Isometries of some classical function spaces among the composition operators

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Dedicated to Professor Joseph Cima on the occasion of his 70th birthday

Abstract. We give a simple and unified proof of the characterizations of all possible composition operators that are isometries of either a general Hardy space or a general weighted Bergman spaces of the disk. We do the same for the isometries of analytic Besov spaces (containing the Dirichlet space) among the composition operators with univalent symbols.

Introduction

Throughout this note, $dm(\theta) = (2\pi)^{-1}d\theta$ will denote the normalized arc length measure on the unit circle $T$. We assume that the reader is familiar with the definition of the standard Hardy spaces $H^p$ of the disk (see [D], for example). We write $dA$ for the normalized Lebesgue area measure on the unit disk $D$: $dA(re^{i\theta}) = \pi^{-1}rdrd\theta$. The weighted Bergman space $A^p_w$ is the space of all $L^p(D, w \, dA)$ functions analytic in the disk, where $w$ is a radial weight function: $w(z) = w(|z|)$, non-negative and integrable with respect to $dA$. Every $H^p$ is a Banach space when $1 \leq p < \infty$, and so is $A^p_w$ when the weight $w$ is “reasonable” (whenever the point evaluations are bounded; roughly speaking, $w$ should not be zero “too often” near the unit circle). The unweighted Bergman space $A^p$ is obtained when $w \equiv 1$ (see [DS] for the theory of these spaces); the standard weighted space $A^p_\alpha$ corresponds to the case $w(z) = (\alpha + 1)(1 - |z|^2)^\alpha$, $-1 < \alpha < \infty$.

Given an analytic function $\varphi$ in the unit disk $D$ such that $\varphi(D) \subset D$, the composition operator $C_\varphi$ with symbol $\varphi$ defined by $C_\varphi f(z) = f(\varphi(z))$ is always bounded on any $H^p$ or $A^p_\alpha$ space, in view of Littlewood’s Subordination Theorem. The monographs [S1] and [CM] are standard sources for the theory of composition operators on such spaces.

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Being a Hilbert space, the Hardy space \( H^2 \) has plenty of isometries. However, the only isometries of \( H^2 \) among the composition operators are the operators induced by inner functions that vanish at the origin. Nordgren ([N], p. 444) showed that if \( \varphi \) is inner and \( \varphi(0) = 0 \) then \( C_\varphi \) is an isometry of \( H^2 \) (alternatively, see p. 321 of [CM]). The converse follows, for example, from a result of Shapiro ([S2], p. 66). According to Cloud [C], this characterization of isometries of \( H^2 \) among the composition operators had already been obtained in the unpublished thesis of Howard Schwarz [S]. Bayart [B] recently showed that every composition operator on \( H^2 \) which is similar to an isometry is induced by an inner function with a fixed point in the disk.

The surjective isometries of the more general \( H^p \) spaces have been described by Forelli [F] as weighted composition operators. A characterization of all isometries of \( H^p \) does not seem to be known. In this note we prove that the only isometries (surjective or not) of \( H^p \), \( 1 \leq p < \infty \), among the composition operators are again induced by inner functions that vanish at the origin (see Theorem 1.3 below). This fact may be known to some experts so our emphasis is on the method of proof, which also works for Bergman spaces.

Kolaski [K] (see also [DS], §2.8) gave a characterization of all surjective isometries of a weighted Bergman space \( A^p_\alpha \) similar to that of Forelli’s. Again, no characterization of all isometries of these spaces seems to be known. The (Hilbert) Bergman space \( A^2_\alpha \), of course, possesses plenty of isometries. In a recent preprint Carswell and Hammond [CH] have shown, among other results, that the only composition operators that are isometries of the weighted (Hilbert) Bergman space \( A^2_\alpha \) are the rotations. We prove an analogous statement (Theorem 1.3) for an arbitrary space \( A^p_\alpha \) with a radial weight, \( p \geq 1 \), where Hilbert space methods no longer work.

