Composition operators in hyperbolic Q-classes

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Abstract

Function theoretic characterizations are given of when a composition operator mapping from a weighted Dirichlet space $\mathcal{D}_q$ into a holomorphic Q_s-space is bounded or compact. If X^* stands for the hyperbolic class corresponding to the space X, it is shown that a composition operator mapping from $\mathcal{D}_q$ into Q_s is bounded if and only it is bounded from $\mathcal{D}_q^*$ into Q_s^*, provided $q \leq 0$ and $s \leq 1$.

1. Introduction and statements of results

Let $\mathbb{D}$ denote the unit disc of the complex plane, and let $\mathbb{T}$ be its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$, and let $B(\mathbb{D})$ be the subset of $H(\mathbb{D})$ consisting of those $h \in H(\mathbb{D})$ for which $|h(z)| < 1$, $z \in \mathbb{D}$. Every $\varphi \in B(\mathbb{D})$ induces a linear composition operator $C_\varphi(f) = f \circ \varphi$ from $H(\mathbb{D})$ or $B(\mathbb{D})$ into itself. For the general theory of composition operators in analytic function spaces, see [4] and [13].

A function $f \in H(\mathbb{D})$ belongs to the α-*Bloch space* $\mathcal{B}_\alpha$, $0 < \alpha < \infty$, if

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

The little α-*Bloch space* $\mathcal{B}_0$ consist of those $f \in H(\mathbb{D})$ for which $|f'(z)|(1 - |z|^2) \to 0$ as $|z| \to 1$. Denoting $h^*(z) = |h'(z)|/(1 - |h(z)|^2)$, the *hyperbolic derivative* of $h \in B(\mathbb{D})$, the *hyperbolic α-Bloch classes* $\mathcal{B}_\alpha^*$ and $\mathcal{B}_{\alpha,0}^*$ are defined as the sets of those $h \in B(\mathbb{D})$ for which

$$\|h\|_{\mathcal{B}_\alpha^*} = \sup_{z \in \mathbb{D}} h^*(z)(1 - |z|^2)^\alpha < \infty$$

and $\lim_{|z|\to 1} h^*(z)(1-|z|^2)^\alpha = 0$, respectively. If $\alpha = 1$, it is simply denoted $\mathcal{B}^* = \mathcal{B}^*_1$ and $\mathcal{B}^*_0 = \mathcal{B}^*_{1,0}$. Clearly $\mathcal{B}^*_\alpha$ and $\mathcal{B}^*_{\alpha,0}$ are not linear spaces. Moreover, the Schwarz-Pick lemma implies $\mathcal{B}^*_\alpha = B(\mathbb{D})$ if $\alpha \geq 1$, and therefore the hyperbolic α-Bloch classes are only considered when $0 < \alpha \leq 1$.

For $s > -1$, the *weighted Dirichlet space* $\mathcal{D}_s$ (resp. *weighted hyperbolic Dirichlet class* $\mathcal{D}^*_s$) consists of those $f \in H(\mathbb{D})$ (resp. $h \in B(\mathbb{D})$) for which

$$\|f\|_{\mathcal{D}_s} = \left(\int_{\mathbb{D}} |f'(z)|^2 \left(\log \frac{1}{|z|} \right)^s dA(z) \right)^{\frac{1}{2}} < \infty$$

(resp.

$$\|h\|_{\mathcal{D}^*_s} = \left(\int_{\mathbb{D}} (h^*(z))^2 \left(\log \frac{1}{|z|} \right)^s dA(z) \right)^{\frac{1}{2}} < \infty),$$

where $dA(z)$ denotes the element of the Lebesgue area measure on $\mathbb{D}$. The Schwarz-Pick lemma implies $\mathcal{D}^*_s = B(\mathbb{D})$ for $s > 1$, and therefore the class $\mathcal{D}^*_s$ is considered only when $-1 < s \leq 1$. In this range the class $\mathcal{D}^*_s$ contains no inner functions by [14, Theorem 1.1].

Let the Green's function of $\mathbb{D}$ be defined as $g(z, a) = -\log |\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is the automorphism of $\mathbb{D}$ which interchanges the points zero and $a \in \mathbb{D}$. For $0 \leq s < \infty$, the Möbius invariant subspace (resp. subclass) Q_s (resp. Q^*_s) of $\mathcal{D}_s$ (resp. $\mathcal{D}^*_s$) consists of those $f \in H(\mathbb{D})$ (resp. $h \in B(\mathbb{D})$) for which

$$\|f\|_{Q_s} = \left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^s(z, a) \, dA(z) \right)^{\frac{1}{2}} < \infty \tag{1.1}$$

(resp.

$$\|h\|_{Q^*_s} = \left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (h^*(z))^2 g^s(z, a) \, dA(z) \right)^{\frac{1}{2}} < \infty). \tag{1.2}$$

The space $Q_{s,0}$ (resp. class $Q^*_{s,0}$) consists of those $f \in H(\mathbb{D})$ (resp. $h \in B(\mathbb{D})$) for which the integral expression in (1.1) (resp. (1.2)) tends to zero as $|a| \to 1$. If $s = 0$, then Q_0 is the classical Dirichlet space $\mathcal{D} = \mathcal{D}_0$. If $s > 1$, then, by [2, Theorem 1], the spaces Q_s and $Q_{s,0}$ coincide with the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$, respectively, and the class Q^*_s reduces to $B(\mathbb{D})$ by Schwarz-Pick lemma.

