# GROMOV HYPERBOLICITY OF DENJOY DOMAINS WITH HYPERBOLIC AND QUASIHYPERBOLIC METRICS

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Abstract. We obtain explicit and simple conditions which in many cases allow one decide, whether or not a Denjoy domain endowed with the Poincaré or quasihyperbolic metric is Gromov hyperbolic. The criteria are based on the Euclidean size of the complement. As a corollary, the main theorem allows to deduce the non-hyperbolicity of any periodic Denjoy domain.

## 1. Introduction

In the 1980s Mikhail Gromov introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors. Initially, the research was mainly centered on hyperbolic group theory, but lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory.

One of the primary questions is naturally whether a metric space $(X, d)$ is hyperbolic in the sense of Gromov or not. The most classical examples, mentioned in every textbook on this topic, are metric trees, the classical Poincaré hyperbolic metric developed in the unit disk and, more generally, simply connected complete Riemannian manifolds with sectional curvature $K \leqslant -k^2 < 0$.

However, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein-Hilbert metric (see [7, 13]) is Gromov hyperbolic (under particular conditions on the domain of definition); that the Gehring-Osgood $j$-metric (see [12]) is Gromov hyperbolic; and that the Vuorinen $j$-metric (see [12]) is not Gromov hyperbolic except in the punctured space. Also, in [14] the hyperbolicity of the conformal modulus metric $\mu$ and the related so-called Ferrand metric $\lambda^*$, is studied.

Since the Poincaré metric is also the metric giving rise to what is commonly known as the hyperbolic metric when speaking about open domains in the complex plane or in Riemann surfaces, it could be expected that there is a connection between the notions of hyperbolicity. For simply connected subdomains $\Omega$ of the complex plane, it follows directly from the Riemann mapping theorem that the metric space $(\Omega, h_\Omega)$ is in fact

*Date*: October 30, 2007.

2000 *Mathematics Subject Classification.* 30F45; 53C23, 30C99.

*Key words and phrases.* Poincaré metric, hyperbolic metric, quasihyperbolic metric, Gromov hyperbolic, Denjoy domain.

* Supported in part by the Academy of Finland.

† Supported in part by three grants from M.E.C. (MTM 2006-13000-C03-02, MTM 2006-11976 and MTM 2006-26627-E), Spain.

‡ Supported in part by a grant from U.C.III M./C.A.M. (CCG06-UC3M/EST-0690), Spain.

Gromov hyperbolic. However, as soon as simple connectedness is omitted, there is no immediate answer to whether the space $h_\Omega$ is hyperbolic or not. The question has lately been studied in [2] and [18]–[24].

The related quasihyperbolic metric has also recently been a topic of interest regarding the question of Gromov hyperbolicity. In [8], Bonk, Heinonen and Koskela found necessary and sufficient conditions for when a planar domain $D$ endowed with the quasihyperbolic metric is Gromov hyperbolic. This was extended by Balogh and Buckley, [4]: they found two different necessary and sufficient conditions which work in Euclidean spaces of all dimensions and also in metric spaces under some conditions.

In this article we are interested in Denjoy domains. In this case either the result of [8] or [4] implies that the domain is Gromov hyperbolic with respect to the quasihyperbolic metric if and only if the domain is inner uniform (see Section 3). Although this is a vrey nice characterization, it is somewhat difficult to check that a domain is inner uniform, since we need to construct uniform paths connecting every pair of points.

In this paper we show that it is necessary to look at paths joining only a very small (countable) number of points when we want to determine the Gromov hyperbolicity. This allows us to derive a simple and very concrete conditions on when the domain is Gromov hyperbolic. Much more importantly, our methods also suggest corresponding results for the hyperbolic metric, which are also proven. To the best of our knowledge, this is the first time that Gromov hyperbolicity of any class of infinitely connected domains has been obtained from conditions on the Euclidean size of the complement of the domain.

The main results in this article are the following:

**Theorem 1.1.** *Let $\Omega$ be a Denjoy domain with $\Omega \cap \mathbb{R} = (-\infty, 0) \cup \bigcup_{n=1}^{\infty}(a_n, b_n)$, $b_n \leqslant a_{n+1}$ for every $n$, and $\lim_{n\to\infty} a_n = \infty$.*

(1) *The metrics $k_\Omega$ and $h_\Omega$ are Gromov hyperbolic if*

$$\liminf_{n\to\infty} \frac{b_n - a_n}{a_n} > 0.$$

(2) *The metrics $k_\Omega$ and $h_\Omega$ are not Gromov hyperbolic if*

$$\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = 0.$$

It is interesting to note that in the case of Denjoy domains many of the results seem to hold for both the hyperbolic and the quasihyperbolic metrics. In fact, we know of no planar domain which is Gromov hyperbolic with respect to one of these metrics, but not the other.

In the previous theorems, the boundary components had a single accumulation point, at $\infty$, and the accumulation happened only from one side. It turns out that if this kind of domain is not Gromov hyperbolic, then we cannot mend the situation by adding some boundary to the other side of the accumulation point, as the following theorem shows.

**Theorem 1.2.** *Let $\Omega$ be a Denjoy domain with $(-\infty, 0) \subset \Omega$ and let $F \subseteq (-\infty, 0]$ be closed. If $k_\Omega$ is not Gromov hyperbolic, then neither is $k_{\Omega\setminus F}$; if $h_\Omega$ is not Gromov hyperbolic, then neither is $h_{\Omega\setminus F}$.*

We also prove the non-hyperbolicity of any periodic Denjoy domain:

**Corollary 1.3.** *Let $E_0 \subset [0,t)$ be closed, $t>0$, set $E_n := E_0 + tn$ for $n \in \mathbb{N}$ or $n \in \mathbb{Z}$, and $\Omega := \mathbb{C} \setminus \cup_n E_n$. Then $h_\Omega$ and $k_\Omega$ are not Gromov hyperbolic.*

## 2. Definitions and notation

By $\mathbf{H}^2$ we denote the upper half plane, $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. For $D \subset \mathbb{C}$ we denote by $\partial D$ and $\overline{D}$ its boundary and closure, respectively. For $z \in D \subsetneq \mathbb{C}$ we denote by $\delta_D(z)$ the distance to the boundary, $\min_{a \in \partial D} |z-a|$. Finally, we denote by $c$, $C$, $c_j$ and $C_j$ generic constants which can change their value from line to line and even in the same line.

Recall that a domain $\Omega \subset \mathbb{C}$ is said to be of *hyperbolic type* if it has at least two finite boundary points. The universal cover of such domain is the unit disk $\mathbb{D}$. In $\Omega$ we can define the Poincaré metric, i.e. the metric obtained by pulling back the metric $ds = 2|dz|/(1-|z|^2)$ of the unit disk. Equivalently, we can pull back the metric $ds = |dz|/\operatorname{Im} z$ of the the upper half plane $\mathbf{H}^2$. Therefore, any simply connected subset of $\Omega$ is isometric to a subset of $\mathbb{D}$. With this metric, $\Omega$ is a geodesically complete Riemannian manifold with constant curvature $-1$, in particular, $\Omega$ is a geodesic metric space. The Poincaré metric is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

The quasihyperbolic metric is the distance induced by the density $1/\delta_\Omega(z)$. By $\lambda_\Omega$ we denote the density of the Poincaré metric in $\Omega$, and by $k_\Omega$ and $h_\Omega$ the quasihyperbolic and Poincaré distance in $\Omega$, respectively. Length (of a curve) will be denoted by the symbol $\ell_{d,\Omega}$, where $d$ is the metric with respect to which length is measured. If it is clear which metric or domain is used, either one or both subscripts in $\ell_{d,\Omega}$ might be left out. The subscript Eucl is used to denote the length with respect to the Euclidean metric. Also, as most of the proofs apply to both the quasihyperbolic and the Poincaré metrics, we will use the symbol $\kappa$ also as a "dummy metric" symbol, where it can be replaced by either $k$ or $h$.

