WEIGHTED WEIERSTRASS' THEOREM WITH FIRST DERIVATIVES

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ABSTRACT

We characterize the set of functions which can be approximated by continuous functions with the norm $\|f\|_{L^\infty(w)}$ for every weight w. We characterize as well the set of functions which can be approximated by smooth functions with the norm

$$\|f\|_{W^{1,\infty}(w_0,w_1)} := \|f\|_{L^\infty(w_0)} + \|f'\|_{L^\infty(w_1)}\,,$$

for a wide range of (even non-bounded) weights w_j's. We allow a great deal of independence among the weights w_j's.

Key words and phrases: Weierstrass' theorem; weight; Sobolev spaces; weighted Sobolev spaces.

1. INTRODUCTION

If I is any compact interval, Weierstrass' Theorem says that $C(I)$ is the largest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere. There are many generalizations of this theorem (see e.g. the monographs [L], [P], and the references therein).

In [R1] and [PQRT] we study the same problem with the norm $L^\infty(I,\, w)$ defined by

$$\|f\|_{L^\infty(I,\,w)} := \operatorname*{ess\,sup}_{x\in I} |f(x)|w(x)\,, \tag{1.1}$$

where w is a weight, i.e. a non-negative measurable function and we use the convention $0\cdot\infty = 0$. Notice that (1.1) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct one when working with weights (see e.g. [BO] and [DMS]). In [PQRT] we improve the theorems in [R1], obtaining sharp results for a large class of weights. Here we also study this problem both with the norm (1.1) for *every* weight w, and with the Sobolev norm $W^{1,\infty}(I, w_0, w_1)$ defined by

$$\|f\|_{W^{1,\infty}(I,w_0,w_1)} := \big\|f\big\|_{L^\infty(I,w_0)} + \big\|f'\big\|_{L^\infty(I,w_1)}\,,$$

since in many situations it is natural to consider the simultaneous approximation of a function and its first derivative.

Considering weighted norms $L^\infty(w)$ has been proved to be interesting mainly because of two reasons: on the one hand, it allows to wider the set of approximable functions (since the functions in $L^\infty(w)$ can have singularities where the weight tends to zero); and, on the other one, it is possible to find functions which approximate f whose qualitative behaviour is similar to the one of f at those points where the weight tends to infinity.

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics, as Approximation Theory, Partial Differential Equations (with or without Numerical Methods), and Quasiconformal and Quasiregular maps (see e.g. [HKM], [IKNS1], [IKNS2], [K], [Ku], [KO] and [KS]). In particular, in [IKNS1] and [IKNS2], the authors showed that the expansions with Sobolev orthogonal polynomials can avoid the Gibbs phenomenon which appears with classical orthogonal series in L^2. In [ELW1], [EL] and [ELW2] the authors study some examples of Sobolev spaces for $p = 2$ with respect to general measures instead of weights,

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in relation with ordinary differential equations and Sobolev orthogonal polynomials. The papers [RARP1], [RARP2], [R1], [R2] and [R3] are the beginning of a theory of Sobolev spaces with respect to general measures for $1 \le p \le \infty$. This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see [LP], [LPP], [RARP2] and [R2]). The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]). The papers [APRR], [BFM], [CM], [FMP], [LPP] and [RY] deal with Sobolev spaces on curves and more general subsets of the complex plane.

In this paper we characterize the set of functions which can be approximated by continuous functions in $L^\infty(I, w)$, for any weight w (see Theorem 2.1); as a consequence of this result, we obtain the set of functions which can be approximated by polynomials in $L^\infty(I, w)$, for any weight w with compact support. We also characterize the set of functions which can be approximated by C^1 functions in $W^{1,\infty}(I, w_0, w_1)$, for a wide range of (possibly unbounded) weights w_0, w_1, which have a great deal of independence among them. It is a remarkable fact that this last characterization depends on the value $L(a) := \operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x)$ at every singular point a of w_1 (see definitions 2.4 and 2.6 below). Depending on the value $L(a) = 0$, $0 < L(a) < \infty$ or $L(a) = \infty$, theorems 4.2, 4.3 and 4.4 describe, respectively, the set of functions which can be approximated by C^1 functions in $W^{1,\infty}(I, w_0, w_1)$, when there is just one singular point of w_1. Furthermore, some of the conditions appearing in the characterizations are not obvious at all. Besides, we would like to remark that our methods of proof are constructive. The main result in Sobolev approximation is Theorem 4.5, which gives the characterization with infinitely many singular points of w_1 (even for non-bounded intervals), combining the results of theorems 4.2, 4.3 and 4.4.