The surjective isometries of the Bloch space have been characterized in a well known work by Cima and Wogen [CW] while the surjective isometries of the general analytic Besov spaces \( B^p \) and some related Dirichlet-type spaces have been described more recently by Hornor and Jamison [HJ]. Recall that an analytic function in the disk is said to belong to the space \( B^p \) if its derivative is in the weighted Bergman space \( A^p_{p-2} \). These spaces form an important scale of Möbius-invariant spaces that includes the Dirichlet space \( (p = 2) \) and the Bloch space (as a limit case as \( p \to \infty \)). They have been studied by many authors (see [AFP], [Z], [DGV] for some details).

The isometries (not necessarily surjective) among the composition operators acting on the Dirichlet space \( B^2 \) have been characterized in [MV]. Here we describe all isometries of Besov spaces \( B^p, 2 < p < \infty \), among the composition operators with univalent symbols (Theorem 1.4). The proof follows a similar pattern to that of the proofs for Hardy and Bergman spaces, with some variations typical of analytic Besov spaces.

1. Main results and their proofs

1.1. Hardy and Bergman spaces. We characterize all isometries among the composition operators on the general \( H^p \) and \( A^p \) spaces by giving an essentially unified proof. The crucial point in both statements is that the symbol \( \varphi \) of any isometry \( C_\varphi \) must fix the origin. Once this has been established, we proceed using a simple idea that is probably known to some experts, at least in the Hilbert space context \( (p = 2) \). In order to prove the claim about fixing the origin, we first establish
an auxiliary result similar to several others that are often used in the theory of best approximation.

**Lemma 1.1.** Let $\mu$ be a positive measure on the measure space $\Omega$, $\mathcal{M}$ a subspace of $L^p(\Omega, d\mu)$, $1 \leq p < \infty$, and let $T$ be a linear isometry of $\mathcal{M}$ (not necessarily onto). Then

$$\int_{\Omega} T|Tf|^p d\mu = \int_{\Omega} |f|^p d\mu$$

for all $f, g$ in the subspace $\mathcal{M}$.

**Proof.** We apply the standard method of variation of (differentiation with respect to) the parameter. Given two arbitrary functions $f, g$ in $\mathcal{M}$, define the function $N_{f,g}(t) = \int_{\Omega} |tf + g|^p d\mu$, $t \in \mathbb{R}$.

Then, as described in Theorem 2.6 of [LL] (with $f$ and $g$ permuted for our convenience),

$$N'_{f,g}(0) = \frac{p}{2} \int_{\Omega} |g|^{p-2}(g\overline{f} + f\overline{g}) d\mu.$$

Since $T$ is a linear isometry of $\mathcal{M}$, we have $N_{Tf, Tg}(t) = N_{f, g}(t)$. After evaluating the derivative of each side at $t = 0$ we get

$$\int_{\Omega} |g|^{p-2}(g\overline{f} + f\overline{g}) d\mu = \int_{\Omega} |Tg|^{p-2}(Tg\overline{Tf} + Tf\overline{Tg}) d\mu.$$

Since this holds for arbitrary $f$ and $g$ we may also replace $g$ by $ig$. After a cancellation, this yields

$$\int_{\Omega} |g|^{p-2}(g\overline{f} - f\overline{g}) d\mu = \int_{\Omega} |Tg|^{p-2}(Tg\overline{Tf} - Tf\overline{Tg}) d\mu.$$

Summing up the last two identities, we get

$$\int_{\Omega} |g|^{p-2} g\overline{f} d\mu = \int_{\Omega} |Tg|^{p-2} g\overline{Tf} d\mu,$$

which implies the desired formula. \qed

From now on we assume that the weight $w$ is not only radial but behaves “reasonably well” in the sense that $A^p_w$ is a complete space.

**Proposition 1.2.** If a composition operator $C_\varphi$ is an isometry (not necessarily onto) of either $H^p$ or $A^p_w$, $1 \leq p < \infty$, then $\varphi(0) = 0$.