We denote by $\lambda_\Omega$ the density of the hyperbolic metric in $\Omega$. It is well known that for every domain $\Omega$

$$\lambda_\Omega(z) \leqslant \frac{2}{\delta_\Omega(z)} \quad \forall\, z \in \Omega, \qquad \ell_{h,\Omega}(\gamma) \leqslant 2\ell_{k,\Omega}(\gamma) \quad \forall\, \gamma \subset \Omega,$$

and that for all domains $\Omega_1 \subset \Omega_2$ we have $\lambda_{\Omega_1}(z) \geqslant \lambda_{\Omega_2}(z)$ for every $z \in \Omega_1$.

If $\Omega_0$ is an open subset of $\Omega$, in $\Omega_0$ we always consider its usual quasihyperbolic or Poincaré metric (independent of $\Omega$). If $D$ is a closed subset of $\Omega$, we always consider in $D$ the inner metric obtained by the restriction of the quasihyperbolic or Poincaré metric in $\Omega$, that is

$$d_{\Omega|D}(z,w) := \inf \big\{ \ell_{\kappa,\Omega}(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w \big\} \geqslant d_\Omega(z,w)\,.$$

It is clear that $\ell_{\Omega|D}(\gamma) = \ell_\Omega(\gamma)$ for every curve $\gamma \subset D$. We always require that $\partial D$ is a union of pairwise disjoint Lipschitz curves; this fact guarantees that $(D, d_{\Omega|D})$ is a geodesic metric space.

A geodesic metric space $(X,d)$ is said to be *Gromov $\delta$-hyperbolic*, if

$$d(w, [x,z] \cup [z,y]) \leqslant \delta$$

for all $x, y, z \in X$; corresponding geodesic segments $[x, y], [y, z]$ and $[x, z]$; and $w \in [x, y]$. If this inequality holds, we also say that the geodesic triangle is *$\delta$-thin*, so Gromov hyperbolicity can be reformulated by requiring that all geodesic triangles are thin.

A *Denjoy domain* $\Omega \subset \mathbb{C}$ is a domain whose boundary is contained in the real axis. Hence, it satisfies $\Omega \cap \mathbb{R} = \cup_{n\in\Lambda}(a_n, b_n)$, where $\Lambda$ is a countable index set, $\{(a_n, b_n)\}_{n\in\Lambda}$ are pairwise disjoint, and it is possible to have $a_{n_1} = -\infty$ for some $n_1 \in \Lambda$ and/or $b_{n_2} = \infty$ for some $n_2 \in \Lambda$.

In order to study Gromov hyperbolicity, we consider the case where $\Lambda$ is countably infinite, since if $\Lambda$ is finite then $h_\Omega$ and $k_\Omega$ are easily seen to be Gromov hyperbolic by Proposition 3.5, below.

## 3. Some classes of Denjoy domains which are Gromov hyperbolic

The quasihyperbolic metric is traditionally defined in subdomains of Euclidean $n$-space $\mathbb{R}^n$, i.e. open and connected subsets $\Omega \subsetneq \mathbb{R}^n$. However, a more abstract setting is also possible, as was shown in the article [8] by Bonk, Heinonen and Koskela. There it is shown that if $(X, d)$ is taken to be any metric space which is locally compact, rectifiably connected and noncomplete, the quasihyperbolic metric $k_X$ can be defined as usual, using the weight $1/\operatorname{dist}(x, \partial X)$.

Given a real number $A \geqslant 1$, a curve $\gamma\colon [0, 1] \to \Omega$ is called *$A$-uniform for the metric $d$* if

$$\begin{aligned} &\ell_d(\gamma) \leqslant A\, d(\gamma(0), \gamma(1)) \quad \text{and} \\ &\min\{\ell_d(\gamma|[0, t]), \ell_d(\gamma|[t, 1])\} \leqslant A\ \operatorname{dist}_d(\gamma(t), \partial\Omega), \qquad \text{for all } t \in [0, 1]. \end{aligned}$$

Moreover, a locally compact, rectifiably connected noncomplete metric space is said to be *$A$-uniform* if every pair of points can be joined by an $A$-uniform curve. The abbreviations "$A$-uniform" and "$A$-inner uniform" (without mention of the metric) mean $A$-uniform for the Euclidean metric and Euclidean inner metric, respectively.

Uniform domains are intimately connected to domains which are Gromov hyperbolic with respect to the quasihyperbolic metric (see [8, Theorems 1.12, 11.3]). Specifically, for a Denjoy domain $\Omega$ these results imply that $k_\Omega$ is Gromov hyperbolic if and only if $\Omega$ is inner uniform.

Here we will use the generalized setting in [8] to show that for Denjoy domains it actually suffices to consider the upper (or lower) intersection with the actual domain, as can be done for the Poincaré metric:

**Lemma 3.1.** *Let $\emptyset \neq E \subset \mathbb{R}$ be a closed set, and denote $D_0 = \mathbb{C} \setminus E$ and $D = D_0 \cap \{z \in \mathbb{C} \mid \operatorname{Im} z \geqslant 0\} = D_0 \cap \overline{\mathbf{H}^2}$. Then the metric space $D$, with the restriction of the Poincaré or the quasihyperbolic metric in $D_0$, is $\delta$-Gromov hyperbolic, with some universal constant $\delta$.*

*Proof.* We deal first with the quasihyperbolic metric. As the upper half-plane is uniform in the classical case, the same curve of uniformity (which is an arc of a circle orthogonal to $\mathbb{R}$) can be shown to be an $A$-uniform curve in the sense of [8] for the set $D$. Hence $D$ is $A$-uniform. By [8, Theorem 3.6] it then follows that the space $(D, k_D)$ is Gromov hyperbolic.

We also have that $D$ is hyperbolic with the restriction of the Poincaré metric $h_{D_0}$, since it is isometric to a geodesically convex subset of the unit disk (in fact, there is just one geodesic in $D$ joining two points in $D$). Therefore, $D$ has $\log\big(1+\sqrt{2}\,\big)$-thin triangles, as does the unit disk (see, e.g. [3, p. 130]). □

**Definition 3.2.** Let $\Omega$ be a Denjoy domain. Then we have $\Omega \cap \mathbb{R} = \cup_{n\geqslant 0}(a_n, b_n)$ for some suitable intervals. We say that a curve in $\Omega$ is a *fundamental geodesic* if it is a geodesic joining $(a_0, b_0)$ and $(a_n, b_n)$, $n > 0$, which is contained in the closed halfplane $\overline{\mathbf{H}^2} = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$. We denote by $\gamma_n$ a fundamental geodesic corresponding to $n$.

The next result was proven for the hyperbolic metric in [2, Theorem 5.1]. In view of Lemma 3.1 one can check that the same proof carries over to the quasihyperbolic metric.

By a *bigon* we mean a closed polygon with two edges.