The following characterization of bounded composition operators mapping from $\mathcal{B}_\alpha$ into Q_s can be found in [17, Theorem 2.2.1(i)].

Theorem A. *Let $0 < \alpha < \infty$, $0 \leq s < \infty$ and $\varphi \in B(\mathbb{D})$. Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_\alpha \to Q_s$ *is bounded;*

(2) $C_\varphi : \mathcal{B}_{\alpha,0} \to Q_s$ is bounded;

(3) $\displaystyle \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} g^s(z,a)\, dA(z) < \infty$;

(4) $\displaystyle \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} (1-|\varphi_a(z)|^2)^s\, dA(z) < \infty.$

To be precise, the case $s = 0$ of Theorem A does not appear in [17, Theorem 2.2.1(i)], but it has been included above since the same proof works also in this case.

A composition operator $C_\varphi : \mathcal{B}_\alpha^* \to Q_s^*$ is said to be *bounded* if there exists a positive constant C such that $\|C_\varphi(h)\|_{Q_s^*} \leq C\|h\|_{\mathcal{B}_\alpha^*}$ for all $h \in \mathcal{B}_\alpha^*$. From hereafter it is agreed the same meaning for the boundedness of C_φ mapping from one hyperbolic class X^* into another hyperbolic class Y^*. This definition is of the same spirit as the definition in [9] of a bounded composition operator mapping from one meromorphic function class into another. However, it would be of interest to find metrics in $\mathcal{B}_\alpha^*$ and Q_s^* such that these classes would become complete metric spaces with respect to these metrics, and the continuity of C_φ would be equivalent to the natural definition of a bounded composition operator given above.

The first result of this paper extends Theorem A to the corresponding hyperbolic classes.

Theorem 1.1. *Let $0 < \alpha \leq 1$, $0 \leq s \leq 1$ and $\varphi \in B(\mathbb{D})$. Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B}_\alpha \to Q_s$ is bounded;

(2) $C_\varphi : \mathcal{B}_\alpha^* \to Q_s^*$ is bounded;

(3) $C_\varphi : \mathcal{B}_{\alpha,0}^* \to Q_s^*$ is bounded.

Keeping the consideration mostly out of meromorphic function classes, it is settled to point out a somewhat surprising phenomenon which occurs here. Namely, by Theorem A, Theorem 1.1 and the theorem in [9], a composition operator C_φ mapping from $\mathcal{B}$ into Q_s is bounded, if and only if, it is bounded from $\mathcal{B}^*$ into Q_s^*, if and only if, it is bounded from $\mathcal{N}$ into $Q_s^\#$, where $\mathcal{N}$ denotes the class of normal functions and $Q_s^\#$ is the meromorphic Q_s-class. See [2] and [9] for necessary definitions. Since the functions in $\mathcal{B}_\alpha$ are bounded if $0 < \alpha < 1$, it is easy to see, by using functions with Hadamard gaps as in the proof of Theorem 1.1, that this result remains also valid when the domain space/classes are $\mathcal{B}_\alpha$, $\mathcal{B}_\alpha^*$ and $\mathcal{N}_\alpha^\#$, $0 < \alpha < 1$, respectively.

Two quantities a and b are said to be *comparable*, denoted by $a \simeq b$, if there exists a positive constant C such that $C^{-1}a \le b \le Ca$.

Example 1.2. For $0 < \beta < 1$, define $\phi_\beta(z) = 1 - (1-z)^\beta$. Then $\phi_\beta(z)$ is a conformal mapping which fixes the points zero and one, and maps $\mathbb{D}$ onto a lens-type region. Since $\phi_\beta^*(z) \simeq |1-z|^{-1}$ in $\mathbb{D}$, Theorem 1.1 and Theorem A imply that $C_{\phi_\beta} : \mathcal{B}^* \to Q_s^*$ is bounded if and only if s is positive. It is now shown that $C_{\phi_\beta} : Q_{s_1}^* \to Q_{s_2}^*$, $0 < s_1 \le 1$, is bounded if and only if s_2 is positive. Since $Q_{s_1}^* \subset \mathcal{B}^*$ with $\|h\|_{\mathcal{B}^*} \le C\|h\|_{Q_{s_1}^*}$, where C is a positive constant, it suffices to show that $C_{\phi_\beta} : Q_{s_1}^* \to \mathcal{D}$ is not bounded. But this easily follows by the fact $(\phi_\beta \circ \phi_\beta)^*(z) \simeq |1-z|^{-1}$ in $\mathbb{D}$, since the Fatou lemma and (3.2) below yield $\phi_\beta \in Q_{s_1}^*$ for $0 < s_1 \le 1$.

If $0 < \alpha < \infty$, $0 < s < \infty$ and $\varphi \in B(\mathbb{D})$, then $C_\varphi : Q_s \to \mathcal{B}_\alpha$ is bounded if and only if $\varphi \in \mathcal{B}_\alpha^*$ by [17, Theorem 2.2.1(iii)]. This result is extended for the corresponding hyperbolic classes in the following theorem.