**Proof.** Consider $\mathcal{M} = H^p$, a subspace of $L^p(\mathbb{T}, dm)$, and $\mathcal{M} = A^p_w$, a subspace of $L^p(\mathbb{D}, w dA)$, respectively. Then set $g \equiv 1$ and $Tf = C_\varphi f$ in Lemma 1.1 and use the standard reproducing property for the origin to get

$$f(\varphi(0)) = \int_\mathbb{T} C_\varphi f dm = \int_\mathbb{T} f dm = f(0)$$

in the case of $H^p$. For the weighted Bergman space $A^p_w$, use the Mean Value Property to get

$$\int_{\mathbb{D}} f w dA = 2 \int_0^1 \left( \int_0^{2\pi} f(re^{i\theta}) dm(\theta) \right) w(r) r dr = 2 \int_0^1 f(0) w(r) r dr = c_w f(0)$$
(for some positive constant $c_w$) and, similarly,
\[
\int_D (f \circ \varphi) w dA = 2 \int_0^1 \left( \int_0^{2\pi} (f \circ \varphi)(re^{i\theta}) d\theta \right) w(r) r dr = c_w f(\varphi(0)),
\]
hence also $f(\varphi(0)) = f(0)$. Finally, choose the identity map: $f(z) \equiv z$ to deduce the statement in both cases. \hfill \Box

Proposition 1.2 could have been established by other methods but we decided to give preference to the application of Lemma 1.1 from approximation theory.

**Theorem 1.3.** Let $1 \leq p < \infty$. Then:

(a) A composition operator $C_\varphi$ is an isometry of $H^p$ if and only if $\varphi$ is inner and $\varphi(0) = 0$.

(b) A composition operator $C_\varphi$ is an isometry of $A^p_w$ if and only if $\varphi$ is a rotation.

**Proof.** (a) Since $C_\varphi$ is an isometry, we have $\|z\|_{H^p} = \|\varphi\|_{H^p}$, hence
\[
0 = \|z\|_{H^p}^p - \|\varphi\|_{H^p}^p = \int_T (1 - |\varphi|^p) \, dm.
\]
Since $|\varphi| \leq 1$ almost everywhere on $T$, it follows that $1 - |\varphi|^p = 0$ almost everywhere on $T$ and, thus, $\varphi$ is inner. We already know from the Corollary that $\varphi(0) = 0$.

(b) In view of the Corollary ($\varphi(0) = 0$) and the Schwarz Lemma, we get that $|\varphi(z)| \leq |z|$ for all $z$ in $D$. Since $w$ is assumed to be a nontrivial weight, it must be strictly positive on a set of positive measure in $D$, hence the equality
\[
0 = \|z\|_{A^p_w}^p - \|\varphi\|_{A^p_w}^p = \int_D \left( |z|^p - |\varphi|^p \right) w(z) \, dA
\]
is only possible if $|\varphi(z)| = |z|$ throughout $D$, that is, when $\varphi$ is a rotation. \hfill \Box

### 1.2. Analytic Besov spaces.

The **analytic Besov space** $B^p$, $1 < p < \infty$, is defined as the set of all analytic functions in the disk such that
\[
\|f\|_{B^p} = |f(0)|^p + s_p(f) = |f(0)|^p + (p - 1) \int_D |f'(z)|^p (1 - |z|^2)^{p-2} \, dA(z) < \infty.
\]
These spaces are Banach spaces with the **Möbius-invariant seminorm** $s_p$, in the sense that $s_p(f \circ \phi) = s_p(f)$ for every disk automorphism $\phi$. It is well known that $B^p \subset B^q$ when $p < q$. For the theory of $B^p$ and other conformally invariant spaces of analytic functions in the disk, we refer the reader to [AFP], [DGV], and [Z], for example.

