Extension of bounded vector-valued functions

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In this paper we consider extensions of bounded vector-valued holomorphic (or harmonic or pluriharmonic) functions defined on subsets of an open set $\Omega \subset \mathbb{R}^N$. The results are based on the description of vector-valued functions as operators. As an application we prove a vector-valued version of Blaschke's theorem.

1 Introduction

If $\mathscr{H}(\Omega)$ is a function space on an open set $\Omega \subset \mathbb{R}^N$ and E is a Banach space (or, more generally, a locally complete locally convex Hausdorff space) one can use L. Schwartz' ε-product $Y\varepsilon E = L(Y'_{co}, E)$ – the space of all continuous linear operators from the dual Y' endowed with the topology of uniform convergence on all convex compact sets into E – to define a space of vector-valued functions

$$\mathscr{H}(\Omega, E) := \{x \mapsto T(\delta_x) : T \in \mathscr{H}(\Omega)\varepsilon E\}.$$

Here δ_x denotes the evaluation at $x \in \Omega$ and $\mathscr{H}(\Omega)$ is endowed with a natural topology.

This abstract definition, which is consistent with the usual definitions of vector-valued continuous, holomorphic, or harmonic functions (see [14, 15, 10]), is the key observation to study in a unified way properties and extension problems for vector-valued functions by using duality theory combined with compactness arguments.

This approach allowed J. Bonet and the two first named authors of the present article to give in [4] a unified treatment with improved results and very transparent proofs of questions considered e.g. by Horvath [9], Bogdanowicz [3], Große-Erdmann [8], Arendt-Nikolski [1], and Bierstedt and Holtmanns [2]. The general question can be described as follows:

> Let $A \subseteq \Omega$, $H \subseteq E'$, and $f : A \to E$ such that for every $u \in H$ the function $u \circ f : A \to \mathbb{C}$ has an extension in $\mathscr{H}(\Omega)$. When does this imply that there is an extension $F \in \mathscr{H}(\Omega, E)$ of f?

For $A = \Omega$ this includes questions about properties of vector-valued functions. For instance, the case $H = E'$ for the space of holomorphic functions describes Grothendieck's theorem about weakly holomorphic functions.

In this article we consider the corresponding question for *bounded* extensions in $\mathscr{H}^\infty(\Omega, E)$, i.e. the space of $f \in \mathscr{H}(\Omega, E)$ such that $f(\Omega) = T(\{\delta_x : x \in \Omega\})$ is bounded in E.

Let us now describe more precisely the setting of our work. We always assume that $\mathscr{H}(\Omega)$ is a space of complex valued infinitely differentiable functions on an open set $\Omega \subseteq \mathbb{R}^N$ which is closed in the space $\mathscr{C}(\Omega)$ of all continuous functions endowed with the Fréchet space topology of uniform convergence on all compact sets. This covers the cases of holomorphic, harmonic and pluriharmonic functions as well as, more generally, spaces of zero solutions of elliptic (or even hypoelliptic) partial differential operators.

The closed graph theorem implies that the topology of $\mathscr{C}(\Omega)$ coincides on $\mathscr{H}(\Omega)$ with that of $\mathscr{C}^\infty(\Omega)$ so that $\mathscr{H}(\Omega)$ is a Fréchet-Schwartz space by the Arzelà-Ascoli theorem.

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We denote by $\mathscr{H}^\infty(\Omega)$ the subspace of all bounded functions in $\mathscr{H}(\Omega)$. Endowed with the supremum norm this is a Banach space continuously embedded in $\mathscr{H}(\Omega)$ and its closed unit ball is compact there. For a locally complete locally convex Hausdorff space E we use the definitions $\mathscr{H}(\Omega, E)$ and $\mathscr{H}^\infty(\Omega, E)$ introduced above. Our notation for locally convex spaces is standard, we refer to [10, 11, 12, 13].

It is clear from the formulation of the general question that one has to balance requirements on $A \subseteq \Omega$ and $H \subseteq E'$. In the second section we consider small sets A and rather restrictive conditions on H, which will be relaxed in section 3.