We use these results in order to study the approximation by C^∞ functions as well (see Theorem 5.2).

The outline of the paper is as follows: In Section 2 we find the closure of continuous functions in $L^\infty(I, w)$. Section 3 is dedicated to definitions and previous results. Section 4 presents the theorems on approximation by C^1 functions in $W^{1,\infty}(I, w_0, w_1)$. We prove the results on approximation by C^∞ functions in Section 5.

2. APPROXIMATION IN $L^\infty(I, w)$

Let us start with some definitions.

Definition 2.1. A *weight* w is a measurable function $w : \mathbf{R} \longrightarrow [0, \infty]$. If w is only defined in $A \subset \mathbf{R}$, we set $w := 0$ in $\mathbf{R} \setminus A$.

Definition 2.2. Given a measurable set $A \subset \mathbf{R}$ and a weight w, we define the space $L^\infty(A, w)$ as the space of equivalence classes of measurable functions $f : A \longrightarrow \mathbf{R}$ with respect to the norm

$$\|f\|_{L^\infty(A,w)} := \operatorname*{ess\,sup}_{x\in A} |f(x)| w(x) .$$

We always consider the space $L^1(A)$, with respect to the restriction of the Lebesgue measure on A.

The theorems in this paper can be applied to functions f with complex values, splitting f into its real and imaginary parts. From now on, if we do not specify the set A, we are assuming that $A = \mathbf{R}$; analogously, if we do not make explicit the weight w, we are assuming that $w \equiv 1$.

Definition 2.3. Given a measurable set A, we define the *essential closure* of A, as the set

$$\operatorname{ess\,cl} A := \big\{ x \in \mathbf{R} :\ |A \cap (x - \delta, x + \delta)| > 0, \ \ \forall\, \delta > 0 \big\} ,$$

where $|E|$ denotes the Lebesgue measure of E.

Definition 2.4. If A is a measurable set, f is a function defined on A with real values and $a \in \operatorname{ess\,cl} A$, we say that $\operatorname{ess\,lim}_{x\in A,\, x\to a} f(x) = l \in \mathbf{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for almost every $x \in A \cap (a - \delta, a + \delta)$. In a similar way we can define $\operatorname{ess\,lim}_{x\in A,\, x\to a} f(x) = \infty$ and $\operatorname{ess\,lim}_{x\in A,\, x\to a} f(x) = -\infty$. We define the *essential superior limit* and the *essential inferior limit* on A as follows:

$$\operatorname*{ess\,lim\,sup}_{x\in A,\, x\to a} f(x) := \inf_{\delta>0} \operatorname*{ess\,sup}_{x\in A\cap(a-\delta,a+\delta)} f(x) ,$$

$$\operatorname*{ess\,lim\,inf}_{x\in A,\, x\to a} f(x) := \sup_{\delta>0} \operatorname*{ess\,inf}_{x\in A\cap(a-\delta,a+\delta)} f(x) .$$

Remarks.

1. The essential superior (or inferior) limit of a function f does not change if we modify f on a set of zero Lebesgue measure.

2. When we say that there exists a essential limit (or essential superior limit or essential inferior limit), we are assuming that it is finite.

3. It is well known that

$$\operatorname*{ess\,lim\,sup}_{x\in A,\, x\to a} f(x) \geq \operatorname*{ess\,lim\,inf}_{x\in A,\, x\to a} f(x)\,,$$
$$\operatorname*{ess\,lim}_{x\in A,\, x\to a} f(x) = l \quad \text{if and only if} \quad \operatorname*{ess\,lim\,sup}_{x\in A,\, x\to a} f(x) = \operatorname*{ess\,lim\,inf}_{x\in A,\, x\to a} f(x) = l\,.$$

4. We impose the condition $a \in \operatorname{ess\,cl} A$ in order to have the unicity of the essential limit. If $a \notin \operatorname{ess\,cl} A$, then every real number is an essential limit for any function f.