**Theorem 3.3.** *Let $\Omega$ be a Denjoy domain and denote by $\kappa_\Omega$ the Poincaré or quasihyperbolic metric. Then the following conditions are equivalent:*

1. *$\kappa_\Omega$ is $\delta$-hyperbolic.*
2. *There exists a constant $c_1$ such that for every choice of fundamental geodesics $\{\gamma_n\}_{n=1}^\infty$ we have $\kappa_\Omega(z, \mathbb{R}) \leqslant c_1$ for every $z \in \cup_{n\geqslant 1}\gamma_n$.*
3. *There exists a constant $c_2$ such that for a fixed choice of fundamental geodesics $\{\gamma_n\}_{n=1}^\infty$ we have $\kappa_\Omega(z, \mathbb{R}) \leqslant c_2$ for every $z \in \cup_{n\geqslant 1}\gamma_n$.*
4. *There exists a constant $c_3$ such that every geodesic bigon in $\Omega$ with vertices in $\mathbb{R}$ is $c_3$-thin.*

*Furthermore, the constants in each condition only depend on the constants appearing in any other of the conditions.*

Note that the case $\Omega \cap \mathbb{R} = \cup_{n=0}^N (a_n, b_n)$ is also covered by the theorem.

**Corollary 3.4.** *Let $\Omega$ be a Denjoy domain and denote by $\kappa_\Omega$ the Poincaré or quasihyperbolic metric. If there exist a constant $C$ and a sequence of fundamental geodesics $\{\gamma_n\}_{n\geqslant 1}$ with $\ell_{\kappa,\Omega}(\gamma_n) \leqslant C$, then $\kappa_\Omega$ is $\delta$-Gromov hyperbolic, and $\delta$ just depends on $C$.*

If $\Omega$ has only finitely many boundary components, then it is always Gromov hyperbolic, in a quantitative way:

**Proposition 3.5.** *Let $\Omega$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=1}^N (a_n, b_n)$, and denote by $\kappa_\Omega$ the Poincaré or quasihyperbolic metric. Then $\kappa_\Omega$ is $\delta$-Gromov hyperbolic, where $\delta$ is a constant which only depends on $N$ and $c_0 = \sup_n \kappa_\Omega\big((a_n, b_n), (a_{n+1}, b_{n+1})\big)$.*

Note that we do not require $b_n \leqslant a_{n+1}$.

*Proof.* Let us consider the shortest geodesics $g_n^*$ joining $(a_n, b_n)$ and $(a_{n+1}, b_{n+1})$ in $\Omega^+ := \Omega \cap \overline{\mathbf{H}^2}$. Then $\ell_\Omega(g_n^*) \leqslant \ell_\Omega(g_n) \leqslant c_0$ for $0 \leqslant n \leqslant N-1$.

By Theorem 3.3, we just need to prove that there exists a constant $c$, which only depends on $c_0$ and $N$, such that $\kappa_\Omega(z, \mathbb{R}) \leqslant c$ for every $z \in \cup_{n=1}^N \gamma_n$.

For each $0 \leqslant n \leqslant N-1$, let us consider the geodesic polygon $P$ in $\Omega^+$, with the following sides: $\gamma_n, g_0^*, \dots, g_{n-1}^*$, and the geodesics joining their endpoints which are contained in $(a_0, b_0), \dots, (a_n, b_n)$. Since $(\Omega^+, \kappa_\Omega)$ is $\delta_0$-Gromov hyperbolic, where $\delta_0$ is a constant which only depends on $c_0$, by Lemma 3.1, and $P$ is a geodesic polygon in $\Omega^+$ with at most $2N+2$ sides, $P$ is $2N\delta_0$-thin. Therefore, given any $z \in \gamma_n$, there exists a point $w \in \cup_{k=0}^{N-1} g_k^* \cup \mathbb{R}$ with $\kappa_\Omega(z, w) \leqslant 2N\delta_0$. Since $\ell_\Omega(g_k^*) \leqslant c_0$ for $0 \leqslant k \leqslant N-1$, there exists $x \in \mathbb{R}$ with $\kappa_\Omega(x,w) \leqslant c_0/2$. Hence, $\kappa_\Omega(z,\mathbb{R}) \leqslant \kappa_\Omega(z,x) \leqslant 2N\delta_0 + c_0/2$, and we conclude that $\kappa_\Omega$ is $\delta$-Gromov hyperbolic. □

**Theorem 3.6.** *Let $\Omega$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n,b_n)$, $(a_0,b_0) = (-\infty,0)$ and $b_n \leqslant a_{n+1}$ for every $n$. Suppose that $b_n \geqslant K a_n$ for a fixed $K > 1$ and every $n$. Then $h_\Omega$ and $k_\Omega$ are $\delta$-Gromov hyperbolic, with $\delta$ depending only on $K$.*

*Proof.* Fix $n$ and consider the domain

$$\Omega_n = \frac{1}{a_n}\,\Omega = \left\{ \frac{x}{a_n} \mid x \in \Omega \right\}.$$

If we define $D := \mathbb{C} \setminus [0,1] \cup [K,\infty)$, then $D \subset \Omega_n$, and $\ell_{k,\Omega_n}(\gamma) \leqslant \ell_{k,D}(\gamma)$ for every curve $\gamma \subset \Omega_n$. The circle $\sigma := S^1(0,(1+K)/2)$ goes around the boundary component $[0,1]$ in $D$ and has finite quasihyperbolic length:

$$\ell_{k,D}(\sigma) \leqslant \int_\sigma \frac{|dz|}{(K-1)/2} = 2\pi\,\frac{K+1}{K-1}\,.$$

Consider the shortest fundamental geodesics joining $(a_0,b_0)$ with $(a_n,b_n)$, with the Poincaré and the quasihyperbolic metrics, $\gamma_n^h$ and $\gamma_n^k$, respectively. Then,

$$\ell_{k,\Omega}(\gamma_n^k) = \ell_{k,\Omega_n}\Big(\frac{1}{a_n}\gamma_n^k\Big) \leqslant \ell_{k,\Omega_n}(\sigma) \leqslant \ell_{k,D}(\sigma) \leqslant 2\pi\,\frac{K+1}{K-1}\,,$$
$$\ell_{h,\Omega}(\gamma_n^h) \leqslant \ell_{h,\Omega}(\gamma_n^k) \leqslant 2\,\ell_{k,\Omega}(\gamma_n^k) \leqslant 4\pi\,\frac{K+1}{K-1}\,.$$

Therefore $h_\Omega$ and $k_\Omega$ are $\delta$-Gromov hyperbolic (and $\delta$ depends only on $K$), by Corollary 3.4. □

*Proof of Theorems 1.1(1).* If $\liminf_{n\to\infty}(b_n-a_n)/a_n > 0$, then we can choose $K > 1$ so that $(b_n-a_n)/a_n > K-1$ for every $n$, whence $b_n > K a_n$. Thus the previous theorem implies the claims. □

## 4. Some classes of Denjoy domains which are not Gromov hyperbolic

The following function was introduced by Beardon and Pommerenke [6].

**Definition 4.1.** For $\Omega \subsetneq \mathbb{C}$, define $\beta_\Omega(z)$ as the function

$$\beta_\Omega(z) := \inf\left\{ \left| \log\left|\frac{z-a}{b-a}\right| \right| :\ a,b \in \partial\Omega,\ |z-a| = \delta_\Omega(z) \right\}.$$

The function $\beta_\Omega$ has a geometric interpretation. We say that an annulus $\{z \in \mathbb{C} : r < |z-a| < R\}$ *separates* $E \subset \mathbb{C}$ if $\{z \in \mathbb{C} : r < |z-a| < R\} \cap E = \emptyset$, $\{z \in \mathbb{C} : |z-a| \leqslant r\} \cap E \neq \emptyset$ and $\{z \in \mathbb{C} : |z-a| \geqslant R\} \cap E \neq \emptyset$. We say that $E$ is *uniformly perfect* if there exists a constant $c_1$ such that $R/r \leqslant c_1$ for every annulus $\{z \in \mathbb{C} : r < |z-a| < R\}$ separating $E$ (see [6, 16, 17]). Now we see that $\beta_\Omega$ is bounded precisely when $\Omega$ is uniformly perfect.