2 Extensions from thin sets

$A \subset \Omega$ is called a *set of uniqueness* for $\mathscr{H}^\infty(\Omega)$ if each function $f \in \mathscr{H}^\infty(\Omega)$ which vanishes on A vanishes on the whole Ω. For the case of one variable holomorphic functions it is clearly enough to have an accumulation point in Ω. For the unit disc $\Omega = \mathbb{D}$ it is a classical result that a sequence $A = \{z_n : n \in \mathbb{N}\}$ is a set of uniqueness for all bounded holomorphic functions iff it satisfies the Blaschke condition $\sum_{n\in\mathbb{N}}(1 - |z_n|) = \infty$.

By standard duality arguments, A is a set of uniqueness if and only if the linear span of $\{\delta_x : x \in A\}$ is $\sigma(\mathscr{H}(\Omega)', \mathscr{H}^\infty(\Omega))$-dense in $\mathscr{H}(\Omega)$.

To formulate our first abstract bounded extension result we say that $H \subseteq E'$ determines boundedness if every subset B of E on which all elements of H are bounded is bounded in E. Obviously, the linear span of such sets is $\sigma(E', E)$-dense.

Proposition 2.1 *Let Z be a Banach space whose closed unit ball B_Z is a compact subset of a locally convex Hausdorff space Y. Let $X \subseteq Y'$ be a $\sigma(Y', Z)$-dense subspace, let E be a locally complete space and let $H \subseteq E'$ be a subspace determining boundedness. If $T : X \to E$ is a $\sigma(X, Z)$-$\sigma(E, H)$ continuous linear mapping, then there exists a unique extension $\hat{T} \in Y\varepsilon E$ such that $\hat{T}(B_Z^{\circ Y'})$ is bounded in E.*

Proof. We take all polars with respect to the dual system $\langle Y, Y'\rangle$ if nothing else is specified. Denote by $\hat{E}_H$ the completion of E for the $\sigma(E, H)$-topology. First we obtain an extension $\hat{T} : Y' \to \hat{E}_H$ which is $\sigma(Y', Z)$-$\sigma(\hat{E}_H, H)$ continuous. It satisfies that $\hat{T}(B_Z^\circ)$ is $\sigma(\hat{E}_H, H)$-bounded and $\hat{T}(X) \subset E$. Since the span of B_Z° is Y', we can find a Banach space V continuously embedded in $\hat{E}_H$ such that $\hat{T}(Y') \subset V$, $\hat{T}(B_Z^\circ)$ is bounded in V and $\hat{T}$ is $\sigma(Y', Z)$-$\sigma(V, H)$ continuous. Since B_Z° is a 0-neighbourhood in Y'_{co} and $\hat{T}(B_Z^\circ)$ is bounded in V, we have $\hat{T} \in Y\varepsilon V$, and, for each $v \in V'$, $v \circ \hat{T}$ belongs to the span of $B_Z^{\circ Y' \circ Y''} = B_Z$ because of the theorem of bipolars and the compactness of B_Z. Therefore $\hat{T}$ is $\sigma(Y', Z)$-$\sigma(V, V')$ continuous. Since H determines boundedness, $(E, \sigma(E, H))$ is locally complete, hence E is a locally closed subspace of $(\hat{E}_H, \sigma(\hat{E}_H, H))$. Thus, $E \cap V$ is closed in V for the norm topology and then $E \cap V$ is $\sigma(V, V')$-closed. Since X is $\sigma(Y', Z)$-dense and $\hat{T}(X) \subset E \cap V$ it follows that $\hat{T}(Y') \subset E$. Again, by the hypothesis on H, $B_V \cap E$ is bounded in E, so the topology on $E \cap V$ inherited from E is weaker than the norm-topology on $V \cap E$, hence the boundedness of $\hat{T}(B_Z^\circ)$ in V implies that $\hat{T}(B_Z^\circ)$ is bounded in E and this shows $\hat{T} \in Y\varepsilon E$.