Theorem 1.3. *Let $0 < \alpha \le 1$, $0 \le s \le 1$ and $\varphi \in B(\mathbb{D})$. Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{B} \to \mathcal{B}_\alpha$ *is bounded;*

(2) $C_\varphi : \mathcal{B}^* \to \mathcal{B}_\alpha^*$ *is bounded;*

(3) $C_\varphi : Q_s^* \to \mathcal{B}_\alpha^*$ *is bounded;*

(4) $\varphi \in \mathcal{B}_\alpha^*$.

Theorem 1.4 generalizes [16, Theorem 4.1(i)] since the term $|\varphi_a'(\varphi(z))|$ in conditions (2) and (3) below can be replaced by $(1-|a|^2)^\tau/|1-\bar{a}\varphi(z)|^{1+\tau}$, $0 < \tau < \infty$, by Lemma B below (in Section 2).

Theorem 1.4. *Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(\mathbb{D})$. Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ *is bounded;*

(2) $\displaystyle\sup_{a,b\in\mathbb{D}} \int_\mathbb{D} |\varphi_a'(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b)\, dA(z) < \infty;$

(3) $\displaystyle\sup_{a,b\in\mathbb{D}}\int_{\mathbb{D}}|\varphi_a'(\varphi(z))|^{2+s_1}|\varphi'(z)|^2(1-|\varphi_b(z)|^2)^{s_2}\,dA(z)<\infty.$

Moreover, if $s_1\leq 0$ and $s_2\leq 1$, then (1)–(3) are equivalent to

(4) $C_\varphi:\mathcal{D}_{s_1}^*\to Q_{s_2}^*$ is bounded.

By Theorem 1.3, $C_\varphi:\mathcal{D}\to\mathcal{B}$ is bounded for all $\varphi\in B(\mathbb{D})$. Moreover, since $Q_s=\mathcal{B}$ for $s>1$, Theorem 1.4 implies that $C_\varphi:\mathcal{D}\to\mathcal{B}$ is bounded if and only if

$$\sup_{a,b\in\mathbb{D}}\int_{\mathbb{D}}|(\varphi_a\circ\varphi)'(z)|^2 g^s(z,b)\,dA(z)<\infty,\quad 1<s<\infty. \tag{1.3}$$

However, (1.3) is equivalent to

$$\sup_{a,z\in\mathbb{D}}|(\varphi_a\circ\varphi)'(z)|(1-|z|^2)=\|\varphi\|_{\mathcal{B}^*}<\infty,$$

which is, of course, satisfied for all $\varphi\in B(\mathbb{D})$.

Example 1.5. For $0\leq p<\infty$, define $\psi_p(z)=\frac{p+z}{p+1}$. Then ψ_p is a conformal mapping which maps $\mathbb{D}$ onto the disc centered at $p/(p+1)$ with radius $1/(p+1)$. Clearly,

$$\|f\circ\varphi\|_{Q_{s_2}}\leq C\|f\circ\varphi\|_{\mathcal{D}}\leq C\|f\|_{\mathcal{D}} \tag{1.4}$$

for all $f\in\mathcal{D}$ and $\varphi\in B(\mathbb{D})$, thus, in particular, $C_{\psi_p}:\mathcal{D}\to Q_{s_2}$ is bounded. A similar reasoning shows that $C_{\psi_p}:\mathcal{D}^*\to Q_{s_2}^*$ is also bounded. However, a geometric argument or a straightforward calculation based on the identity

$$1-\overline{a}\psi_p(z)=\frac{p(1-\overline{a})+1-\overline{a}z}{p+1}$$

shows that $|1-a\psi_p(z)|\leq|1-az|$ for all $z\in\mathbb{D}$ and $a\in(0,1)$, and therefore

$$\begin{aligned}
&\sup_{a,b\in\mathbb{D}}\int_{\mathbb{D}}|\varphi_a'(\psi_p(z))|^{2+s_1}|\psi_p'(z)|^2(1-|\varphi_b(z)|^2)^{s_2}\,dA(z)\\
&\geq\lim_{a\to 1}\frac{(1-a^2)^{2+s_1+s_2}}{(p+1)^2}\int_{\mathbb{D}}\frac{(1-|z|^2)^{s_2}}{|1-a\psi_p(z)|^{2(2+s_1)}|1-az|^{2s_2}}\,dA(z)\\
&\geq\lim_{a\to 1}\frac{(1-a^2)^{2+s_1+s_2}}{(p+1)^2}\int_{\mathbb{D}}\frac{(1-|z|^2)^{s_2}}{|1-az|^{2(2+s_1+s_2)}}\,dA(z)\\
&\simeq\lim_{a\to 1}(1-a^2)^{-s_1},
\end{aligned}$$

from which it follows by Theorem 1.4 that $C_{\psi_p}:\mathcal{D}_{s_1}\to Q_{s_2}$ is not bounded if s_1 is positive.

