$

a property which entire functions generally fail to possess. It is chiefly this reverse estimate that interests us now in the framework of K^p_θ spaces.

We mention in passing that Theorem A remains true for the endpoint exponent $p = \infty$, as does the sufficiency part for $p = 1$ (see [8]), the definitions of K^1_θ and K^∞_θ being similar to (1.2). On the other hand, the necessity fails for $p = 1$, as was recently shown by Baranov [4]. Also, there are various 'weighted' versions and extensions of the Bernstein-type inequality (1.5); for these, see [2, 3] and [9, Sections 10–11]. Finally, let us mention our recent study of some further properties (compactness, membership in the Schatten–von Neumann ideals) of the differentiation operator on K^p_θ, as carried out in [11, 12]. In particular, it was proved in [11] that $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$ is a compact operator if and only if $\theta' \in C_0(\mathbb{R})$.

Our current topic, the two-sided estimate $\|f\|_p \asymp \|f'\|_p$ on K^p_θ, can be viewed as a certain "anti-compactness" feature of the differentiation operator $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$. Precisely speaking, we now want this operator to be an isomorphism onto its range. Equivalently, we want it to be both bounded and *left-invertible* (or *bounded from below*). Yet another reformulation of the same condition is that $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$ be a bounded operator with closed range; to see why, note that the kernel of $\frac{d}{dx}$ in K^p_θ is trivial and apply the open mapping theorem.

Our first result, Theorem 1.1 below, will provide a criterion for all this to happen. Before stating it, let us recall that a sequence of points $\{z_j\} \subset \mathbb{C}_+$ is called an *interpolating sequence* if the restriction map $f \mapsto \{f(z_j)\}$, going from H^∞ to ℓ^∞, is surjective. (Sometimes we shall view $\{z_j\}$ as a set and speak of *interpolating* *sets.*) By Carleson's interpolation theorem [14, Chapter VII], $\{z_j\}$ is an interpolating sequence if and only if

$$\inf_j \prod_{k:\, k\neq j} \rho(z_j, z_k) > 0,$$

where $\rho(\cdot,\cdot)$ is the noneuclidean metric on $\mathbb{C}_+$ given by $\rho(z,w) := |z-w|/|z-\bar{w}|$. Finally, an *interpolating Blaschke product* is, by definition, a Blaschke product whose zeros are simple and form an interpolating sequence.

Theorem 1.1. *Let $1 < p < \infty$ and let θ be an inner function. The following are equivalent.*

(i.1) *The operator $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$ is an isomorphism onto its range.*

(ii.1) $\theta' \in L^\infty(\mathbb{R})$ *and* $\inf_\phi \|\phi'\|_\infty > 0$*, where ϕ ranges over the nonconstant inner divisors of θ.*

(iii.1) *θ is a finite product of interpolating Blaschke products, and*

$$0 < \inf\{\Im z : z \in \theta^{-1}(0)\} \le \sup\{\Im z : z \in \theta^{-1}(0)\} < \infty. \tag{1.6}$$

Remarks. (1) It should be noted that if $\{z_j\}$ is a sequence lying in some horizontal strip

$$\{z : a < \Im z < b\}, \qquad \text{with } 0 < a < b < \infty, \tag{1.7}$$

then $\{z_j\}$ is an *interpolating* sequence if and only if it is *separated*, in the sense that $\inf_{j\neq k} \rho(z_j, z_k) > 0$, or equivalently, $\inf_{j\neq k} |z_j - z_k| > 0$; see [16, pp. 259–260]. Therefore, the 'interpolating Blaschke products' in Theorem 1.1 (and those in Theorem 1.3 below) can be replaced by 'Blaschke products with separated zeros'.

(2) We also observe, for future reference, that if θ is an interpolating Blaschke product whose zeros $\{z_j\}$ are bounded away from $\mathbb{R}$, so that $\Im z_j \ge c > 0$, then θ satisfies (1.3) with $\delta = c/2$ (because the ρ-distance between $\{z_j\}$ and the strip $\{0 < \Im z < c/2\}$ is positive), and hence $\theta' \in L^\infty(\mathbb{R})$. Of course, a similar fact is true for finite products $\theta = \theta_1 \dots \theta_n$, where each θ_k is an interpolating Blaschke product with $\inf\{\Im z : z \in \theta_k^{-1}(0)\} > 0$.