The uniqueness also follows from the above arguments, since each continuous linear mapping $S : Y' \to E$ such that $S(B_Z^\circ)$ is bounded in E is $\sigma(Y', Z)$-$\sigma(E, E')$ continuous and hence such an S is determined by its restriction to X. □

Theorem 2.2 *Let A be a set of uniqueness for $\mathscr{H}^\infty(\Omega)$, let E be a locally complete space and let $H \subset E'$ be a subspace which determines boundedness in E. If $f : A \to E$ is a function such that $u \circ f$ admits an extension $g_u \in \mathscr{H}^\infty(\Omega)$ for each $u \in H$, then f admits a unique extension $g \in \mathscr{H}^\infty(\Omega, E)$.*

Proof. Take $Y := \mathscr{H}(\Omega)$, $Z := \mathscr{H}^\infty(\Omega)$, $X = span\{\delta_x : x \in A\}$ and $T : X \to E$, $\sum_{k=1}^n \lambda_k \delta_{x_k} \mapsto \sum_{k=1}^n \lambda_k f(x_k)$. In particular, H is separating, so that T is well-defined. Moreover, T satisfies the assumptions of Proposition 1 and hence there exists a unique continuous extension $\hat{T} \in Y\varepsilon E$ such that $\hat{T}(B_Z^{\circ Y'})$ is bounded in E. We set $g(x) := \hat{T}(\delta_x),\ x \in \Omega$. Then $g \in \mathscr{H}(\Omega, E)$ and $\{\delta_x : x \in \Omega\} \subset B_Z^\circ$ implies that g is bounded. □

As a corollary we obtain the following uniqueness result.

Corollary 2.3 *Let A be a set of uniqueness for $\mathscr{H}^\infty(\Omega)$, let E be a locally complete space and let $F \subset E$ be a locally closed subspace of E. If $f \in \mathscr{H}^\infty(\Omega, E)$ is a function such that $f(A) \subset F$, then $f(\Omega) \subset F$.*

P r o o f. The restriction of f to A clearly satisfies the hypothesis of Theorem 2.2 with range space F. Then there exists $g \in \mathscr{H}^\infty(\Omega, F)$ which agrees with f in A. Now, for each $u \in E'$, $u \circ f$ and $u \circ g$ are functions with belong to $\mathscr{H}^\infty(\Omega)$ and they coincide on A. This yields $u \circ f(z) = u \circ g(z)$ for each $z \in \Omega$ and for each $u \in E'$, hence $f = g$. $\square$

Corollary 2.4 *Let $A \subseteq \Omega$ be a set of uniqueness for $\mathscr{H}^\infty(\Omega)$, let E be a Fréchet space and let $(B_n)_n \subset E'$ be a sequence of bounded subsets such that $(B_n^\circ)_n$ is a fundamental system of zero neighbourhoods of E. If $f : A \to E$ is a function such that $u \circ f$ admits an extension $g_u \in \mathscr{H}^\infty(\Omega)$ for each $u \in \cup_n B_n$ and $\{g_u : u \in B_n\}$ is bounded in $\mathscr{H}^\infty(\Omega)$ for each $n \in \mathbb{N}$, then f admits an extension $g \in \mathscr{H}^\infty(\Omega, E)$.*

P r o o f. From the hypothesis it follows that for each $n \in \mathbb{N}$, $\alpha \in l_1$ and $(u_k)_{k \in \mathbb{N}} \in B_n$, $(\sum_{k \in \mathbb{N}} \alpha_k u_k) \circ f$ admits an extension in $\mathscr{H}^\infty(\Omega)$. The result is a consequence of Theorem 2 together with the fact that the span of these functionals determines boundedness in E (see [6, Proposition 7]). $\square$

If E is Banach and $B \subset E'$ is a bounded set such that B° defines an equivalent norm in E (i.e. if B° is bounded), B is called an *almost norming* subset. If $B \subset E'$ is *not* almost norming, as a consequence of [6, Proposition 5], if $\Omega \subset \mathbb{C}$ is a relatively compact domain then there exists a non-continuous function such that $u \circ f \in \mathscr{H}^\infty(\Omega)$ for each $u \in B$ and $\{u \circ f : u \in B\}$ is bounded in $\mathscr{H}^\infty(\Omega)$. The assumptions of Corollary 4 thus cannot be relaxed.

3 Extensions from fat sets

Now we study the problem of extending functions which admit extensions for functionals in a subspace H of E' which we only assume to be $\sigma(E', E)$-dense. In this case we will require that A (the set on which the function is defined) is quite large. This is symmetric with the problem studied by Gramsch [7], Große-Erdmann [8], and Bonet and the first two authors [4, 6] in the non-bounded case.