Thus it follows from the next theorem, that $\lambda_\Omega$ and $1/\delta_\Omega$ are comparable if and only if $\Omega$ is uniformly perfect.

**Theorem 4.2** (Theorem 1, [6])**.** *For every domain $\Omega \subset \mathbb{C}$ of hyperbolic type and for every $z \in \Omega$, we have that*

$$2^{-3/2} \leqslant \lambda_\Omega(z)\,\delta_\Omega(z)\,(k_0 + \beta_\Omega(z)) \leqslant \pi/4\,,$$

*where $k_0 = 4 + \log(3 + 2\sqrt{2})$.*

**Lemma 4.3.** *Let $\gamma$ be a curve in a domain $D \subset \mathbb{R}^n$ from $a \in D$ with Euclidean length $s$. Then:*

(1) $\ell_{k,D}(\gamma) \geqslant \log\left(1 + \frac{s}{d_D(a)}\right)$.

(2) *If $D$ is a Denjoy domain and $a \in (a_n, b_n)$, with $b_n - a_n \leqslant r$, then $\ell_{h,D}(\gamma) \geqslant 2^{-3/2}\log\left(1 + k_0^{-1}\log\left(1+\frac{s}{r}\right)\right)$, with $k_0$ as in Theorem 4.2.*

*Proof.* Let $z \in \partial D$ be a point with $\delta_D(a) = |a - z|$. Without loss of generality we assume that $z = 0$. By monotonicity $\ell_{k,D}(\gamma) \geqslant \ell_{k,\mathbb{R}^n\setminus\{0\}}(\gamma)$. Further, it is clear that $\ell_{k,\mathbb{R}^n\setminus\{0\}}(\gamma) \geqslant \ell_{k,\mathbb{R}^n\setminus\{0\}}([|a|, |a|+s])$, whence the first estimate by integrating the density $1/|x|$.

We then prove the second estimate. Without loss of generality we assume that $b_n = 0$. By monotonicity $\ell_{h,D}(\gamma) \geqslant \ell_{h,\mathbb{C}\setminus\{a_n,0\}}(\gamma)$. By [15, Theorem 4.1(ii)] we have that $\lambda_{\mathbb{C}\setminus\{a_n,0\}}(z) \geqslant \lambda_{\mathbb{C}\setminus\{a_n,0\}}(|z|)$ and by [15, Theorem 4.1(i)] that $\lambda_{\mathbb{C}\setminus\{a_n,0\}}(r)$ is a decreasing function in $r \in (0,\infty)$; hence, $\ell_{h,\mathbb{C}\setminus\{a_n,0\}}(\gamma) \geqslant \ell_{h,\mathbb{C}\setminus\{a_n,0\}}([|a_n|, |a_n|+s]) = \ell_{h,\mathbb{C}\setminus\{-1,0\}}([1, 1+s/|a_n|])$. By Theorem 4.2

$$\ell_{h,D}(\gamma) \geqslant \ell_{h,\mathbb{C}\setminus\{-1,0\}}([1, 1+s/|a_n|]) \geqslant \int_1^{1+s/|a_n|} \frac{2^{-3/2}\,dx}{x\left(k_0 + \log x\right)}$$
$$= 2^{-3/2}\log\left(1 + k_0^{-1}\log\left(1 + \frac{s}{|a_n|}\right)\right) \geqslant 2^{-3/2}\log\left(1 + k_0^{-1}\log\left(1+\frac{s}{r}\right)\right). \quad \square$$

*Proof of Theorem 1.1(2), for the quasihyperbolic metric.* We use the characterization of Bonk, Heinonen and Koskela [8]. Hence it suffices to show that the domain in not inner uniform. So, suppose for a contradiction that the domain is $A$-inner uniform for some fixed $A > 0$.

We define $s_n := \max_{1\leqslant m\leqslant n}(b_m - a_m)$. It is clear that $s_n$ is an increasing sequence and $\lim_{n\to\infty} s_n/a_n = 0$. If we define $g_n := \sqrt{s_n/a_n}$, then $b_m - a_m \leqslant a_n g_n^2$ for every $1 \leqslant m \leqslant n$ and $\lim_{n\to\infty} g_n = 0$.

Since $g_n > 0$, we can choose a subsequence $\{g_{n_k}\}$ with $g_{n_k} \geqslant g_m$ for every $m \geqslant n_k$; consider a fixed $n$ from the sequence $\{n_k\}$. Set $c_n = \frac{b_n+a_n}{2}$, the mid-point of $(a_n, b_n)$. We define $x_n = c_n + ic_ng_n$ and $y_n = c_n - ic_ng_n$. Since $[x_n, y_n] \subset \Omega$, we have $\ell_{\text{Eucl},\Omega}([x_n, y_n]) = 2c_ng_n$. Let $\gamma$ be an $A$-inner uniform curve joining $x_n$ and $y_n$, and let $z \in \gamma \cap \mathbb{R}$. Since $|x_n - z|, |y_n - z| \geqslant c_ng_n$, we conclude by the uniformity of the curve that $\delta_\Omega(z) \geqslant \frac{c_ng_n}{A}$. On the other hand, the uniformity of $\gamma$ also implies that $|z - c_n| \leqslant 2Ac_ng_n$.

We may assume that $n$ is so large that $c_n > 2Ac_ng_n$. Then $z$ lies in the positive real axis, which means that $z \in (a_m, b_m)$ for some $m \geqslant 1$. If $m \leqslant n$, then we have $b_m - a_m \leqslant s_n = a_ng_n^2 < c_ng_n^2$. For $m > n$ we have $b_m - a_m \leqslant g_m^2 a_m \leqslant g_n^2 a_m$. However, since $a_m < z \leqslant c_n + 2Ac_ng_n < 2c_n$, so for every $m$ we have $b_m - a_m < 2c_ng_n^2$.

Since $\delta_\Omega(z) < \frac{b_m - a_m}{2}$, we conclude that $\frac{c_ng_n}{A} < c_ng_n^2$. Since $g_n \to 0$ and $A$ is a constant, this is a contradiction. Hence the assumption that an $A$-inner uniform curve exists was false, and we can conclude that the domain is not Gromov hyperbolic. $\square$

For the proof in the hyperbolic case we need the following concepts. A function between two metric spaces $f : X \longrightarrow Y$ is an $(a,b)$-*quasi-isometry*, $a \geqslant 1$, $b \geqslant 0$, if

$$\frac{1}{a}\, d_X(x_1,x_2) - b \leqslant d_Y(f(x_1), f(x_2)) \leqslant a d_X(x_1,x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

An $(a,b)$-*quasigeodesic* in $X$ is an $(a,b)$-quasi-isometry between an interval of $\mathbb{R}$ and $X$.