As a byproduct of our proof of Theorem 1.1, we are able to point out some other spaces Y, in addition to $Y = H^p$ $(1 < p < \infty)$, where the estimate $\|f'\|_Y \asymp \|f\|_Y$ holds, under the same conditions, in the appropriate classes of meromorphic functions with fixed poles. Below, we state such a result for $Y = \mathrm{BMOA}$ and $Y = A^\alpha$, with $0 < \alpha < \infty$. Here, BMOA is the analytic subspace of $\mathrm{BMO} := \mathrm{BMO}(\mathbb{R})$, the space of functions that have bounded mean oscillation on $\mathbb{R}$, endowed with the usual BMO-norm $\|\cdot\|_*$ (see [14, Chapter VI]). By A^α we denote the analytic Lipschitz–Zygmund spaces on $\mathbb{C}_+$. An analytic function f is thus in A^α if and only if

$$f^{(n)}(z) = O\left((\Im z)^{\alpha - n}\right), \qquad z \in \mathbb{C}_+,$$

for some, or any, integer $n > \alpha$; the (semi)norm $\|f\|_{A^\alpha}$ is then taken to be the best constant in this O-condition, say with $n = [\alpha] + 1$.

Proposition 1.2. *Under condition* (iii.1)*, we have*

$$\|f'\|_* \asymp \|f\|_*$$

and

$$\|f'\|_{A^\alpha} \asymp \|f\|_{A^\alpha} \qquad (0 < \alpha < \infty)$$

for every rational function $f \in K^2_\theta$*.*

Of course, it is understood that the constants in these equivalence relations are independent of f, so the estimates extend to the appropriate closed subspaces of BMOA and A^α (the ones spanned by the rational functions in question).

While Theorem 1.1 tells us that the existence of a left inverse to the (bounded) operator $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$ is equivalent to the fact that θ is a *finite product* of interpolating Blaschke products with zeros in some strip (1.7), our next result shows that the case of a *single* interpolating Blaschke product is also describable in terms of the left inverse to the differentiation operator. Namely, this is precisely the case when the left inverse can be realized as a coanalytic Toeplitz operator, to be defined in a moment.

Recalling the M. Riesz decomposition $L^p(\mathbb{R}) = H^p \oplus \overline{H^p}$ for $1 < p < \infty$, we write $P_+ : L^p(\mathbb{R}) \to H^p$ for the canonical projection that arises, and then define the *Toeplitz operator* T_φ *with symbol* $\varphi \in L^\infty(\mathbb{R})$ by the formula

$$T_\varphi f := P_+(\varphi f), \qquad f \in H^p.$$

When $\varphi \in \overline{H^\infty}$, we say that T_φ is a *coanalytic Toeplitz operator.*

Theorem 1.3. *Let* $1 < p < \infty$ *and let* θ *be an inner function. The following are equivalent.*

(i.2) *The operator* $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$ *is bounded, and there exists a function* $\psi \in H^\infty$ *such that* $T_{\bar\psi}\frac{d}{dx} = I$ *on* K^p_θ *(here* I *is the identity map).*

(ii.2) θ *is an interpolating Blaschke product satisfying* (1.6)*.*

Remark. A close look at the proof will reveal that Theorem 1.3 remains valid if the Toeplitz operator $T_{\bar\psi} : f \mapsto P_+(\bar\psi f)$ in (i.2) gets replaced by its 'restricted version' $f \mapsto P_\theta(\bar\psi f)$, where P_θ is the canonical projection onto K^p_θ given by $P_\theta = P_+ - \theta P_+ \bar\theta$. This answers a question raised by the referee.