A set $A \subseteq \Omega$ is said to be *sampling* for $\mathscr{H}^\infty(\Omega)$ if there exists $C \geq 1$ such that

$$\sup_{z \in \Omega} |f(z)| \leq C \sup_{z \in A} |f(z)|$$

for each $f \in \mathscr{H}^\infty(\Omega)$.

For $M := \{\delta_x : x \in A\} \subset \mathscr{H}(\Omega)'$ this definition precisely states that the Minkowski gauge of M° defines an equivalent norm on $\mathscr{H}^\infty(\Omega)$.

In the one variable holomorphic case on the unit disc a theorem of Brown, Shields, and Zeller [5] states that a set A is sampling if and only if almost every boundary point is a non-tangential limit of a sequence contained in A.

Again we formulate our result first in terms of operators.

Proposition 3.1 *Let Z be a Banach space, whose closed unit ball B_Z is a compact subset of a locally convex Hausdorff space Y. Let $M \subset B_Z^{\circ Y'}$ such that $M^{\circ Z}$ is bounded in Z. If $T : spanM \to E$ is a linear map with values in a locally complete space E, bounded on M, such that there is a $\sigma(E', E)$-dense subspace H of E' with $u \circ T \in Z$ for all $u \in H$, then there is a unique extension $\hat{T} \in Y \varepsilon E$ of T such that $\hat{T}(B_Z^{\circ Y'})$ is bounded in E.*

P r o o f. We take all the polars in the dual system $\langle Y, Y' \rangle$. Without loss of generality we may assume that E is a Banach space and that M is absolutely convex. So, we equip $spanM$ with the seminorm induced by M. Since there is $C \geq 1$ such that $Z \cap M^\circ \subset CB_Z = CB_Z^{\circ\circ} \subset CM^\circ$, the linear map $Z \to spanM'$, $z \mapsto z|_{spanM}$, is injective, open onto its range and continuous, hence we may consider Z as a topological subspace of $spanM'$.
Let $W := \{u \in E' : u \circ T \in Z\} \supset H$ and let $(u_\iota)_{\iota \in I}$ be a net in $W \cap B_{E'}$ which $\sigma(E', E)$-converges to some $u \in B_{E'}$. Since $T(M)$ is bounded in E and since $Z \cap M^\circ$ bounded in Z, we obtain that there is $K \geq 1$ such that $u_\iota \circ T \subset KB_Z$ for all $\iota \in I$. Now KB_Z is $\sigma(Z, spanM)$-compact, hence there is a subnet $(u_{\tau(\kappa)} \circ T)_{\kappa \in J}$ converging pointwise on $spanM$ to some $v \in KB_Z$. On the other hand $(u_{\tau(\kappa)} \circ T)_{\kappa \in J}$ converges pointwise on $spanM$ to $u \circ T$, so $u \circ T \in Z$, i.e. $u \in W$. The Krein-Šmulyan theorem implies that W is $\sigma(E', E)$-closed, and since it is dense, we have $W = E'$. But this means that T is $\sigma(spanM, Z)$-$\sigma(E, E')$ continuous. Applying

Proposition 1 to $X := spanM$ (which is $\sigma(Y', Z)$-dense) and $\tilde{H} := E'$ we obtain an extension $\hat{T} \in Y\varepsilon E$ such that $\hat{T}(B_Z^\circ)$ is bounded in E.

Let $S \in Y\varepsilon E$ be another extension of T with $S(B_Z^\circ)$ bounded. Then $u \circ S \in B_Z^{\circ\circ} \subset Z$ for all $u \in E'$, hence S is also $\sigma(Y', Z)$-$\sigma(E, E')$ continuous and it coincides with $\hat{T}$ on the $\sigma(Y', Z)$-dense subspace $spanM$. This implies $S = \hat{T}$. □

Theorem 3.2 *Let A be a sampling set for $\mathscr{H}^\infty(\Omega)$, let E be a locally complete space and let H be a $\sigma(E', E)$-dense subspace of E'. If $f : A \to E$ is a bounded function such that $u \circ f$ admits an extension $g_u \in \mathscr{H}^\infty(\Omega)$ for each $u \in H$ then there exists a unique extension $g \in \mathscr{H}^\infty(\Omega, E)$ of f.*

Proof. This is a consequence of Proposition 3.1 applied to $Y = \mathscr{H}(\Omega)$, $Z = \mathscr{H}^\infty(\Omega)$, $M = \{\delta_z : z \in A\}$, and $T\colon \mathrm{span}M \to E$ defined by $T(\sum_{k=1}^n \lambda_k \delta_{z_k}) := \sum_{k=1}^n \lambda_k f(z_k)$. □

Proposition 3.1 also gives a Wolff type description of the dual of $\mathscr{H}(\Omega)$.