Example 1.6. Let $0 < \beta < 1$, $-1 < s_1 < \infty$ and $0 \leq s_2 < \infty$, and consider the map $\phi_\beta(z) = 1 - (1 - z)^\beta$. It is proved that C_{ϕ_β} admits the same behavior as C_{ψ_p} does in the sense that $C_{\phi_\beta} : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded if and only if $-1 < s_1 \leq 0$. In view of (1.4) it suffices to show that $C_{\phi_\beta} : \mathcal{D}_{s_1} \to Q_{s_2}$ is not bounded if $s_1 > 0$. To this end, choose $f_a(z) = (1 - z)^{-a}$, $0 < a < \infty$. Now, by [5, p. 65 Lemma], there is a positive constant C_1 such that

$$\|f_a\|^2_{\mathcal{D}_{s_1}} = a^2 \int_{\mathbb{D}} \frac{(1 - |z|^2)^{s_1}}{|1 - z|^{2(1+a)}} \, d\sigma(z) \leq C_1 a^2 \int_0^1 \frac{r dr}{(1 - r)^{2a+1-s_1}},$$

and therefore $f_a \in \mathcal{D}_{s_1}$ for $0 < a < \frac{s_1}{2}$. Moreover, denote $\omega_{a,\beta}(z) = (1 - z)^{-a\beta} = f_a \circ \phi_\beta$. Then, by [2, Proposition 1], there is a positive constant C_2 such that

$$\|f_a \circ \phi_\beta\|_{Q_{s_2}} = \|\omega_{a,\beta}\|_{Q_{s_2}} \geq C_2 \|\omega_{a,\beta}\|_{\mathcal{B}} = \infty,$$

and therefore $f \circ \phi_\beta \notin Q_{s_2}$. Thus $C_{\phi_\beta} : \mathcal{D}_{s_1} \to Q_{s_2}$ is not bounded if $s_1 > 0$.

The following result generalizes [16, Theorem 4.1(ii)].

Theorem 1.7. *Let* $-1 < s_1 < \infty$, $0 \leq s_2 < \infty$ *and* $\varphi \in B(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ *is compact;*

(2) $\displaystyle \limsup_{\substack{|a| \to 1 \\ b \in \mathbb{D}}} \int_{\mathbb{D}} |\varphi'_a(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z, b) \, dA(z) = 0;$

(3) $\displaystyle \limsup_{\substack{|a| \to 1 \\ b \in \mathbb{D}}} \int_{\mathbb{D}} |\varphi'_a(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 (1 - |\varphi_b(z)|^2)^{s_2} \, dA(z) = 0.$

Since, by the general definition of a bounded (resp. compact) operator mapping from one Banach space into another, $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2,0}$ is bounded (resp. compact), if and only if, $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded (resp. compact) and $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$, the operator $C_\varphi : \mathcal{D}^*_{s_1} \to Q^*_{s_2,0}$ is said to be *bounded*, if $C_\varphi : \mathcal{D}^*_{s_1} \to Q^*_{s_2}$ is bounded and $C_\varphi(\mathcal{D}^*_{s_1}) \subset Q^*_{s_2,0}$.

Theorem 1.8. *Let* $-1 < s_1 < \infty$, $0 < s_2 < \infty$ *and* $\varphi \in B(\mathbb{D})$. *Then* $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2,0}$ *is bounded if and only if* $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ *is bounded and the following two conditions are satisfied:*

(1) $\varphi \in Q_{s_2,0};$

(2) $\displaystyle \lim_{|a|,|b|,t \to 1} \int_{|\varphi(z)| \geq t} |\varphi'_a(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z, b) \, dA(z) = 0.$

Similarly, if $s_1 \leq 0$ and $s_2 \leq 1$, then $C_\varphi : \mathcal{D}_{s_1}^* \to Q_{s_2,0}^*$ is bounded if and only if $C_\varphi : \mathcal{D}_{s_1}^* \to Q_{s_2}^*$ is bounded, $\varphi \in Q_{s_2,0}^*$ and (2) is satisfied.

It is easy to show that the conditions (1) and (2) in Theorem 1.8 together are equivalent to

$$(3) \quad \lim_{|b| \to 1} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b) \, dA(z) = 0,$$

and hence the first part of Theorem 1.8 implies the following result.

Theorem 1.9. *Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(\mathbb{D})$. Then $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2,0}$ is compact if and only if $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is compact and the condition (3) above is satisfied.*

The remains of the paper is organized as follows. In Section, 2, some background material and auxiliary results needed later on are recalled, and Section 3 contains the proofs of the results presented in this section.

2. Background material

A positive Borel measure μ on $\mathbb{D}$ is a *bounded s-Carleson measure*, if

$$\sup_I \frac{\mu(S(I))}{|I|^s} < \infty, \quad 0 < s < \infty,$$

where $|I|$ denotes the arc length of a subarc I of $\mathbb{T}$,

$$S(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, 1 - |I| \leq |z| \right\}$$

is the Carleson box based on I, and the supremum is taken over all subarcs I of $\mathbb{T}$ such that $|I| \leq 1$. Moreover, if

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^s} = 0, \quad 0 < s < \infty,$$

then μ is a *compact s-Carleson measure*. If $s = 1$, then a bounded (resp. compact) 1-Carleson measure is just a standard bounded (resp. compact) Carleson measure.

Some well-known and useful characterizations of bounded s-Carleson measures are gathered in the following lemma. For the proof, see [1, Theorem 13], [3, Lemma 2.1], [10, pp. 89–90], and [11, Proposition 2.1].