In conclusion, we supplement Theorem 1.3 with a weighted norm estimate for K^p_θ functions and their derivatives. The weight w will satisfy the Muckenhoupt (A_p) condition on $\mathbb{R}$, to be written as $w \in (A_p)$; see [13, Chapter 3] or [14, Chapter VI] for the definition and discussion of (A_p) weights. The corresponding L^p_w-norm will be denoted by $\|\cdot\|_{p,w}$, so that

$$\|f\|_{p,w} := \left(\int_{\mathbb{R}} |f|^p w\, dx\right)^{1/p}.$$

Proposition 1.4. *If* $1 < p < \infty$ *and* $w \in (A_p)$*, and if* θ *satisfies* (ii.2)*, then*

$$\|f'\|_{p,w} \asymp \|f\|_{p,w}$$

for all rational functions f *in* K^2_θ*.*

The rest of the paper contains some preliminary material, collected in Section 2, and the proofs of our results, given in Sections 3 and 4.

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2. Preliminaries on finite unions of interpolating sets

The proofs in the next section will rely on an explicit characterization, due to Tolokonnikov [17, Theorem 2], of the trace space $H^p\big|_E$, where E is a *finite union of interpolating sets* (FUIS, for short) in $\mathbb{C}_+$. Strictly speaking, [17] deals with H^p-spaces of the disk, but the case of a half-plane is similar.

Suppose E is a FUIS in $\mathbb{C}_+$, and fix a number $\delta \in (0,1)$. Then, for some $n \in \mathbb{N}$, there is a numbering

$$E = \{z_{jk} : j = 1, 2, \dots, 1 \le k \le k_j\}$$

such that

- $k_j \le n$ for all j,
- for $j = 1, 2, \dots$, the noneuclidean diameters of the sets $E_j := \{z_{jk} : 1 \le k \le k_j\}$ are all bounded by δ,
- for $k = 1, \dots, n$, each of the sets $E^{(k)} := \{z_{jk} : j \text{ satisfies } k_j \ge k\}$ is an interpolating set.

Further, given a pair of indices (j,k) and a function $w : E \to \mathbb{C}$, we put $R_{jk}(z) := \prod_{l=1}^{k-1}(z - z_{jl})$ and let P_{jk}^w stand for the polynomial of degree $\le k-2$ that interpolates w on the set $\{z_{jl} : 1 \le l \le k-1\}$. It is understood that $P_{j1}^w \equiv 0$ and $R_{j1} \equiv 1$. Finally, following [17], we write $X^p(E)$ for the set of functions $w : E \to \mathbb{C}$ with

$$\sum_{j,k} \left| \frac{w - P_{jk}^w}{R_{jk}}(z_{jk}) \right|^p y_{j1}^{(k-1)p+1} < \infty,$$

where $y_{jk} := \Im z_{jk}$; when $p = \infty$, the sum gets replaced by the corresponding supremum. (To keep on the safe side, let us observe that $y_{jk} \asymp y_{j1}$ for $j \ge 1$ and $1 \le k \le k_j$.) Tolokonnikov's theorem now states that

$$H^p\big|_E = X^p(E), \qquad 0 < p \le \infty. \tag{2.1}$$

Although derivatives of H^p-functions are not explicitly mentioned in [17], while the points z_{jk} above are (tacitly) assumed to be pairwise distinct, it is not hard to see what happens to (2.1) when some of the points are allowed to "merge together". In this case, one deals with a multiple interpolation problem, so that the values $f(\lambda)$, $f'(\lambda)$, $\dots$, $f^{(s-1)}(\lambda)$ should be considered for a point $\lambda \in E$ of multiplicity s (i.e., for one which is included in s copies). The admissible multiplicities s are, of course, uniformly bounded (by n). The trace space that arises is then obtained from $X^p(E)$ by means of a limiting argument.

In this way, (2.1) yields a description of the trace space $(H^p)'\big|_E$, where $(H^p)' := \{f' : f \in H^p\}$ and $E = \{z_{jk}\}$ is again a FUIS. Namely, it turns out that the operator

$\mathcal{J}_E$ given by

$$(\mathcal{J}_E f)(z_{jk}) = y_{j1} f'(z_{jk}) \qquad (j = 1, 2, \dots,\ 1 \le k \le k_j) \tag{2.2}$$

maps H^p onto $X^p(E)$; thus

$$\mathcal{J}_E\,(H^p) = X^p(E), \qquad 0 < p \le \infty. \tag{2.3}$$

In particular, comparing (2.1) and (2.3), we arrive at the following result.