Theorem 3.3 *Let $(z_\nu)_{\nu\in\mathbb{N}} \subset \Omega$ be sampling for $\mathscr{H}^\infty(\Omega)$. Then there is $0 < \lambda \in \ell_1$ such that for every bounded $B \subset \mathscr{H}(\Omega)'$ there exists $C \geq 1$ with*

$$\{\mu|_{\mathscr{H}^\infty(\Omega)} : \mu \in B\} \subset \{\sum_{\nu=1}^\infty \alpha_\nu \delta_{z_\nu} \in \mathscr{H}^\infty(\Omega)' : |\alpha| \leq C\lambda\}.$$

Proof. We consider $M := \{\delta_{z_\nu} : \nu \in \mathbb{N}\} \subset \mathscr{H}(\Omega)'$, and we set $E := \{\sum_{\nu=1}^\infty \alpha_\nu \delta_{z_\nu} \in \mathscr{H}^\infty(\Omega)' : \alpha \in \ell_1\}$. Let $T : spanM \to E$ denote the restriction to $\mathscr{H}^\infty(\Omega)$. We apply Proposition 3.1 to obtain a continuous linear extension $\hat{T} : \mathscr{H}(\Omega)' \to E$ of T and $C \geq 1$ such that

$$\hat{T}(\{\mu : \|\mu|_{\mathscr{H}^\infty(\Omega)}\|_{\mathscr{H}^\infty(\Omega)'} \leq 1\}) \subset CB_E.$$

Obviously, we have $\hat{T}(\mu) = \mu|_{\mathscr{H}^\infty(\Omega)}$. Let B be an absolutely convex, closed, and bounded subset of $\mathscr{H}(\Omega)'$. Let$X := spanB$ be endowed with the Minkowski functional of B. The Fréchet space $\mathscr{H}(\Omega)$ is nuclear, hence there exist an absolutely convex, closed, and bounded subset S of $\mathscr{H}(\Omega)'$, $(x'_k)_{k\in\mathbb{N}} \subset B_{X'}$, $(\mu_k)_{k\in\mathbb{N}} \subset S$, and $0 \leq \gamma \in \ell_1$ such that

$$\mu = \sum_{k=1}^\infty \gamma_k x'_k(\mu)\mu_k,\ \mu \in B.$$

$\hat{T}(S)$ is bounded in E, hence there is a bounded sequence $(\beta^{(k)})_{k\in\mathbb{N}} \in \ell_1$ with

$$\mu_k|_{\mathscr{H}^\infty(\Omega)} = \sum_{\nu=1}^\infty \beta_\nu^{(k)} \delta_{z_\nu},\ k \in \mathbb{N}.$$

We set $\rho_\nu := \sum_{k=1}^\infty \gamma_k |\beta_\nu^{(k)}|,\ \nu \in \mathbb{N}$. Then $\rho := (\rho_\nu)_{\nu\in\mathbb{N}} \in \ell_1$ and for all $\mu \in B$ there is $|\alpha| \leq \rho$ with

$$\mu|_{\mathscr{H}^\infty(\Omega)} = \sum_{\nu=1}^\infty \alpha_\nu \delta_{z_\nu}. \tag{1}$$

Let now $(B_l)_l$ be a fundamental sequence of the bounded subsets of $\mathscr{H}(\Omega)'$. For each $l \in \mathbb{N}$ we obtain $0 \leq \rho^{(l)} \in \ell_1$ with (1). Choose $0 < \lambda \in \ell_1$ such that each $\rho^{(l)}$ is componentwise smaller than a multiple of λ. □

Corollary 3.4 *Let $(z_\nu)_{\nu\in\mathbb{N}} \subset \Omega$ be sampling for $\mathscr{H}^\infty(\Omega)$. Then there is a decreasing zero sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that for all compact $K \subset \Omega$ there is $C \geq 1$ with*

$$\sup_K |f| \leq C \sup_{\nu\in\mathbb{N}} |f(z_\nu)|\varepsilon_\nu,\ f \in \mathscr{H}^\infty(\Omega).$$