Definition 2.5. Given a weight w, the *support* of w, denoted by $\operatorname{supp} w$, is the complement of the largest open set $G \subset \mathbf{R}$ with $w = 0$ a.e. on G.

Definition 2.6. Given a weight w we say that $a \in \operatorname{supp} w$ is a *singularity* of w (or *singular* for w) if

$$\operatorname*{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a} w(x) = 0\,.$$

We say that a singularity a of w is of *type* 1 if $\operatorname{ess\,lim}_{x\to a} w(x) = 0$.

We say that a singularity a of w is of *type* 2 if $0 < \operatorname{ess\,lim\,sup}_{x\to a} w(x) < \infty$.

We denote by $S(w)$ and $S_i(w)$ $(i = 1, 2)$, respectively, the set of singularities of w and the set of singularities of w of type i.

We say that $a \in S^+(w)$ (respectively $a \in S^-(w)$) if $\operatorname{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a^+} w(x) = 0$ (respectively $\operatorname{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a^-} w(x) = 0$).

We say that $a \in S_i^+(w)$ (respectively $a \in S_i^-(w)$) if a verifies the property in the definition of $S_i(w)$ when we take the limit as $x \to a^+$ (respectively $x \to a^-$).

Definition 2.7. Given a weight w, we define the *right regular* and *left regular* points of w, respectively, as

$$R^+(w) := \big\{ a \in \operatorname{supp} w : \operatorname*{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a^+} w(x) > 0 \big\}\,, \qquad R^-(w) := \big\{ a \in \operatorname{supp} w : \operatorname*{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a^-} w(x) > 0 \big\}\,.$$

The following result characterizes the set of functions which can be approximated by continuous functions in $L^\infty(w)$, for any weight w.

Theorem 2.1. *Let w be any weight and*

$$\begin{aligned} H_0 := \big\{ f \in L^\infty(w) : \, & f \ \text{ is continuous to the right at every point of } R^+(w),\\ & f \ \text{ is continuous to the left at every point of } R^-(w),\\ & \text{for each } a \in S^+(w), \ \operatorname*{ess\,lim}_{x\to a^+} |f(x) - f(a)|\, w(x) = 0\,,\\ & \text{for each } a \in S^-(w), \ \operatorname*{ess\,lim}_{x\to a^-} |f(x) - f(a)|\, w(x) = 0 \,\big\}\,. \end{aligned}$$

Then:

(a) The closure of $C(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is H_0.

(b) If $w \in L^\infty_{loc}(\mathbf{R})$, then the closure of $C^\infty(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also H_0.

(c) If $\operatorname{supp} w$ is compact and $w \in L^\infty(\mathbf{R})$, then the closure of the space of polynomials is H_0 as well.

(d) If $f \in H_0 \cap L^1(\operatorname{supp} w)$, $S_1^+(w) \cup S_2^+(w) \cup S_1^-(w) \cup S_2^-(w)$ is countable and $|S(w)| = 0$, then f can be approximated by functions in $C(\mathbf{R})$ with the norm $\| \cdot \|_{L^\infty(w)} + \| \cdot \|_{L^1(\operatorname{supp} w)}$.

Remark. Recall that we identify functions which are equal almost everywhere.

As a consequence of this result and Theorem A below, we characterize the set of functions which can be approximated by polynomials in $L^\infty(w)$, for any weight w with compact support.

Definition 2.8. Given a weight w with compact support, a polynomial $p \in L^\infty(w)$ is said to be *a minimal polynomial for* w if every polynomial in $L^\infty(w)$ is a multiple of p. A minimal polynomial for w is said to be *the minimal polynomial for* w (and we denote it by p_w) if it is 0 or it is monic.

It is clear that there always exists a minimal polynomial for w (although it can be 0): it is sufficient to consider a polynomial in $L^\infty(w)$ of minimal degree. Minimal polynomials for w are unique except for a constant factor; this fact allows to define p_w.

Theorem A. [PQRT, Theorem 2.2] *Let us consider a weight w with compact support. If $p_w \equiv 0$, then the closure of the space of polynomials in $L^\infty(w)$ is $\{0\}$. If p_w is not identically 0, the closure of the space of polynomials in $L^\infty(w)$ is the set of functions f such that f/p_w is in the closure of the space of polynomials in $L^\infty(|p_w|w)$.*

Remark. The weight $|p_w|w$ is bounded (since $p_w \in L^\infty(w)$) and has compact support. Then we know which is the closure of the space of polynomials in $L^\infty(|p_w|w)$ by Theorem 2.1.