Theorem B. *Let $0 < p \le \infty$, and suppose E is a FUIS contained in some strip $\{z : \, a < \Im z < b\}$ with $0 < a < b < \infty$. Then $H^p\big|_E = (H^p)'\big|_E$.*

Indeed, the factors y_{j1} appearing in (2.2) are in this case harmless and can be safely dropped, so the two trace spaces coincide with $X^p(E)$. In fact, under the hypotheses of Theorem B, one also has $(H^p)^{(l)}\big|_E = X^p(E)$, where $l \in \mathbb{N}$ and $(H^p)^{(l)}$ is formed by the lth order derivatives of H^p-functions.

Finally, we remark that the identity between the two trace spaces in Theorem B is accompanied by an equivalence relation between their respective norms (quasinorms, when $0 < p < 1$). These are defined, for a function $w : E \to \mathbb{C}$, as

$$\inf\left\{\|f\|_p : \, f \in H^p,\ f\big|_E = w\right\}$$

and

$$\inf\left\{\|f\|_p : \, f \in H^p,\ f'\big|_E = w\right\},$$

so the two quantities are comparable to each other and to the (quasi)norm $\|w\|_{X^p(E)}$, defined in the natural way. Moreover, the constants in the corresponding inequalities can be taken to depend only on a, b, p and on the Carleson norm of the measure μ_E given by

$$\mu_E(\sigma) := \sum_{z \in \sigma \cap E} \Im z, \qquad \sigma \subset \mathbb{C}_+. \tag{2.4}$$

Here, the 'Carleson norm' of μ_E is understood as

$$\sup_I |I|^{-1} \mu_E(Q_I) =: \|\mu_E\|_{\mathrm{carl}},$$

where I ranges over the real intervals, $|I|$ is the length of I, and Q_I denotes the square $I \times (0, |I|)$. It is worth recalling at this point that a (generic) set $E \subset \mathbb{C}_+$ will be a FUIS if and only if μ_E is a Carleson measure (i. e., $\|\mu_E\|_{\mathrm{carl}} < \infty$); see [14, p. 314] or [16, p. 158].

Remark. There is an alternative characterization, involving the so-called divided differences, of the trace space $H^p\big|_E$ for E a FUIS. This was given by Vasyunin [18, 19] in the case $p = \infty$ and subsequently extended by Hartmann [15] to the range $1 < p \le \infty$. For these values of p (but not for $0 < p \le 1$) one can also arrive at Theorem B via that alternative approach.

3. Proofs of Theorem 1.1 and Proposition 1.2

Proof of Theorem 1.1. (i.1) $\implies$ (ii.1). We know from Theorem A that $\theta' \in L^\infty(\mathbb{R})$. Now if ϕ is a nonconstant divisor of θ, then K^p_ϕ is a non-null subspace of K^p_θ, and we have

$$c\|f\|_p \le \|f'\|_p \le C\|\phi'\|_\infty \|f\|_p \qquad \text{for all } f \in K^p_\phi.$$

(Here, the former inequality is due to the hypothesis that $\frac{d}{dx}$ is bounded from below, while the latter relies on Theorem A.) Consequently, $\|\phi'\|_\infty \ge c/C$, and we arrive at (ii.1).

(ii.1) $\implies$ (iii.1). First we observe that θ must be a Blaschke product. Indeed, the only singular factors compatible with the condition $\theta' \in L^\infty(\mathbb{R})$ are of the form $S_a(z) := e^{iaz}$, with $a > 0$. However, these are ruled out by the assumption that

$$\inf\{\|\phi'\|_\infty : \phi \text{ divides } \theta\} > 0, \tag{3.1}$$

since S_a is divisible by S_ε for $0 < \varepsilon < a$, and $\|S'_\varepsilon\|_\infty = \varepsilon$.