For future reference we record the following lemma:

**Lemma 4.4.** *Let us consider a geodesic metric space $X$ and a geodesic $\gamma : I \longrightarrow X$, with $I$ any interval, and $g : I \longrightarrow X$, with $d(g(t), \gamma(t)) \leqslant \varepsilon$ for every $t \in I$. Then $g$ is a $(1, 2\varepsilon)$-quasigeodesic.*

*Proof.* We have for every $s, t \in I$

$$d(g(s), g(t)) \geqslant d(\gamma(s), \gamma(t)) - d(\gamma(s), g(s)) - d(\gamma(t), g(t)) \geqslant |t-s| - 2\varepsilon.$$

The upper bound is similar. □

*Proof of Theorem 1.1(2), for the hyperbolic metric.* We consider two cases: either $\{b_m - a_m\}_m$ is bounded or unbounded. We start with the latter case.

As in the previous proof, we define $s_n := \max_{1 \leqslant m \leqslant n}(b_m - a_m)$ and $g_n := \sqrt{s_n/a_n}$. Then $b_m - a_m \leqslant a_n g_n^2$ for every $1 \leqslant m \leqslant n$ and $\lim_{n\to\infty} g_n = 0$. Since $g_n > 0$, we can choose a subsequence $\{g_{n_k}\}$ with $g_{n_k} \geqslant g_m$ for every $m \geqslant n_k$. Since $\{b_m - a_m\}_m$ is not bounded we may, moreover, choose the sequence so that $g_n^2 = (b_n - a_n)/a_n$ for every $n \in \{n_k\}$. Fix now $n$ from the sequence $\{n_k\}$. As before, we conclude that $b_m - a_m \leqslant a_n g_n^2$ for $m \leqslant n$ and $b_m - a_m \leqslant a_m g_m^2 \leqslant a_m g_n^2$ for $m > n$.

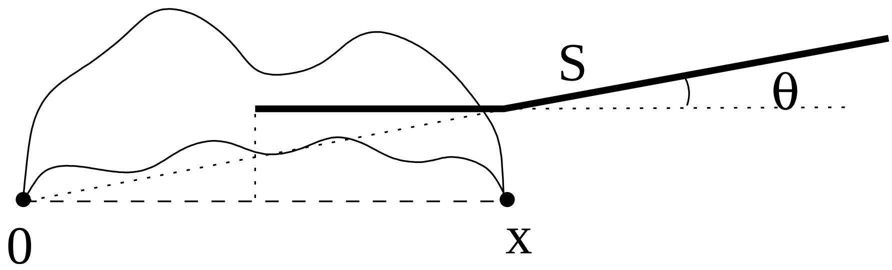


FIGURE 1. The set $S$

Consider $x \in (a_n, b_n)$ which lies on the shortest fundamental geodesic $\gamma_n$ joining $(-\infty, 0)$ with $(a_n, b_n)$. Define an angle $\theta = \arctan g_n \in (0, \pi/2)$ and a set

$$S = [\tfrac{1}{2}x + ixg_n, x + ixg_n] \cup \{x + ixg_n + te^{\pi i\theta} \mid t \geqslant 0\}.$$

The set $S$ is shown in Figure 1. Notice that any point $\zeta \in S$ satisfies $g_n \operatorname{Re}\zeta \leqslant \operatorname{Im}\zeta \leqslant 2g_n \operatorname{Re}\zeta$. It is clear that $\gamma_n$ hits the set $S \cup [\frac{1}{2}x + ixg_n, \frac{1}{2}x]$. We claim that it in fact hits $S$. Assume to the contrary that this is not the case. Then it hits $[\frac{1}{2}x + ixg_n, \frac{1}{2}x]$. Let $\gamma'$ denote a part of $\gamma_n$ connecting $x$ and this segment which does not intersect $S$. Since $\Omega$ is a Denjoy domain, we conclude that $b \mapsto \lambda_\Omega(a + ib)$ is decreasing for $b > 0$ (see [15, Theorem 4.1(i)]). Hence $\ell_{h,\Omega}(\gamma') \geqslant \ell_{h,\Omega}([\frac{1}{2}x + ixg_n, x + ixg_n])$. Since the gap size in $[\frac{1}{2}x, x]$ is at most $a_n g_n^2$, we have $\delta_\Omega(w) \leqslant \sqrt{x^2 {g_n}^2 + a_n^2 g_n^4} \leqslant \sqrt{2}\, xg_n$. Since the gap size is smaller than the distance to the boundary, it follows from Theorem 4.2 that

$$\lambda_\Omega(w) \geqslant \frac{C}{\delta_\Omega(w)} \geqslant \frac{C}{xg_n}$$

for $w \in [\frac{1}{2}x + ixg_n, x + ixg_n]$. Multiplying this with the Euclidean length $\frac{1}{2}x$ of the segment gives

$$\ell_{h,\Omega}(\gamma_n) \geqslant \ell_{h,\Omega}([\tfrac{1}{2}x + ixg_n, x + ixg_n]) \geqslant \frac{C}{g_n}.$$

We next construct another path $\sigma$ and show that it is in the same homotopy class as the supposed geodesic, only shorter. Let $z$ be the midpoint of gap $n$ and let $\sigma$ be the curve $[z, z+iz] \cup [z+iz, -z+iz] \cup [-z+iz, -z]$. Using $b_n - a_n = a_n g_n^2$ we easily calculate

$$\ell_{h,\Omega}(\sigma) \leqslant 2\ell_{k,\Omega}(\sigma) \leqslant 2\log\Big(\frac{2z}{a_n g_n^2}\Big) + C \leqslant 4\log\Big(\frac{1}{g_n}\Big) + C$$

with an absolute constant $C$. The curve $\sigma$ joins $(-\infty, 0)$ and $(a_n, b_n)$; therefore $\ell_{h,\Omega}(\gamma_n) \leqslant \ell_{h,\Omega}(\sigma)$. But this contradicts the previously derived bounds for the lengths as $g_n \to 0$.

Therefore the supposition that $\gamma_n$ does not intersect $S$ was wrong, so we conclude that $\gamma_n \cap S \neq \emptyset$. Let now $\zeta \in S \cap \gamma_n$. We claim that $h_\Omega(\zeta, \mathbb{R}) \to \infty$, which means the domain is not Gromov hyperbolic, by Theorem 3.3. Let $\xi \in \Omega \cap \mathbb{R}$; chose $m$ so that $\xi \in (a_m, b_m)$. Let $\alpha$ be a curve joining $\xi$ and $\zeta$.

If $0 < m \leqslant n$, then the size of $(a_m, b_m)$ is at most $a_n g_n^2$, so $\delta_\Omega(\xi) \leqslant a_n g_n^2$. Then $\alpha$ has Euclidean length at least $\operatorname{Im}\zeta \geqslant x g_n$, so by Lemma 4.3, $\ell_{h,\Omega}(\alpha) \geqslant c\log\log(C/g_n)$. As $g_n \to 0$, this bound tends to $\infty$. If, on the other hand, $m > n$, then the Euclidean length of $\alpha$ is at least

$$d(\xi,\zeta) \geqslant d(\xi, S) \geqslant \xi\sin\theta \geqslant \tfrac{1}{2}\,\xi\tan\theta = \tfrac{1}{2}\,\xi g_n,$$

and the size of the gap is at most $a_m g_n^2$. By Lemma 4.3 this implies that $\ell_{h,\Omega}(\alpha) \geqslant c\log\log(C/g_n)$. As $g_n \to 0$, this bound again tends to $\infty$.