In the proof of Theorem 2.1 we need the following lemma.

Lemma 2.1. *Let us consider a weight w with $a \in S_1^+(w) \cup S_2^+(w)$. Let us fix $\eta > 0$ and $f \in L^\infty(w)$ such that* $\operatorname{ess\,lim}_{x\to a^+} |f(x) - f(a)|w(x) = 0$. *Then, there exists $b_3 \in (a, a+1)$ such that for any $a < b_1 < b_2 < b_3$ there exist $b_0 \in (b_1, b_2)$ and a function $g \in L^\infty(w) \cap C([a, b_0])$, with $g = f$ in $\mathbf{R} \setminus (a, b_0)$, $\|f - g\|_{L^\infty(w)} < \eta$ (and $\|f - g\|_{L^1(\operatorname{supp} w)} < \eta$ if $f \in L^1(\operatorname{supp} w)$).*

Remark. A similar result is true if $a \in S_1^-(w) \cup S_2^-(w)$.

Proof. Let us fix $\varepsilon > 0$. Since $a \in S_1^+(w) \cup S_2^+(w)$, $\operatorname{ess\,lim\,sup}_{x\to a^+} w(x) = m \in [0, \infty)$. It follows that there exists $\delta_1 > 0$ such that $w(x) \le m + 1$, *a.e.* $x \in (a, a + \delta_1)$.

If $f \in L^1(\operatorname{supp} w)$, there exists $\delta_2 > 0$, such that $\|f - f(a)\|_{L^1([a,a+\delta_2]\cap \operatorname{supp} w)} < \varepsilon$. If $f \notin L^1(\operatorname{supp} w)$, we take $\delta_2 := 1$.

By hypothesis, there exists $0 < \delta < \min\{\delta_1, \delta_2, 1\}$ such that $|f(x) - f(a)|w(x) < \varepsilon$, *a.e.* $x \in (a, a + \delta)$.

Let us define $b_3 := a + \delta$ and let us consider $a < b_1 < b_2 < b_3$. Let us consider $c := \inf_{x\in(b_1,b_2)} |f(x) - f(a)|$. Then, there exists $b_0 \in (b_1, b_2)$ such that $|f(b_0) - f(a)| < \varepsilon + c \le \varepsilon + |f(x) - f(a)|$ for every $x \in (b_1, b_2)$. Let us choose $s > 0$ small enough such that $(b_0 - s, b_0) \subseteq (b_1, b_2)$. Then, we define the function g as

$$g(x) := \begin{cases} f(a)\,, & \text{if } x \in (a, b_0 - s]\,, \\ f(b_0) + (f(b_0) - f(a))(x - b_0)/s\,, & \text{if } x \in (b_0 - s, b_0)\,, \\ f(x)\,, & \text{if } x \notin (a, b_0)\,. \end{cases}$$

Let us remark that g is continuous in $[a, b_0]$ and $g = f$ in $\mathbf{R} \setminus (a, b_0)$.

It is obvious that $|f(a) - g(x)| \le |f(a) - g(b_0)| = |f(a) - f(b_0)|$ for every $x \in [a, b_0]$.

$$\begin{aligned} \|f - g\|_{L^\infty(w)} &= \|f - g\|_{L^\infty([a,b_0],w)} \le \|f - f(a)\|_{L^\infty([a,b_0],w)} + \|f(a) - g\|_{L^\infty([a,b_0],w)} \\ &\le \|f - f(a)\|_{L^\infty([a,b_0],w)} + \|f(a) - f(b_0)\|_{L^\infty([a,b_0],w)} \\ &\le 2\|f - f(a)\|_{L^\infty([a,b_0],w)} + \|\varepsilon\|_{L^\infty([a,b_0],w)} \le 2\varepsilon + (m+1)\varepsilon \ = (3+m)\varepsilon\,. \end{aligned}$$