P r o o f. We choose $(\lambda_\nu)_{\nu\in\mathbb{N}} \in \ell_1$ according to the previous Theorem and a decreasing zero sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that $(\frac{\lambda_\nu}{\varepsilon_\nu})_{\nu\in\mathbb{N}}$ is still in ℓ_1. If $K \subset \Omega$ is compact, there is a bounded $B \subset \mathscr{H}(\Omega)'$ with $\hat{T}(B)^\circ \subset \{f \in \mathscr{H}^\infty(\Omega) : \sup_K |f| \leq 1\}$. Hence there is $C \geq 1$ such that

$$\begin{aligned}\sup_K |f| &\leq \sup_{\mu\in B} | < \hat{T}(\mu), f > | \leq C \sup_{|\alpha|\leq\lambda} |\sum_{\nu=1}^{\infty} \alpha_\nu f(z_\nu)| \\ &\leq C\|(\frac{\lambda_\nu}{\varepsilon_\nu})_\nu\|_1 \sup_{\nu\in\mathbb{N}} |f(z_\nu)|\varepsilon_\nu\end{aligned}$$

for all $f \in \mathscr{H}^\infty(\Omega)$. □

Remark 3.5 If we assume, in addition, that $(z_\nu)_{\nu\geq n}$ is sampling for all $n \in \mathbb{N}$ then we obtain that there is a decreasing zero sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that for all $K \subset \Omega$ compact and $n \in \mathbb{N}$ there is $C \geq 1$ with

$$\sup_K |f| \leq C \sup_{\nu\geq n} |f(z_\nu)|\varepsilon_\nu,\ f \in \mathscr{H}^\infty(\Omega).$$

Indeed, applying the previous corollary to each member K_n of a fundamental sequence of compacts $(K_n)_{n\in\mathbb{N}}$ and $(z_\nu)_{\nu\geq n}$ we obtain decreasing zero sequences $(\varepsilon_\nu^{(n)})_{\nu\geq n}$. Choose now a decreasing zero sequence $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that each $(\varepsilon_\nu^{(n)})_{\nu\geq n}$ is componentwise smaller then a multiple of $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$.

4 A vector-valued Blaschke theorem

As an application of our results we want to prove a vector-valued analogue of the classical Blaschke theorem in the spirit of [1, 2.4, 2.5]. The Blaschke theorem asserts that if $(z_k)_{k\in\mathbb{N}}$ is a sequence in the unit disc $\mathbb{D} \subset \mathbb{C}$ such that $\sum_{k=1}^\infty (1 - |z_k|) = \infty$ and if $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded holomorphic functions on $\mathbb{D}$ such that $(f_n(z_k))_n$ converges for each $k \in \mathbb{N}$, then there exists a holomorphic function f, bounded on $\mathbb{D}$, such that $(f_n)_{n\in\mathbb{N}}$ converges to f for the compact open topology. In this section we prove a vector-valued version of this theorem also valid in the general situation considered here.

To prove the next proposition we need the following observation of Arendt and Nikolski [1]: *For a Banach space E and a directed index set I let*

$$\ell_\infty(I, E) := \{(y_\iota)_{\iota\in I} \in E^I : \sup_{\iota\in I} \|(y_\iota)\| < \infty\}.$$

Then $\ell_\infty(I, E)$ is a Banach space and the subspace

$$c(I, E) := \{(y_\iota)_{\iota\in I} \in \ell_\infty(I, E) : \lim_\iota y_\iota \textit{ exists } \}$$

is closed. Moreover,

$$\ell_1(I, E') := \{(u_\iota)_{\iota\in I} \in E'^I : \sum_{\iota\in I} \|u_\iota\|' < \infty\} \subset \ell_\infty(I, E)'$$

determines boundedness in $\ell_\infty(I, E)$.