Now let $\{z_j = x_j + iy_j\}$ be the zeros of θ. We know that

$$m := \inf_j y_j > 0$$

(this is guaranteed by the fact that $\theta' \in L^\infty(\mathbb{R})$) and we further claim that

$$M := \sup_j y_j < \infty. \tag{3.2}$$

This, again, is a consequence of (3.1), since θ is divisible by

$$b_j(z) := \frac{z - z_j}{z - \bar{z}_j}$$

and $\|b'_j\|_\infty = 2/y_j$.

Finally, we use (3.2) and (1.4) (with $a = 0$) to obtain

$$\sum_k \frac{y_j y_k}{|z_j - \bar{z}_k|^2} \le M \sum_k \frac{y_k}{|x_j - z_k|^2} = \frac{1}{2} M |\theta'(x_j)| \le \frac{1}{2} M \|\theta'\|_\infty,$$

for every j. Thus,

$$\sup_j \sum_k \frac{y_j y_k}{|z_j - \bar{z}_k|^2} < \infty.$$

This in turn means (see, e. g., [16, p. 151]) that the measure

$$\mu_{\{z_j\}} := \sum_j y_j \delta_{z_j}, \tag{3.3}$$

where δ_{z_j} denotes the unit point mass at z_j, is a Carleson measure. Equivalently (see [16, p. 158]), the sequence $\{z_j\}$ splits into finitely many interpolating ones. We have already seen that this sequence is contained in the strip $\{m \le \Im z \le M\}$, so (iii.1) is now established.

(iii.1) $\Longrightarrow$ (i.1). Let $\theta^{-1}(0) = \{z_j\}$, so that $\{z_j\}$ is a FUIS lying in some strip (1.7), and assume for the sake of simplicity that θ has no multiple zeros. Consider a rational function of the form

$$f(z) = \sum_j \frac{\lambda_j}{z - \bar{z}_j}, \tag{3.4}$$

so that only finitely many λ_j's are nonzero. Further, put $q = p/(p-1)$, and let $g_0 \in H^q$ be a function with $\|g_0\|_q = 1$ such that

$$I_0 := \left| \int_{\mathbb{R}} \bar{f} g_0 \, dx \right| \ge \frac{1}{2} \sup \left\{ \left| \int_{\mathbb{R}} \bar{f} g \, dx \right| : \, g \in H^q, \|g\|_q = 1 \right\}. \tag{3.5}$$

We now notice that

$$I_0 = 2\pi \left| \sum_j \bar{\lambda}_j g_0(z_j) \right|$$

(by Cauchy's formula) and then invoke Theorem B, or rather the part that says $H^q\big|_{\{z_j\}} \subset (H^q)'\big|_{\{z_j\}}$, to find a function $h_0 \in H^q$ with the properties that

$$h_0'(z_j) = g_0(z_j) \quad (j = 1, 2, \dots) \quad \text{and} \quad \|h_0\|_q \le C.$$

This last constant C depends only on p, on the Carleson norm of the measure (3.3), on $\inf_j y_j$ and $\sup_j y_j$ (recall the discussion following Theorem B). We have then

$$\begin{aligned} I_0 = 2\pi \left| \sum_j \bar{\lambda}_j h_0'(z_j) \right| &= \left| \int_{\mathbb{R}} \bar{f}' h_0 \, dx \right| \\ &\le C \sup \left\{ \left| \int_{\mathbb{R}} \bar{f}' h \, dx \right| : \, h \in H^q, \|h\|_q = 1 \right\}. \end{aligned}$$

The latter supremum is comparable to $\|f'\|_p$, while the one in (3.5) is comparable to $\|f\|_p$, so we conclude that

$$\|f\|_p \le \text{const} \cdot \|f'\|_p.$$

To prove the reverse inequality

$$\|f'\|_p \le \text{const} \cdot \|f\|_p,$$

we may either proceed in a similar fashion, using this time the inclusion $(H^q)'\big|_{\{z_j\}} \subset H^q\big|_{\{z_j\}}$ from Theorem B, or apply Theorem A instead. (It does apply because (iii.1) guarantees that $\theta' \in L^\infty(\mathbb{R})$; see Remark (2) following the statement of Theorem 1.1.) The two estimates actually hold on all of K_θ^p, since the rational functions with poles in $\{\bar{z}_j\}$ are dense therein.