Lemma B. *Let μ be a positive Borel measure on $\mathbb{D}$, $1 < s < \infty$, $0 < r < 1$ and $0 < \tau < \infty$. Then the following statements are equivalent:*

(1) $K_1 = \sup\limits_{I} \dfrac{\mu(S(I))}{|I|^s} < \infty\,;$

(2) $K_2 = \sup\limits_{z \in \mathbb{D}} \dfrac{\mu(D(z,r))}{(1 - |z|^2)^s} < \infty\,;$

(3) $K_3 = \sup\limits_{z \in \mathbb{D}} \displaystyle\int_{\mathbb{D}} \left(\dfrac{(1 - |z|^2)^\tau}{|1 - \bar{z}w|^{1+\tau}} \right)^s d\mu(w) < \infty\,.$

Moreover, K_1, K_2 and K_3 are comparable.

Here $D(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$ is the pseudo-hyperbolic disc of center $a \in \mathbb{D}$ and radius $0 < r < 1$. $D(a,r)$ is an Euclidean disc centered at $(1 - r^2)a/(1 - |a|^2 r^2)$ with radius $(1 - |a|^2)r/(1 - |a|^2 r^2)$, see [8, p. 3].

The following change of variables formula by C. S. Stanton, [6] and [15], was apparently first used by J. H. Shapiro [12] in the study of composition operators, and it also plays a key role in some of the proofs in this paper.

Lemma C. *Let g and u be positive measurable functions on $\mathbb{D}$, and let $\varphi \in B(\mathbb{D})$. Then*

$$\int_{\mathbb{D}} (g \circ \varphi)(z)|\varphi'(z)|^2 u(z)\, dA(z) = \int_{\mathbb{D}} g(w) U(\varphi, w)\, dA(w),$$

where

$$U(\varphi, w) = \sum_{z \in \varphi^{-1}\{w\}} u(z), \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

If $u(z) = (-\log|z|)^s$, then $U(\varphi, w)$ is the *generalized Nevanlinna counting function*

$$N_{\varphi,s}(w) = \sum_{z \in \varphi^{-1}\{w\}} \left(\log \frac{1}{|z|} \right)^s.$$

For the study of compactness, the following well-known result is needed. See [4, Proposition 3.11] for a similar result.

Lemma D. *Let $-1 < s_1 < \infty$, $0 \le s_2 < \infty$ and $\varphi \in B(\mathbb{D})$. Then $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{D}_{s_1}$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $\|f_n \circ \varphi\|_{Q_{s_2}} \to 0$ as $n \to \infty$.*

3. Proofs

Proof of Theorem 1.1. It is enough to prove the implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (1)$ since $(2) \Rightarrow (3)$ is clearly true.

Suppose $C_\varphi : \mathcal{B}_\alpha \to Q_s$ is bounded, that is, (1) is satisfied. If $h \in \mathcal{B}_\alpha^*$, then

$$\int_{\mathbb{D}} ((h \circ \varphi)^*(z))^2 g^s(z, a) \, dA(z) \leq \|h\|_{\mathcal{B}_\alpha^*}^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2\alpha}} g^s(z, a) \, dA(z),$$

where the last integral is uniformly bounded for all $a \in \mathbb{D}$ by Theorem A, and therefore $C_\varphi : \mathcal{B}_\alpha^* \to Q_s^*$ is bounded. Thus (1) implies (2) is proved.

To prove $(3) \Rightarrow (1)$, let first $\alpha = 1$, and suppose that $C_\varphi : \mathcal{B}_0^* \to Q_s^*$ is bounded. If $h_b(z) = bz$, $b \in \mathbb{D}$, then $h^*(z) = |b|(1 - |bz|^2)^{-1}$ and $h_b \in \mathcal{B}_0^*$ for all $b \in \mathbb{D}$, and

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|b|^2 |\varphi'(z)|^2}{(1 - |b\varphi(z)|^2)^2} g^s(z, a) \, dA(z) \leq C\|h\|_{\mathcal{B}^*}^2 \leq C|b|^2$$

for some positive C by the assumption. Taking limit as $|b| \to 1$, $b \in \mathbb{D}$, Lebesgue's dominated convergence theorem with Theorem A implies that $C_\varphi : \mathcal{B} \to Q_s$ is bounded, that is, (1) with $\alpha = 1$ holds.

If $0 < \alpha < 1$, functions with Hadamard gaps may be used. Define

$$g_n(z) = \sum_{k=0}^{\infty} 2^{k(\alpha-1)} (b_n z)^{2^k},$$

where $\{b_n\} \subset \mathbb{D}$ and $|b_n| \to 1$, as $n \to \infty$. Then $g_n \in \mathcal{B}_{\alpha,0}$ by [18, Theorem 1]. Since $|g_n(z)| \leq \sum_{k=0}^{\infty} 2^{k(\alpha-1)}$, there is a positive constant C, depending only on α, such that $h_n = C^{-1} g_n$ satisfies $|h_n(z)| \leq \frac{1}{2}$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$, and therefore $h_n^*(z) \simeq |h_n'(z)|$ in $\mathbb{D}$. Now one may argue as in [1, p. 133] and use the Lebesgue's dominated convergence theorem to conclude that the condition (3) with $0 < \alpha < 1$ in Theorem A is satisfied, and hence (1) with $0 < \alpha < 1$ holds. $\square$

Remark. To characterize bounded composition operators from $Q_{s_1}^*$ to $Q_{s_2}^*$ when $0 < s_1, s_2 < 1$ appears to be more complicated. However, Example 1.2 shows that there is a function φ for which $C_\varphi : \mathcal{B}^* \to Q_{s_2}^*$ is bounded if and only $C_\varphi : Q_{s_1}^* \to Q_{s_2}^*$ is bounded, provided $0 < s_1 \leq 1$.