Proposition 4.1 *Let Z be a Banach space, whose closed unit ball B_Z is a compact subset of a locally convex Hausdorff space Y, let E be a Banach space, and let I be a directed set and $(T_\iota)_{\iota\in I} \subset Y\varepsilon E$ a net such that*

$$\sup_{\iota\in I}\{\|T_\iota(y)\| : y \in B_Z^{\circ Y'}\} < \infty.$$

Assume further that there exists a $\sigma(Y', Z)$-dense subspace $X \subset Y'$ such that $\lim_\iota T_\iota(x)$ exists for each $x \in X$. Then there is $T \in Y\varepsilon E$ with $T(B_Z^{\circ Y'})$ bounded and $\lim_\iota T_\iota = T$ uniformly on the equicontinuous subsets of Y', i.e. for all equicontinuous $B \subset Y'$ and $\varepsilon > 0$ there exists $\kappa \in I$ such that

$$\sup_{y\in B} \|T_\iota(y) - T(y)\| < \varepsilon$$

for each $\iota \geq \kappa$.

P r o o f. The map

$$S : Y' \to \ell_\infty(I, E),\ y \mapsto (T_\iota(y))_{\iota \in I},$$

belongs to $Y \varepsilon \ell_\infty(I, E)$ since, by assumption, it maps $B_Z^{\circ Y'}$ into a bounded subset of $\ell_\infty(I, E)$. If $(u_\iota)_{\iota \in I} \in \ell_1(I, E')$ then $\sum_{\iota \in I} u_\iota \circ T_\iota \in B_Z^{\circ\circ} = B_Z \subset Z$, hence S is also $\sigma(Y', Z)$-$\sigma(l_\infty(I, E), \ell_1(I, E'))$ continuous. This shows that $S|_X : X \to c(I, E)$ is $\sigma(X, Z)$-$\sigma(c(I, E), \ell_1(I, E'))$ continuous. We can therefore apply Proposition 1 to obtain an extension $\widehat{S|_X} \in Y \varepsilon\, c(I, E)$ of $S|_X$ with $\widehat{S|_X}(B_Z^{\circ Y'})$ bounded. The uniqueness part of Proposition 1 shows $S = \widehat{S|_X}$ and we have, in particular, $S(Y') \subset c(I, E)$. Since the map $R : c(I, E) \to E$, $(y_\iota)_{\iota \in I} \mapsto \lim_\iota y_\iota$ is linear and continuous we obtain that $T := R \circ S \in Y \varepsilon E$ and $\lim_\iota T_\iota = T$ pointwise on Y'.

The statement about the uniform convergence is obtained from the fact the equicontinuous subsets of Y' are precompact with respect to the seminorm with unit ball B_Z° which follows from the theorem of bipolars (or an appropriate version of Schauder's theorem about the transposed of compact operators, see e.g. [12, 15.3]). □

We now obtain the above mentioned generalization of Blaschke's theorem. The one variable holomorphic case is due to Arendt and Nikolski [1, 2.5].

Corollary 4.2 *Let E be a Banach space and let $(f_\iota)_{\iota \in I} \subset \mathscr{H}^\infty(\Omega, E)$ be a bounded net. If $A \subset \Omega$ is a set of uniqueness for $\mathscr{H}^\infty(\Omega)$, and if $\lim_\iota f_\iota(z)$ exists for all $z \in A$, then there is $f \in \mathscr{H}^\infty(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges locally uniformly to f.*

P r o o f. Take $Y := \mathscr{H}(\Omega)$, $Z := \mathscr{H}^\infty(\Omega)$, $X := \operatorname{span}\{\delta_x : x \in A\}$, and $T_\iota : X \to E$ defined by $T_\iota(\delta_x) := f_\iota(x)$, $x \in A$. Since the closed absolutely convex hull of $\{\delta_x : x \in A\}$ is $B_Z^{\circ Y'}$ we may apply Proposition 4.1 and we get $T \in Y \varepsilon E$ with $T(B_Z^{\circ Y'})$ bounded and $\lim_\iota T_\iota = T$ uniformly on the equicontinuous subsets of Y'. If we set $f(x) := T(\delta_x)$, $x \in \Omega$, then $f \in \mathscr{H}^\infty(\Omega, E)$ and $\lim_\iota \sup_{x \in K} \|f_\iota(x) - f(x)\| = \lim_\iota \sup_{x \in K} \|T_\iota(\delta_x) - T(\delta_x)\| = 0$ for all compact subsets $K \subset \Omega$. □

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