A general composition operator on $B^p$ is not necessarily bounded, roughly speaking because “too many points of the disk can be covered too many times” by $\varphi$ (for an exact condition for the boundedness in terms of the counting function, see [AFP]). However, the boundedness of $C_{\varphi}$ is guaranteed for every univalent symbol $\varphi$ when $p \geq 2$. Indeed, after applying the Schwarz-Pick Lemma and the change of
variable \( w = \varphi(z) \), we get

\[
s_p^p(f \circ \varphi) = (p - 1) \int_D |f'(\varphi(z))|^p|\varphi'(z)|^p(1 - |z|^2)^{p-2}dA(z)
\]
\[
\leq (p - 1) \int_D |f'(\varphi(z))|^p(1 - |\varphi(z)|^2)^{p-2}|\varphi'(z)|^2dA(z)
\]
\[
= (p - 1) \int_{\varphi(D)} |f'(w)|^p(1 - |w|^2)^{p-2}dA(w)
\]
\[
\leq (p - 1) \int_D |f'(w)|^p(1 - |w|^2)^{p-2}dA(w),
\]
showing that \( s_p(f \circ \varphi) \leq s_p(f) \) for all \( f \) in \( B^p \), whenever \( \varphi \) is univalent and \( p \geq 2 \).

**Theorem 1.4.** Let \( \varphi \) be a univalent self-map of the disk and \( 2 < p < \infty \). Then the induced composition operator \( C_\varphi \) is an isometry of \( B^p \) if and only if \( \varphi \) is a rotation.

**Proof.** The sufficiency of the condition is trivial.

For the necessity, suppose \( C_\varphi \) is an isometry of \( B^p \), \( 2 < p < \infty \). We first show that again we must have \( \varphi(0) = 0 \). Since \( s_p(f \circ \varphi) \leq s_p(f) \) in this case and

\[
||f \circ \varphi||_{B^p} = |f(\varphi(0))|^p + s_p(f \circ \varphi) = |f(0)|^p + s_p(f) = ||f||_{B^p}^p,
\]

it follows that \( |f(\varphi(0))| \geq |f(0)| \) for all \( f \) in \( B^p \). Writing \( \varphi(0) = a \) and choosing \( f \) to be the standard disk automorphism

\[
\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}
\]

that interchanges the origin and the point \( a \), we get

\[
0 = |\varphi_a(a)| = |\varphi_a(\varphi(0))| \geq |\varphi_a(0)| = |a|,
\]
hence \( \varphi(0) = 0 \). This proves the claim.

Thus, if \( C_\varphi \) is an isometry of \( B^p \), it must satisfy \( s_p(f \circ \varphi) = s_p(f) \) for all \( f \) in \( B^p \). Choose \( f(z) \equiv z \). Using the definition of \( s_p \), the Schwarz-Pick Lemma (note that \( p - 2 > 0 \) by assumption), and the change of variable \( \varphi(z) = w \), we get

\[
0 = s_p(z)^p - s_p(\varphi)^p
\]
\[
= (p - 1) \int_D (1 - |z|^2)^{p-2}dA(z) - (p - 1) \int_D |\varphi'(z)|^p(1 - |z|^2)^{p-2}dA(z)
\]
\[
\geq 1 - (p - 1) \int_D (1 - |\varphi(z)|^2)^{p-2}|\varphi'(z)|^2dA(z)
\]
\[
= 1 - (p - 1) \int_{\varphi(D)} (1 - |w|^2)^{p-2}dA(w)
\]
\[
\geq 1 - (p - 1) \int_D (1 - |w|^2)^{p-2}dA(w)
\]
\[
= 0,
\]
hence equality must hold throughout. In particular, it must hold in the Schwarz-Pick Lemma (in the third line of the chain above) and so \( \varphi \) must be a disk automorphism. Since it also fixes the origin, it follows that \( \varphi \) is actually a rotation. \( \square \)
Note that the requirement that $\varphi$ be univalent was crucial near the end of the proof. Also, the requirement that $p > 2$ was fundamental because the key inequality above becomes reversed when $p < 2$. The above proof still works when $p = 2$ (the Dirichlet space) as we no longer need the Schwarz lemma but in this case it only follows that the normalized area of $\varphi(\mathbb{D})$ equals the area of $\mathbb{D}$ itself, that is, $\varphi$ must be a (univalent) map of $\mathbb{D}$ onto a subset of full measure. All such maps that fix the origin generate isometries, a fact already proved without the univalence assumption in [MV]. It would be interesting to know which composition operators acting on $B^p$ are isometries when $1 < p < 2$.

References


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