It remains to consider $m = 0$, i.e., $\xi < 0$. We consider only the case $\zeta \in [\frac{1}{2}x + ixg_n, x + ixg_n]$, since the other case is similar. Now the Euclidean length of $\alpha$ is at least $\frac{1}{2}x$. Since the gap size in $[0, \frac{1}{2}x]$ is at most $a_n g_n^2$, we see that the boundary satisfies the separation condition when $|\operatorname{Im} z| \geqslant a_n g_n^2$ in which case also $\delta_\Omega(z) \geqslant |\operatorname{Im} z| \geqslant a_n g_n^2$. Since $\lambda_\Omega(z)$ is decreasing in $|\operatorname{Im} z|$ (see [15, Theorem 4.1(i)]), we conclude that

$$\lambda_\Omega(z) \geqslant \frac{C}{\max\{|\operatorname{Im} z|, a_n g_n^2\}} \geqslant \frac{C}{\max\{\delta_\Omega(z), a_n g_n^2\}} \tag{4.5}$$

for the points on the curve with $\operatorname{Re} z \in (0, x/2)$. Let $\alpha^-$ be the part of $\alpha$ on which $\delta_\Omega(z) < a_n g_n^2$. If $\ell_{\mathrm{Eucl}}(\alpha^-) > x g_n^{3/2}$, then

$$\ell_{h,\Omega}(\alpha) \geqslant \ell_{h,\Omega}(\alpha^-) \geqslant \frac{x g_n^{3/2}}{a_n g_n^2} > g_n^{-1/2}.$$

If $\ell_{\mathrm{Eucl}}(\alpha^-) \leqslant x g_n^{3/2}$, then $\ell_{\mathrm{Eucl}}(\alpha \setminus \alpha^-) > \frac{1}{2}x - x g_n^{3/2}$. Hence we conclude (as in the proof of part (1) in Lemma 4.3) that

$$\int_\alpha \lambda_\Omega(z)\,|dz| \geqslant C\int_{\delta_\Omega(\zeta)+x g_n^{3/2}}^{x/2} \frac{dt}{t} \geqslant C\log\Big(\frac{x/2}{\sqrt{2}\,a_n g_n + x g_n^{3/2}}\Big) \geqslant C\log\Big(\frac{1}{g_n}\Big) - C.$$

Hence in either case we get a lower bound which tends to infinity as $g_n \to 0$.

This takes care of the case when $\{b_m - a_m\}_m$ is unbounded. Assume next that $\sup_m (b_m - a_m) = M < \infty$. In this case it is difficult to work with bigons, since we do not get a good control on what the gedesics look like; the problem with the previous argument is that we cannot choose $g_{n_k}^2 = (b_{n_k} - a_{n_k})/a_{n_k}$ in our sequence, and consequently we do not get a good bound on the length of the curve $\sigma$, as defined above.

To get around this we consider a geodesic triangle. Assume for a contradiction that $h_\Omega$ is $\delta$-Gromov hyperbolic. By geodesic stability [9], there exists a number $\delta'$ so that every $(\sqrt{2},0)$-quasigeodesic triangle is $\delta'$-thin.

Fix $R \gg M^2$ and set $w_\pm = \pm iR$. Let $\gamma_0$ be the geodesic segment joining $w_+$ and $w_-$. Choose $t>0$ so large that $h_\Omega(\gamma_0, H_t) > \delta'$, where $H_t = \{z \in \mathbb{C} \mid \operatorname{Re} z > t\}$. Let $w \in \Omega \cap \mathbb{R}$ be a point in $H_{2\max\{t,R\}}$, and let $\gamma_+ \subset \overline{\mathbf{H}^2}$ be a geodesic joining $w$ and $w_+$.

If $\gamma_+$ dips below the ray from $w$ through $w_+$, then we replace the part below the ray by a part of the ray. The resulting curve is denoted by $\tilde\gamma_+$. Let us show that $\tilde\gamma_+$ is a quasigeodesic. We define a mapping $f\colon \gamma_+ \to \tilde\gamma_+$ as follows. If $x \in \gamma_+ \cap \tilde\gamma_+$, then $f(x) = x$. If $x \in \gamma_+ \setminus \tilde\gamma_+$ then we set $f(x)$ to equal the point on $\tilde\gamma_+$ with real part equal to $\operatorname{Re} x$.

Since $\Omega$ is a Denjoy domain, the function $b \mapsto \lambda_\Omega(a+ib)$ is decreasing for $b>0$ (see [15, Theorem 4.1(i)]). Hence $\lambda_\Omega(f(x)) \leqslant \lambda_\Omega(x)$. The arc-length distance element is the vertical projection of the distance element at $x$ to the line through $w$ and $w_+$: specifically, the distance element $(dx, dy)$ becomes $(dx, \theta dx)$, where $\theta$ is the slope of the line. Thus the maximal increase in the distance element is $\sqrt{1+\theta^2}$. Since the slope of the line lies in the range $[-1,0)$, we conclude from these facts that $\tilde\gamma_+$ is a $(\sqrt{2},0)$-quasigeodesic.

Similarly, we construct $\tilde\gamma_-$ and conclude that it is a $(\sqrt{2},0)$-quasigeodesic. Choose now $\zeta \in \tilde\gamma_+ \cap H_{\max\{t,R\}}$ with $\operatorname{Im}\zeta = \sqrt{R}$. Since $\gamma_0 \cup \tilde\gamma_+ \cup \tilde\gamma_-$ is a $(\sqrt{2},0)$-quasigeodesic triangle, it should be possible to to connect $\zeta$ with some point in $\gamma_0 \cup \tilde\gamma_-$ using a path of length $\delta'$. By the definition of $t$, $h_\Omega(\zeta,\gamma_0) > \delta'$. If $\alpha$ is a path connecting $\zeta$ and $\gamma_-$, then it crosses the real axis at some point $\xi$. If $\xi$ lies in $(a_m, b_m)$, $m>0$, then $\ell_{h,\Omega}(\alpha) \geqslant C \log\log \frac{\sqrt{R}}{M}$, by Lemma 4.3. Otherwise, $\xi \in (-\infty, 0)$. This case is handled as in the first case of the proof, see the paragraph around (4.5). In each case we see that $h_\Omega(\zeta,\gamma_-) > \delta'$ provided $R$ is large enough. But this means that $\Omega$ is not Gromow hyperbolic, as was to be shown. □

In Theorem 1.1(2) the gaps $(a_n,b_n)$ and $(a_{n+1},b_{n+1})$ are separated by a boundary component $[b_n, a_{n+1}]$. We easily see from the proofs that it would have made no difference if this boundary component had some gaps, as long as they at most comparable to the lengths of the adjecent gaps, $(a_n,b_n)$ and $(a_{n+1},b_{n+1})$. Thus we get the following stronger theorem by the same proofs. (In the proofs we can assume that $(-\infty,0) \subset \Omega$, by using Theorem 1.2).

**Theorem 4.6.** *Let $\Omega$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \bigcup (a_n,b_n)$ and $\limsup_{n\to\infty} a_n = \infty$. Suppose $G\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function with $\lim_{x\to\infty} G(x) = 0$. If $b_n - a_n \leqslant a_n G(a_n)$ for every $a_n > 0$, then $\kappa_\Omega$, the hyperbolic or quasihyperbolic metric, is not Gromov hyperbolic.*

The function $G$ plays the role of $g_n^2$ in the proofs of Theorem 1.1(2).

*Remark* 4.7. The condition $\Omega \cap \mathbb{R} = \bigcup (a_n,b_n)$ (without the hypothesis $b_n \leqslant a_{n+1}$ for every $n$) allows any topological behaviour; for instance, $\partial\Omega$ can contain a countable sequence of Cantor sets.