Finally, in the case that θ has multiple zeros, one needs to introduce some routine changes to the argument above. In particular, the denominators in (3.4) should be raised to suitable powers, so as to allow for multiple poles, and the appropriate "higher order" version of Theorem B should be employed. We omit the details. $\square$

Remark. Alternatively, once the 'simple zeros' case is established, the remaining step towards the general case could be replaced by an approximation argument.

That would involve passing from a generic rational function $f \in K^p_\theta$ to a new rational function f_ε, with simple poles only, that satisfies $\|f - f_\varepsilon\|_p < \varepsilon$, $\|f' - f'_\varepsilon\|_p < \varepsilon$ and $\|f'_\varepsilon\|_p \asymp \|f_\varepsilon\|_p$. (A suitable perturbation of the original function's poles would do the job.) I owe this observation to the referee.

Proof of Proposition 1.2. This is similar to what we did in the final part of the preceding proof. In fact, the BMO-estimate is derived by duality from the $p = 1$ case of Theorem B, while the A^α-estimate corresponds to the range $0 < p < 1$ (in view of the duality relation $(H^p)^* = A^\alpha$, with $\alpha = p^{-1} - 1$; see [7]). $\square$

4. Proofs of Theorem 1.3 and Proposition 1.4

Proof of Theorem 1.3. (i.2) $\Longrightarrow$ (ii.2). Since $\frac{d}{dx} : K^p_\theta \to L^p(\mathbb{R})$ is bounded and left-invertible, we know from Theorem 1.1 that θ is a Blaschke product whose zeros, say $\{z_k\}$, satisfy

$$0 < c < y_k < C < \infty; \tag{4.1}$$

here $y_k := \Im z_k$ and c, C are suitable constants.

The functions

$$f_k(x) := (x - \bar{z}_k)^{-1} \qquad (k = 1, 2, \dots) \tag{4.2}$$

are in K^p_θ, and so (i.2) yields

$$T_{\bar\psi} f'_k = f_k. \tag{4.3}$$

In order to compute $T_{\bar\psi} f'_k$, we first note that

$$\overline{\psi(x)} f'_k(x) = -\frac{\overline{\psi(x)}}{(x - \bar{z}_k)^2} = -\overline{g_k(x)} - h_k(x), \qquad x \in \mathbb{R}, \tag{4.4}$$

where

$$g_k(x) := \frac{\psi(x) - \psi(z_k) - \psi'(z_k) \cdot (x - z_k)}{(x - z_k)^2}$$

and

$$h_k(x) := \frac{\overline{\psi(z_k)} + \overline{\psi'(z_k)} \cdot (x - \bar{z}_k)}{(x - \bar{z}_k)^2}.$$

Since $g_k \in H^p$ and $h_k \in H^p$, it follows from (4.4) that

$$T_{\bar\psi} f'_k = P_+ \left(\overline{\psi} f'_k\right) = -h_k = \overline{\psi(z_k)} \cdot f'_k - \overline{\psi'(z_k)} \cdot f_k.$$

By (4.3), we therefore have

$$\overline{\psi(z_k)} \cdot f'_k - \overline{\psi'(z_k)} \cdot f_k = f_k,$$

and this implies that

$$\psi(z_k) = 0, \qquad k = 1, 2, \dots, \tag{4.5}$$

and

$$\psi'(z_k) = -1, \qquad k = 1, 2, \dots \tag{4.6}$$

(because f_k and f'_k are linearly independent).

Our next step will be to check that θ has no multiple zeros. Assume to the contrary that θ has a zero of multiplicity ≥ 2 at z_k, for some k. The function

$$F_k(x) := (x - \bar{z}_k)^{-2}$$

is then in K^p_θ, and (i.2) tells us that

$$T_{\bar\psi} F'_k = F_k. \tag{4.7}$$

Proceeding in the same spirit as above, we find that

$$\overline{\psi(x)} F'_k(x) = -2 \frac{\overline{\psi(x)}}{(x - \bar{z}_k)^3} = -2 \left\{ \overline{G_k(x)} + H_k(x) \right\}, \qquad x \in \mathbb{R}, \tag{4.8}$$