If $f \in L^1(\operatorname{supp} w)$, we also have

$$\begin{aligned} \|f - g\|_{L^1(\operatorname{supp} w)} =& \|f - f(a)\|_{L^1([a,b_0-s]\cap \operatorname{supp} w)} + \|f - f(b_0) - (f(b_0) - f(a))(x - b_0)/s\|_{L^1([b_0-s,b_0]\cap \operatorname{supp} w)} \\ \le& \|f - f(a)\|_{L^1([a,b_0-s]\cap \operatorname{supp} w)} + \|f - f(a)\|_{L^1([b_0-s,b_0]\cap \operatorname{supp} w)} \\ &\quad + 2\|f(a) - f(b_0)\|_{L^1([b_0-s,b_0]\cap \operatorname{supp} w)} \\ \le& \|f - f(a)\|_{L^1([a,b_0]\cap \operatorname{supp} w)} + 2\|f - f(a)\|_{L^1([b_0-s,b_0]\cap \operatorname{supp} w)} + 2\|\varepsilon\|_{L^1([b_0-s,b_0]\cap \operatorname{supp} w)} \\ \le& 3\|f - f(a)\|_{L^1([a,b_0]\cap \operatorname{supp} w)} + 2\varepsilon s < 3\varepsilon + 2\varepsilon s < 5\varepsilon\,. \end{aligned}$$

This finishes the proof of the lemma. $\sharp$

Proof of Theorem 2.1. Items (b), (c) and (d) are direct consequences of (a) (see the proof in [PQRT, Proposition 2.1 and Theorem 2.1]). The proof of the inclusion of the closure of $C(\mathbf{R}) \cap L^\infty(w)$ in H_0 is not difficult (see the proof in [PQRT, Proposition 2.1 and Theorem 2.1]). In order to prove the other inclusion, let us fix $f \in H_0$. The proof has several ingredients: Lemma 2.1 allows to modify f in a neighborhood of each singular pint of w; then we need to paste these modifications in an appropriate way.

Fix $\eta > 0$. Let us assume that $a \in (S_1^-(w) \cup S_2^-(w)) \cap (S_1^+(w) \cup S_2^+(w))$. Then Lemma 2.1 gives intervals $[b_0^-, a]$, $[a, b_0^+]$ and functions $g^- \in L^\infty(w) \cap C([b_0^-, a])$, $g^+ \in L^\infty(w) \cap C([a, b_0^+])$, with $g^- = f$ in $\mathbf{R} \setminus (b_0^-, a)$, $\|f - g^-\|_{L^\infty(w)} < \eta$, $g^+ = f$ in $\mathbf{R} \setminus (a, b_0^+)$, $\|f - g^+\|_{L^\infty(w)} < \eta$. Without loss of generality we can assume that $r^- := a - b_0^- \le b_0^+ - a$. If $b_0^+ - a \le 21r^-/20$, we define $r^+ := b_0^+ - a$ and $g_0 := g^+$. If $b_0^+ - a > 21r^-/20$, Lemma 2.1 allows to find $r^+ \in [r^-, 21r^-/20]$ and a function $g_0 \in L^\infty(w) \cap C([a, a + r^+])$, with $g_0 = f$ in $\mathbf{R} \setminus (a, a + r^+)$, $\|f - g_0\|_{L^\infty(w)} < \eta$. Hence, the function g defined by

$$g(x) := \begin{cases} g^-(x)\,, & \text{if } x \in [a - r^-, a]\,, \\ g_0(x)\,, & \text{if } x \in [a, a + r^+]\,, \\ f(x)\,, & \text{in other case}\,, \end{cases}$$

verifies $g \in L^\infty(w) \cap C([a - r^-, a + r^+])$, $g = f$ in $\mathbf{R} \setminus (a - r^-, a + r^+)$ and $\|f - g\|_{L^\infty(w)} < \eta$.

If $a \in (S_1^-(w) \cup S_2^-(w)) \cap R^+(w)$ (or if $a \in (S_1^+(w) \cup S_2^+(w)) \cap R^-(w)$), we can also obtain such an interval and such an approximating function. Using this result, we can follow the arguments of the proofs of [PQRT, Proposition 2.1 and Theorem 2.1] in order to obtain a way to "paste" the approximations to f in each singular point (in these arguments it is crucial to have $20/21 \le r^+/r^- \le 21/20$). This finishes the proof of the theorem. $\sharp$

3. SOBOLEV SPACES AND PREVIOUS RESULTS

We state here an useful technical result which was proved in [PQRT].