Proof of Theorem 1.3. It suffices to prove the implications $(3) \Rightarrow (4)$ and $(4) \Rightarrow (2)$ since $(2) \Rightarrow (3)$ is clearly true and (1) is equivalent to (4) by [17, Theorem 2.2.1(iii)].

Suppose $C_\varphi : Q_s^* \to \mathcal{B}_\alpha^*$ is bounded, that is (3) holds, and let first $0 < s \leq 1$. If $\phi_{\beta,a}(z) = 1 - (1 - \bar{a}z)^\beta$, where $0 < \beta < 1$ and $a \in \mathbb{D}$, then

$$\phi_{\beta,a}^*(z) = \frac{\beta|a||1 - \bar{a}z|}{1 - |1 - (1 - \bar{a}z)^\beta|} \simeq \frac{\beta|a|}{|1 - \bar{a}z|}, \quad z \in \mathbb{D}. \tag{3.1}$$

By [7, Lemma 2.5], there is a positive constant C_1 such that

$$\int_\mathbb{D} (\phi_{\beta,a}^*(z))^2 (1 - |\varphi_b(z)|^2)^s \, dA(z) \simeq \beta^2 |a|^2 (1 - |b|^2)^s \int_\mathbb{D} \frac{(1 - |z|^2)^s}{|1 - \bar{b}z|^{2s}|1 - \bar{a}z|^2} \, dA(z)$$
$$\leq C_1^2 \beta^2 |a|^2 \frac{(1 - |b|^2)^s}{|1 - \bar{a}b|^s}, \tag{3.2}$$

and it follows that $\|\phi_{\beta,a}\|_{Q_s^*} \leq C_1 2^{\frac{s}{2}} \beta|a|$ for all $a \in \mathbb{D}$, $0 < \beta < 1$ and $0 < s \leq 1$. By the assumption there exists a positive constant C_2 such that

$$\frac{\beta|a|}{2} \frac{|\varphi'(z)|}{|1 - \bar{a}\varphi(z)|}(1 - |z|^2)^\alpha \leq (\phi_{\beta,a} \circ \varphi)^*(z)(1 - |z|^2)^\alpha \leq C_2\|\phi_{\beta,a}\|_{Q_s^*} \leq C_2 C_1 2^{\frac{s}{2}} \beta|a|,$$

and the assertion $\varphi \in \mathcal{B}_\alpha^*$ follows by choosing $a = \varphi(z)$. The case $s = 0$ can be proved in a similar manner by choosing the test function $\varphi_a(z)/2$.

If $h \in \mathcal{B}^* = B(\mathbb{D})$, then

$$(h \circ \varphi)^*(z)(1 - |z|^2)^\alpha \leq \|h\|_{\mathcal{B}^*} \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2}(1 - |z|^2)^\alpha \leq \|h\|_{\mathcal{B}^*}\|\varphi\|_{\mathcal{B}_\alpha^*},$$

and $(4) \Rightarrow (2)$ follows. $\square$

Proof of Theorem 1.4. To prove that (2) implies (1), the reasoning in the proof of [10, Theorem 2.2] is followed. By Lemma C and the subharmonicity of $|f'(z)|^2$, there is a positive constant C_1 such that

$$\int_\mathbb{D} |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) = \int_\mathbb{D} |f'(w)|^2 d\mu_{a,s_2}(w)$$
$$\leq C_1 \int_\mathbb{D} \left(\frac{1}{(1 - |w|^2)^2} \int_{D(w,1/2)} |f'(z)|^2 dA(z) \right) d\mu_{a,s_2}(w),$$

where $d\mu_{a,s_2}(w) = N_{\varphi \circ \varphi_a, s_2}(w)dA(w)$. Then the symmetry $\chi_{D(z,r)}(w) = \chi_{D(w,r)}(z)$ of the characteristic functions of pseudohyperbolic discs and Fubini's theorem yield

$$\int_\mathbb{D} |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) \leq C_1 \int_\mathbb{D} |f'(z)|^2 \left(\int_{D(z,1/2)} \frac{d\mu_{a,s_2}(w)}{(1 - |w|^2)^2} \right) dA(z). \tag{3.3}$$

By Lemmas B and C, the assumption (2) is equivalent to

$$\sup_{a \in \mathbb{D}} \int_{D(z,1/2)} d\mu_{a,s_2}(w) \leq C_2(1 - |z|^2)^{2+s_1}, \quad z \in \mathbb{D}, \tag{3.4}$$

for some positive constant C_2. Since $1 - |w| \simeq 1 - |z|$ for $w \in D(z, 1/2)$, it follows by (3.3) and (3.4) that $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded.