where

$$G_k(x) := \frac{1}{(x - z_k)^3} \left\{ \psi(x) - \psi(z_k) - \psi'(z_k) \cdot (x - z_k) - \frac{1}{2} \psi''(z_k) \cdot (x - z_k)^2 \right\}$$

and

$$H_k(x) := \frac{1}{(x - \bar{z}_k)^3} \left\{ \overline{\psi(z_k)} + \overline{\psi'(z_k)} \cdot (x - \bar{z}_k) + \frac{1}{2} \overline{\psi''(z_k)} \cdot (x - \bar{z}_k)^2 \right\}.$$

Since G_k and H_k are both in H^p, it follows from (4.8) that

$$\begin{aligned} T_{\bar\psi} F'_k &= P_+ \left(\overline{\psi} F'_k \right) = -2 H_k \\ &= -2 \overline{\psi(z_k)} \cdot (x - \bar{z}_k)^{-3} - 2 \overline{\psi'(z_k)} \cdot (x - \bar{z}_k)^{-2} - \overline{\psi''(z_k)} \cdot (x - \bar{z}_k)^{-1}. \end{aligned}$$

By (4.7), the latter expression must be identical to $(x - \bar{z}_k)^{-2}$, which yields

$$\psi'(z_k) = -\frac{1}{2} \tag{4.9}$$

and

$$\psi(z_k) = \psi''(z_k) = 0.$$

And since (4.9) contradicts (4.6), we see that each z_k is necessarily a simple zero for θ, as claimed. Thus θ is a Blaschke product *with simple zeros.*

Recalling (4.5), we now deduce that $\psi = \theta \varphi$ for some $\varphi \in H^\infty$. Consequently, (4.6) takes the form

$$\theta'(z_k) \varphi(z_k) = -1, \qquad k = 1, 2, \dots,$$

whence

$$|\theta'(z_k)| \cdot \|\varphi\|_\infty \geq 1, \qquad k = 1, 2, \dots.$$

Combining this with (4.1), we finally conclude that

$$\inf_k y_k |\theta'(z_k)| > 0, \tag{4.10}$$

which means (see [14, p. 314]) that θ is an interpolating Blaschke product.

(ii.2) $\Longrightarrow$ (i.2). Now we have (4.10) and (4.1) at our disposal. (As before, we write $\{z_k\}$ for the zero sequence of θ, and y_k for $\Im z_k$.) First of all, these conditions imply $\theta' \in L^\infty(\mathbb{R})$, and so $\frac{d}{dx}$ maps K^p_θ boundedly into $L^p(\mathbb{R})$. Furthermore, the two conditions yield

$$\sup_k |\theta'(z_k)|^{-1} < \infty,$$

and we can solve the interpolation problem

$$\varphi(z_k) = -1/\theta'(z_k), \qquad k = 1, 2, \dots,$$

with a function $\varphi \in H^\infty$. Putting $\psi := \theta\varphi$, we arrive at (4.5) and (4.6), and hence also at (4.3), where f_k is again defined by (4.2). Indeed, the passage from (4.3) to (4.5) and (4.6) can be reversed.

Finally, since K^p_θ is spanned by the f_k's, (4.3) actually gives

$$T_{\bar\psi} f' = f \qquad \text{for all } f \in K^p_\theta, \tag{4.11}$$

and we are done. $\square$

Proof of Proposition 1.4. The estimate

$$\|f'\|_{p,w} \le \text{const} \cdot \|f\|_{p,w}, \qquad f \in K^p_\theta,$$

valid when $\theta' \in L^\infty(\mathbb{R})$ and $w \in (A_p)$, can be found in [8, Section 4]. To verify the reverse inequality

$$\|f\|_{p,w} \le \text{const} \cdot \|f'\|_{p,w},$$

one uses (4.11), with ψ as above, in conjunction with the fact that the Riesz projection P_+ (and hence the Toeplitz operator $T_{\bar\psi}$) acts boundedly on $L^p_w(\mathbb{R})$; see [13, Chapter 3] or [14, Chapter VI]. $\square$

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