Lemma A. [PQRT, Lemma 2.1] *Let us consider a weight w and $a \in \operatorname{supp} w$. If $\operatorname{ess\,lim\,sup}_{x\to a} w(x) = l \in (0, \infty]$, then for every function f in the closure of $C(\mathbf{R}) \cap L^\infty(w)$ with the norm $L^\infty(w)$, we have that*

$$\operatorname*{ess\,lim}_{x\to a,\, w(x)\ge\eta} f(x) = f(a)\,, \qquad \textit{for every} \quad 0 < \eta < l\,.$$

Remark. A similar result is true if we change both limits when $x \to a$ by $x \to a^+$ (or $x \to a^-$).

In order to control a function from its derivative, we need the following version (see a proof in [RARP1, Lemma 3.2]) of Muckenhoupt inequality (see [Mu], [M, p.44]).

Lemma B. *Let us consider w_0, w_1 weights in $[\alpha, \beta]$ and $a \in [\alpha, \beta]$. Then there exists a positive constant c such that*

$$\Big\| \int_a^x g(t)\, dt \Big\|_{L^\infty([\alpha,\beta],w_0)} \le c\, \|g\|_{L^\infty([\alpha,\beta],w_1)}$$

for any measurable function g in $[\alpha, \beta]$, if and only if

$$\operatorname*{ess\,sup}_{\alpha<x<\beta} w_0(x) \Big| \int_a^x 1/w_1 \Big| < \infty\,.$$

We deal now with the definition of Sobolev spaces $W^{1,\infty}(w_0, w_1)$.

We follow the approach in [KO]. First of all, notice that the distributional derivative of a function f in Ω is a function belonging to $L^1_{loc}(\Omega)$. If $f' \in L^\infty(\Omega, w_1)$, in order to get the inclusion

$$L^\infty(\Omega, w_1) \subseteq L^1_{loc}(\Omega)\,,$$

a sufficient condition, is that the weight w_1 satisfies $1/w_1 \in L^1_{loc}(\Omega)$ (see e.g. the proof of Proposition 4.3 below). Consequently, $f \in AC_{loc}(\Omega)$, i.e. f is an absolutely continuous function on every compact interval contained in Ω, if $1/w_1 \in L^1_{loc}(\Omega)$.

Given two weights w_0, w_1, let us denote by Ω the largest set (which is a union of intervals) such that $1/w_1 \in L^1_{loc}(\Omega)$. We always require that $\operatorname{supp} w_1 = \overline{\Omega}$. We define *the Sobolev space* $W^{1,\infty}(w_0, w_1)$, as the set of all (equivalence classes of) functions $f \in L^\infty(w_0) \cap AC_{loc}(\Omega)$ such that their weak derivative f' in Ω belongs to $L^\infty(w_1)$.

With this definition, the weighted Sobolev space $W^{1,\infty}(w_0, w_1)$ is a Banach space (see [KO, Section 3]). In general, this is not true without our hypotheses (see some examples in [KO]).

4. APPROXIMATION BY C^1 FUNCTIONS IN $W^{1,\infty}(I, w_0, w_1)$

The main result of this section is Theorem 4.5, which characterizes the functions which can be approximated by C^1 functions in $W^{1,\infty}(w_0, w_1)$, under very weak hypotheses on w_0, w_1. We obtain it by means of some previous lemmas and theorems.

Lemma 4.1. *Let us consider $\lambda \in \mathbf{R}$ and $u \in C([a-\delta_0, a))$. For each $0 < \delta < \delta_0$ there exists $v \in C([a-\delta_0, a])$ with $v(x) = u(x)$ if $x \notin (a-\delta, a)$, $|v(x) - u(a)| \le 2|u(x) - u(a)|$ for every $x \in [a-\delta_0, a)$, and there exists $\eta > 0$ with $v(x) = u(a)$ if $x \in [a-\eta, a]$. Furthermore, if we define $U(x) := \int_{a-\delta_0}^x u$, $V(x) := \int_{a-\delta_0}^x v$, we also have:*

(i) $V(a) = U(a-)$ and $|V(x) - U(a-)| \le |U(x) - U(a-)| + 2|u(a)||x-a|$ for every $x \in [a-\delta_0, a)$, if there exists $U(a-) := \lim_{x\to a^-} U(x)$,

(ii) $V(a) = \lambda$ and $|V(x) - \lambda| \le |U(x) - \lambda| + 2|u(a)||x-a|$ for every $x \in [a-\delta_0, a)$, if $\lim_{x\to a^-} U(x)$ does not exist.