Suppose then that $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded. For $a \in \mathbb{D}$, define $f_a(z) = \int_0^z (\varphi_a'(w))^{1+\frac{s_1}{2}} \, dw$. Then, by Forelli-Rudin estimates [19, Lemma 4.2.2] there is a positive constant C_1, depending only on s_1, such that

$$\|f_a\|_{\mathcal{D}_{s_1}}^2 = (1 - |a|^2)^{1+\frac{s_1}{2}} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{s_1}}{|1 - \overline{a}z|^{2+s_1}} \, d\sigma(z) \leq C_1$$

for all $a \in \mathbb{D}$, and thus the family $\{f_a : a \in \mathbb{D}\}$ is norm bounded uniformly in $\mathcal{D}_{s_1}$. Since $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded, there is a positive constant C_2 such that

$$\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 (1 - |\varphi_b(z)|^2)^{s_2} \, d\sigma(z) = \|f_a \circ \varphi\|_{Q_{s_2}}^2 \leq C_2 \|f_a\|_{\mathcal{D}_{s_1}}^2 \leq C_1 C_2$$

for all $a \in \mathbb{D}$, and the condition (2) follows.

Since (2) and (3) are clearly equivalent, it is proceeded to consider the hyperbolic case. Suppose that $s_1 \leq 0$ and $s_2 \leq 1$. If (2) is satisfied, then the same reasoning as in the first part of the proof shows that $C_\varphi : \mathcal{D}_{s_1}^* \to Q_{s_2}^*$ is bounded since also $(h^*)^2$ is a subharmonic function in $\mathbb{D}$.

Suppose then that $C_\varphi : \mathcal{D}_{s_1}^* \to Q_{s_2}^*$ is bounded. For $a \in \mathbb{D}$ and $\frac{1}{2} < \gamma \leq 1$ let

$$f_{a,\gamma}(z) = \int_0^z (\varphi_a'(w))^\gamma \, dw = \begin{cases} \frac{(1-|a|^2)^\gamma}{\overline{a}(1-2\gamma)} \left((1 - \overline{a}z)^{1-2\gamma} - 1 \right) , & a \in \mathbb{D} \setminus \{0\} \\ z & , \ a = 0 \end{cases} \tag{3.5}$$

and

$$h_{a,\gamma}(z) = \begin{cases} \frac{(2\gamma-1)\overline{a}}{6} f_{a,\gamma}(z) , & a \in \mathbb{D} \setminus \{0\} \\ \frac{z}{2} & , \ a = 0. \end{cases} \tag{3.6}$$

Then $\|h_{a,\gamma}\|_\infty \leq \frac{1}{2}$ for all $a \in \mathbb{D}$, and therefore $h_{a,\gamma}^*(z) \simeq |h_{a,\gamma}'(z)|$ in $\mathbb{D}$. The reasoning in the proof of the implication $(1) \Rightarrow (2)$ with the functions

$$h_{a,1+s_1/2}(z) = \frac{(1+s_1)\overline{a}}{6} \int_0^z (\varphi_a'(w))^{1+\frac{s_1}{2}} \, dw$$

yields the assertion (2). $\qquad \square$

Proof of Theorem 1.7. Suppose first that $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is compact and consider the functions $f_a(z) = \int_0^z (\varphi_a'(w))^{1+\frac{s_1}{2}} \, dw$. Since, by the proof of Theorem 1.4, there is a positive constant C such that $\|f_a\|_{\mathcal{D}_{s_1}} \leq C$ for all $a \in \mathbb{D}$, and further $f_a \to 0$ uniformly on compact subsets as $|a| \to 1$, Lemma D gives (2).

Suppose now that (2) holds. Let $\{f_n\} \subset \mathcal{D}_{s_1}$ such that $\|f_n\|_{\mathcal{D}_{s_1}} \leq C_1 < \infty$ for all $n \in \mathbb{N}$, and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma D, it suffices to show that $\|f_n \circ \varphi\|_{Q_{s_2}} \to 0$ as $n \to \infty$. For $0 < \delta < 1$, let $\Delta(0, \delta)$ denote the Euclidean disc centered at the origin and of radius δ. A similar reasoning as in the proof of Theorem 1.4 with the fact $1 - |w|^2 \simeq 1 - |z|^2 \simeq |1 - \bar{z}w|$, $w \in D(z, 1/2)$, yields

$$
\begin{aligned}
\|f_n \circ \varphi\|_{Q_{s_2}}^2 &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_n'(w)|^2 \, d\mu_{a,s_2}(w) \\
&\leq C_2 \int_{\mathbb{D}} |f_n'(z)|^2 (1 - |z|^2)^{s_1} \left(\sup_{a \in \mathbb{D}} \int_{D(z,1/2)} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,s_2}(w) \right) dA(z) \\
&\leq C_2 \int_{\mathbb{D} \setminus \Delta(0,\delta)} |f_n'(z)|^2 (1 - |z|^2)^{s_1} \left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,s_2}(w) \right) dA(z) \\
&\quad + \frac{C_2 4^{1+s_1}}{(1-\delta)^2} \|\varphi\|_{Q_{s_2}}^2 \int_{\Delta(0,\delta)} |f_n'(z)|^2 dA(z) \\
&= I_1(\delta) + I_2(\delta),
\end{aligned}
$$

where C_2 is a positive constant. For a given $\varepsilon > 0$, by the assumption (2) and Lemma C, there exists a $\delta_0 \in (0, 1)$ such that, for all $|z| > \delta_0$,

$$
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,s_2}(w) \leq \frac{\varepsilon^2}{2C_1^2 C_2},
$$

and it follows that $I_1(\delta_0) < \varepsilon^2/2$. In view of Theorem 1.4, the assumption (2) implies that $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded, and hence $\varphi \in Q_{s_2}$. Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, in particular, in $\Delta(0, \delta_0)$, there exists an $N \in \mathbb{N}$ such that, for $n \geq N$,

$$
\int_{\Delta(0,\delta_0)} |f_n'(z)|^2 dA(z) < \frac{\varepsilon^2 (1 - \delta_0)^2}{2C_2 4^{1+s_1} \|\varphi\|_{Q_{s_2}}^2},
$$

and therefore $I_2(\delta_0) < \varepsilon^2/2$. Thus, as $n \geq N$, $\|f_n \circ \varphi\|_{Q_{s_2}} < \varepsilon$, and $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is compact by Lemma D.