Let $E_0 \subset [0,t)$ be closed, $t>0$, set $E_n := E_0 + tn$ for $n \in \mathbb{N}$, and $\Omega := \mathbb{C} \setminus \cup_n E_n$. Then $\Omega$ satisfies the hypotheses of Theorem 4.6 for $G(x) = t/x$. From this we deduce Corollary 1.3, the non-hyperbolicity of periodic Denjoy domain, in the case the index set is $\mathbb{N}$. The case with index set $\mathbb{Z}$ follows from this and Theorem 1.2.

## 5. On the far side of the accumulation point

**Lemma 5.1.** *Let $\Omega$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$ and $a_0 = -\infty$. If $h_\Omega$ is not Gromov hyperbolic, then for every $N > 0$ there exist fundamental geodesics $\gamma_{n_k}$, $n_k > N$, such that the hyperbolic distance of the endpoints of $\gamma_{n_k}$ to $(-\infty, b_0)$ is greater than $N$, and points $z_k \in \gamma_{n_k}$ with $\lim_{k\to\infty} h_\Omega(z_k, \mathbb{R}) = \infty$.*

*Proof.* Let us choose fundamental geodesics $\{\gamma_n^0\}$. Since $h_\Omega$ is not Gromov hyperbolic, by Theorem 3.3 there exists points $w_k \in \gamma_{n_k}^0$ with $n_k > N$ and $\lim_{k\to\infty} h_\Omega(w_k, \mathbb{R}) = \infty$. Since $\lim_{x\to b_n} h_\Omega(x, (-\infty, b_0)) = \infty$ for every $n$, there exist $x_0 \in (a_0, b_0)$ and $x_{n_k} \in (a_{n_k}, b_{n_k})$, with $h_\Omega(x_0, (-\infty, b_0)), h_\Omega(x_{n_k}, (-\infty, b_0)) > N$.

Let us consider the fundamental geodesics $\gamma_{n_k}$ joining $x_0$ and $x_{n_k}$, as well as the bordered Riemann surface $X := \Omega \cap \overline{\mathbf{H}^2}$, which as in the proof of Theorem 3.1 can be shown to have $\log\big(1+\sqrt{2}\,\big)$-thin triangles.

Let $Q_k$ be the geodesic quadrilateral given by $\gamma_{n_k}^0$, $\gamma_{n_k}$ and the two geodesics (contained in $(a_0, b_0)$ and $(a_{n_k}, b_{n_k})$) joining their endpoints. Since $Q_k \subset X$, it is $2\log\big(1+\sqrt{2}\,\big)$-thin, and there exists $z_k \in \gamma_{n_k} \cup \mathbb{R}$ with $h_\Omega(z_k, w_k) \leqslant 2\log\big(1+\sqrt{2}\,\big)$.

Since $\lim_{k\to\infty} h_\Omega(w_k, \mathbb{R}) = \infty$, we deduce that $z_k \in \gamma_{n_k}$ for every $k \geqslant k_0$ and $\lim_{k\to\infty} h_\Omega(z_k, \mathbb{R}) = \infty$. □

**Lemma 5.2** (Lemma 3.1, [1])**.** *Consider an open Riemann surface $S$ of hyperbolic type, a closed non-empty subset $C$ of $S$, and set $S^* := S \setminus C$. For $\epsilon > 0$ we have $1 < \ell_{S^*}(\gamma)/\ell_S(\gamma) < \coth(\varepsilon/2)$, for every curve $\gamma \subset S$ with finite length in $S$ such that $h_S(\gamma, C) \geqslant \varepsilon$.*

Given a Riemann surface $S$, a geodesic $\gamma$ in $S$, and a continuous unit vector field $\xi$ along $\gamma$ orthogonal to $\gamma$, we define *Fermi coordinates* based on $\gamma$ as the map $Y(r,t) := \exp_{\gamma(r)} t\xi(r)$.

It is well known that if the curvature is $K \equiv -1$, then the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dt^2 + \cosh^2 t\, dr^2$ (see e.g. [10, p. 247–248]).

**Corollary 5.3.** *Consider an open Riemann surface of hyperbolic type $S$, a closed non-empty subset $C$ of $S$, and set $S^* := S \setminus C$. For $\epsilon > 0$ and $C_\varepsilon := \{z \in S :\, h_S(z, C) \geqslant \varepsilon\}$ we have*

$$h_S(z,w) \leqslant h_{S^*}(z,w), \qquad \text{for every } z, w \in S^*,$$

$$h_{S^*}(z,w) \leqslant \coth(\varepsilon/2)\, h_{S|C_\varepsilon}(z,w), \qquad \text{for every } z, w \in C_\varepsilon\,.$$

*Furthermore, if $S$ is a Denjoy domain and $C$ is a component of $S \cap \mathbb{R}$ then*

$$h_{S^*}(z,w) \leqslant \cosh\varepsilon\, \coth(\varepsilon/2)\, h_S(z,w),$$

*for every $z, w$ in the same component of $C_\varepsilon$ with* $\operatorname{Im} z, \operatorname{Im} w \geqslant 0$.

*Proof.* The first and second inequalities are direct consequences of Lemma 5.2. In order to prove the third one, it is sufficient to prove that

$$h_{S|C_\varepsilon}(z,w) \leqslant (\cosh\varepsilon)\, h_S(z,w), \tag{5.4}$$

for every $z, w$ in the same component of $C_\varepsilon$ with $\operatorname{Im} z, \operatorname{Im} w \geqslant 0$.

Fix $z, w$ in the same component $\Gamma$ of $C_\varepsilon$. Since $\operatorname{Im} z, \operatorname{Im} w \geqslant 0$ there exists a unique geodesic $\gamma \subset S \cap \overline{\mathbf{H}^2}$ joining $z$ with $w$.

If $\gamma \subset \Gamma$, then $h_{S|C_\varepsilon}(z,w) = h_S(z,w)$. If $\gamma$ is not contained in $\Gamma$, then it is sufficient to show that there exists a curve $\eta$ joining $z$ and $w$ in $\Gamma$, with $\ell_{h,S}(\eta) \leqslant (\cosh\varepsilon)\, \ell_{h,S}(\gamma)$.

In order to prove this, consider the geodesics $\gamma_z, \gamma_w \subset S \cap \overline{\mathbf{H}^2}$ joining $z$ and $w$ with $C$, and the geodesic $\gamma_0 \subset C$ joining the endpoints of $\gamma_z, \gamma_w$ (which are in $C$).

We denote by $P$ the simply connected closed region with boundary $\gamma \cap \gamma_z \cap \gamma_w \cap \gamma_0$. Since $P$ is simply connected, we can identify it with a domain $P_0 \subset \overline{\mathbf{H}^2}$ using Fermi coordinates based on $C$.

If $g$ is the lift of $\gamma$, then $g_1 := g \cap \{(r,t) :\ 0 \leqslant t \leqslant \varepsilon\}$ is the lift of $\gamma \setminus C_\varepsilon$. If $g \cap \{(r,t) :\ t = \varepsilon\} = \{(r_1,\varepsilon),(r_2,\varepsilon)\}$ (with $r_1 < r_2$), then we define $g_2 := \{(r,\varepsilon) :\ r_1 \leqslant r \leqslant r_2\}$ and $g_0 := \{(r,0) :\ r_1 \leqslant r \leqslant r_2\}$. Notice that in order to prove (5.4) it is sufficient to show that $\ell(g_2) \leqslant (\cosh \varepsilon)\, \ell(g_1)$. But this is a direct consequence of the facts $\ell(g_0) \leqslant \ell(g_1)$ and $\ell(g_2) = (\cosh \varepsilon)\, \ell(g_0)$. □

*Proof of Theorem 1.2.* Since $\kappa_\Omega$ is not Gromov hyperbolic, by Proposition 3.5, we conclude that $\Omega$ has countably infinitely many boundary components: $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$. Without loss of generality we can assume that $(-\infty, 0) \subseteq (a_1, b_1)$.