Since (2) and (3) are clearly equivalent, the proof is complete. $\qquad\square$

Proof of Theorem 1.8. Suppose first that $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2,0}$ is bounded, that is, $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded and $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$. Then, by using the functions $f(z) = z \in \mathcal{D}_{s_1}$ and $f_a(z) = \int_0^z (\varphi_a'(w))^{1+\frac{s_1}{2}} \, dw \in \mathcal{D}_{s_1}$, the inclusion $C_\varphi(\mathcal{D}_{s_1}) \subset Q_{s_2,0}$ implies that (1) and (2) are satisfied.

Suppose now that $C_\varphi : \mathcal{D}_{s_1} \to Q_{s_2}$ is bounded, and the conditions (1) and (2) are satisfied. It suffices to show that $f \circ \varphi \in Q_{s_2,0}$ for $f \in \mathcal{D}_{s_1}$. A similar reasoning as in the

proof of Theorem 1.7 yields

$$
\begin{aligned}
\int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) \leq C &\int_{\mathbb{D} \setminus \Delta(0,t)} |f'(z)|^2 (1 - |z|^2)^{s_1} \\
&\cdot \left(\int_{\mathbb{D}} |\varphi_z'(w)|^{2+s_1} d\mu_{a,s_2}(w) \right) dA(z) \\
&+ C \int_{\Delta(0,t)} |f'(z)|^2 \left(\int_{D(z,1/2)} \frac{d\mu_{a,s_2}(w)}{(1 - |z|^2)^2} \right) dA(z) \\
&= I_1(t) + I_2(t),
\end{aligned}
\tag{3.7}
$$

where C is a positive constant and $0 < t < 1$. To deal with $I_1(t)$, write

$$
\begin{aligned}
\int_{\mathbb{D}} |\varphi_z'(w)|^{2+s_1} d\mu_{a,s_2}(w) &= \int_{\Delta(0,t)} |\varphi_z'(w)|^{2+s_1} d\mu_{a,s_2}(w) + \int_{\mathbb{D} \setminus \Delta(0,t)} |\varphi_z'(w)|^{2+s_1} d\mu_{a,s_2}(w) \\
&\leq \frac{2^{2+s_1}}{(1 - t)^{2+s_1}} \int_{\mathbb{D}} |\varphi'(u)|^2 g^{s_2}(u, a) \, d\sigma(u) \\
&+ \int_{\mathbb{D} \setminus \Delta(0,t)} |\varphi_z'(w)|^{2+s_1} d\mu_{a,s_2}(w) \\
&= I_3(t) + I_4(t).
\end{aligned}
$$

By the assumption (2), for a given $\varepsilon > 0$, there exists a $\delta_1 \in (0, 1)$ such that

$$
I_4(t) < \frac{\varepsilon^2}{3C\|f\|_{\mathcal{D}_{s_1}}^2}
\tag{3.8}
$$

for all $|a|, |z|, t \geq \delta_1$. Let $t \geq \delta_1$ be fixed. Since $\varphi \in Q_{s_2,0}$ by the assumption (1), there exists a $\delta_2 \in [\delta_1, 1)$ such that

$$
I_3(t) < \frac{\varepsilon^2 (1 - t)^{2+s_1}}{3 \cdot 2^{2+s_1} C \|f\|_{\mathcal{D}_{s_1}}^2}
\tag{3.9}
$$

for all $|a| \geq \delta_2$. Since $|z| \geq t$ in the term $I_1(t)$, it follows by combining (3.7), (3.8) and (3.9) that $I_1(t) \leq 2\varepsilon^2/3$ for $|a| \geq \delta_2$. Further, since $\varphi \in Q_{s_2,0}$ by the assumption (1), there exists a $\delta_3 \in [\delta_2, 1)$ such that

$$
I_2(t) \leq \frac{C\pi t^2}{(1 - t^2)^2} \sup_{|z|=t} |f'(z)|^2 \int_{\mathbb{D}} |\varphi'(z)|^2 g^{s_2}(z, a) \, dA(z) < \frac{\varepsilon^2}{3}
$$

for all $|a| \geq \delta_3$. Therefore one finally concludes

$$
\int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 g^{s_2}(z, a) \, dA(z) \leq I_1(t) + I_2(t) < \frac{2\varepsilon^2}{3} + \frac{\varepsilon^2}{3} = \varepsilon^2
$$

for all $|a| \geq \delta_3$, that is, $f \circ \varphi \in Q_{s_2,0}$ as one wished to prove. $\qquad \square$

Proof of Theorem 1.9. As it was pointed out in Section 1, Theorem 1.9 follows by Theorem 1.8. $\qquad \square$

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