We first prove that $(\Omega \setminus F, k_{\Omega\setminus F})$ is not Gromov hyperbolic. Let us consider fundamental geodesics $\gamma_n$ of $k_\Omega$ joining the midpoint $c_0$ of $(a_0, b_0)$ with the midpoint $c_n$ of $(a_n, b_n)$ for $n \geqslant 2$ which are shortest possible. Since $\gamma_n$ is contained in $\{z \in \mathbb{C} :\ c_0 \leqslant \operatorname{Re} z \leqslant c_n\}$, and $k_{\Omega\setminus F} = k_\Omega$ in $\{z \in \mathbb{C} :\ \operatorname{Re} z \geqslant \inf_{n\geqslant 2} a_n\}$, we deduce that $\gamma_n$ is also a fundamental geodesic with the metric $k_{\Omega\setminus F}$.

Since $k_\Omega$ is not Gromov hyperbolic, there exist points $z_k \in \gamma_{n_k}$ with $\lim_{k\to\infty} k_\Omega(z_k, \mathbb{R}) = \infty$ by Theorem 3.3. Since $\gamma_{n_k}$ are also fundamental geodesics with the metric $k_{\Omega\setminus F}$, we deduce that $\lim_{k\to\infty} k_{\Omega\setminus F}(z_k, \mathbb{R}) \geqslant \lim_{k\to\infty} k_\Omega(z_k, \mathbb{R}) = \infty$. Consequently, $(\Omega \setminus F, k_{\Omega\setminus F})$ is not Gromov hyperbolic.

We now prove that $(\Omega \setminus F, h_{\Omega\setminus F})$ is not Gromov hyperbolic. Choose $\varepsilon_0 > 0$. Since $h_\Omega$ is not Gromov hyperbolic, by Lemma 5.1 there exist fundamental geodesics $\gamma_{n_k}$ of $h_\Omega$, such that the hyperbolic distance of the endpoints of $\gamma_{n_k}$ to $(-\infty, b_1)$ is greater than $\varepsilon_0$, and points $z_k \in \gamma_{n_k}$ with $\lim_{k\to\infty} h_\Omega(z_k, \mathbb{R}) = \infty$.

Fix $\varepsilon \in \big(0, \min\{\varepsilon_0, \min_k h_\Omega(z_k, \mathbb{R})\}\big)$. If we define

$$U_\varepsilon := \{z \in \Omega :\ h_\Omega(z, (-\infty, b_1)) \geqslant \varepsilon\},$$

we see that $z_k \in \gamma_{n_k} \cap U_\varepsilon$ for every $k$. (Notice that $\gamma_{n_k} \cap \partial U_\varepsilon$ has at most two points.) If $\gamma_{n_k} \cap \partial U_\varepsilon$ is empty or a one-point set, we define $g_{n_k} := \gamma_{n_k}$. Since the endpoints of $\gamma_{n_k}$ are in $U_\varepsilon$, we conclude that $g_{n_k} \subset U_\varepsilon$.

Then assume that $\gamma_{n_k} \cap \partial U_\varepsilon = \{w^1, w^2\}$. If there is an arc $\alpha$ in $\partial U_\varepsilon$ joining $w^1$ and $w^2$, we define a curve $g_{n_k}$ joining $(a_0, b_0)$ with $(a_{n_k}, b_{n_k})$ in $U_\varepsilon$, by $g_{n_k} := (\gamma_{n_k} \cap U_\varepsilon) \cup \alpha$. Then $\gamma_{n_k}$ and $g_{n_k}$ have the same endpoints and are homotopic. If there is not an arc in $\partial U_\varepsilon$ joining $w^1$ and $w^2$, there are still maximal arcs $\alpha, \beta$ in $\partial U_\varepsilon$ joining $w^1$ and $\omega^1 \in (a_{m^1}, b_{m^1})$, and $w^2$ and $\omega^2 \in (a_{m^2}, b_{m^2})$, respectively, and a geodesic $\eta$ (with respect to $h_\Omega$) in $\Omega \setminus U_\varepsilon$ joining $\omega^1$ and $\omega^2$, such that if $\gamma_{n_k} \cap U_\varepsilon = [z^1, w^1] \cup [z^2, w^2]$, then $[z^1, w^1] \cup \alpha \cup \eta \cup \beta \cup [z^2, w^2]$ has the same endpoints as $\gamma_{n_k}$, and they are homotopic.

Since $\varepsilon < h_\Omega(z_k, \mathbb{R})$, we have either $z_k \in [z^1, w^1]$ or $z_k \in [z^2, w^2]$. Without loss of generality we can assume that $z_k \in [z^2, w^2]$. Then we define $g_{n_k} := \beta \cup [z^2, w^2] \subset U_\varepsilon$, which is a curve joining $(a_{m^2}, b_{m^2})$ with $(a_{n_k}, b_{n_k})$.

In any case, Lemma 4.4 gives that $g_{n_k}$ is a $(1, 2\varepsilon)$-quasigeodesic with respect to $h_\Omega$. Hence, for every $t, s$, we have

$$|t-s| - 2\varepsilon \leqslant h_\Omega\big(g_{n_k}(t), g_{n_k}(s)\big) \leqslant |t-s| + 2\varepsilon.$$

Since $g_{n_k}$ is contained in $U_\varepsilon$, Corollary 5.3 implies that

$$\begin{aligned}|t-s|-2\varepsilon &\leqslant h_\Omega\big(g_{n_k}(t),g_{n_k}(s)\big) < h_{\Omega\setminus F}\big(g_{n_k}(t),g_{n_k}(s)\big)\\ &\leqslant h_{\Omega\setminus(-\infty,0]}\big(g_{n_k}(t),g_{n_k}(s)\big)\\ &\leqslant \cosh\varepsilon\ \coth(\varepsilon/2)\, h_\Omega\big(g_{n_k}(t),g_{n_k}(s)\big)\\ &\leqslant \cosh\varepsilon\ \coth(\varepsilon/2)\,\big(|t-s|+2\varepsilon\big),\end{aligned}$$

and hence $g_{n_k}$ is a $\big(\cosh\varepsilon\coth(\varepsilon/2), 2\varepsilon\cosh\varepsilon\coth(\varepsilon/2)\big)$-quasigeodesic with respect to $h_{\Omega\setminus F}$.

To get a contradiction, assume that $(\Omega\setminus F, h_{\Omega\setminus F})$ is Gromov hyperbolic. Consider the fundamental geodesic $\eta_{n_k}$ of $h_{\Omega\setminus F}$ with the same endpoints as $g_{n_k}$. Then there is a constant $C$ such that the Hausdorff distance of $g_{n_k}$ and $\eta_{n_k}$ is less than $C$. Hence, there exist points $w_k\in\eta_{n_k}$ with $h_{\Omega\setminus F}(z_k,w_k)\leqslant C$, and thus

$$\lim_{k\to\infty} h_{\Omega\setminus F}(w_k,\mathbb{R}) \geqslant \lim_{k\to\infty} h_{\Omega\setminus F}(z_k,\mathbb{R}) - C \geqslant \lim_{k\to\infty} h_\Omega(z_k,\mathbb{R}) - C = \infty,$$

which contradicts $h_{\Omega\setminus F}$ being Gromov hyperbolic. □

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