Remarks.

1. Notice that the value $u(a)$ does not need to have any relation with the values of u in $[a-\delta_0, a)$.

2. A similar result is true for $u \in C((a, a+\delta_0])$.

Proof. Our goal is to construct a function V which approximates U, which is equal to U far away from a and whose graph is a stright line r near a. In order to do this, we will make two changes on u: the first one, v_1, will have a primitive intersecting r, and the second one, v_2, will make smooth the connection with r.

It is clear that we can assume that $a = 0$. We only consider the case $u(0) > 0$; the case $u(0) < 0$ is similar and the case $u(0) = 0$ is easier.

(i) Let us assume that there exists $U(0-) := \lim_{x\to 0^-} U(x)$.

(1) Consider first the case $U(x) > r(x) := U(0-) + u(0)x$, for every point in some interval $(-\delta', 0)$, with $\delta' < \delta_0$. If $u(x) = u(0)$ for every x in a left neighborhood of 0, it is sufficient to take $v := u$. If this is not so, it is possible to choose $0 < \delta_2 < \delta_1 < \min\{\delta, \delta'\}$ and a function $v_1 \in C([-\delta_0, 0))$ with $v_1(x) = u(x)$ if $x \notin (-\delta_1, -\delta_2)$, $v_1(x) < u(x)$ if $x \in (-\delta_1, -\delta_2)$, $|v_1(x) - u(0)| \le 2|u(x) - u(0)|$ for every x; then $V_1(x) \le U(x)$ for every x, if $V_1(x) := \int_{-\delta_0}^x v_1$. It is clear that $\lim_{x\to 0^-} V_1(x) < U(0-)$, and consequently there exists a minimum $-\delta_3 \in (-\delta_1, 0)$ with $V_1(-\delta_3) = r(-\delta_3)$; this implies that $V_1'(-\delta_3) = v_1(-\delta_3) \le u(0) = r'(-\delta_3)$, since $V_1(-\delta_1) = U_1(-\delta_1) > r(-\delta_1)$.

(1.1) If $v_1(-\delta_3) < u(0)$, let us choose $0 < \varepsilon_1 < \delta_1 - \delta_3$ and $0 < \varepsilon_2 < \delta_3/2$ with $v_1(x) < u(0)$ for $x \in [-\delta_3 - \varepsilon_1, -\delta_3 + \varepsilon_2]$. Let us consider a function $v_2 \in C([-\delta_3 - \varepsilon_1, -\delta_3 + \varepsilon_2])$ with $v_1 \le v_2 \le u(0)$, $v_2(-\delta_3 - \varepsilon_1) = v_1(-\delta_3 - \varepsilon_1)$, $v_2(-\delta_3 + \varepsilon_2) = u(0)$, and $\int_{-\delta_3-\varepsilon_1}^{-\delta_3} (v_2 - v_1) = \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) \le u(0)\delta_3/2$. We define $v(x) := v_1(x)$ if $x < -\delta_3 - \varepsilon_1$, $v(x) := v_2(x)$ if $x \in [-\delta_3 - \varepsilon_1, -\delta_3 + \varepsilon_2]$, and $v(x) := u(0)$ if $x > -\delta_3 + \varepsilon_2$. It is clear that $v \in C([-\delta, 0])$ and $|v(x) - u(0)| \le |v_1(x) - u(0)| \le 2|u(x) - u(0)|$ for every x.

If $V(x) := \int_{-\delta_0}^x v$, notice that $V(x) = V_1(x) = U(x)$ if $x \le -\delta_1$, and $V(x) = V_1(x)$ if $x \in [-\delta_1, -\delta_3 - \varepsilon_1]$. It is obvious that $r(x) \le V_1(x) \le U(x)$ if $x \in [-\delta_1, -\delta_3]$; consequently

$$u(0)x \le V_1(x) - U(0-) \le U(x) - U(0-)\,,$$
$$|V_1(x) - U(0-)| \le \max\{|U(x) - U(0-)|, |u(0)x|\} \le |U(x) - U(0-)| + |u(0)x|\,,$$

if $x \in [-\delta_1, -\delta_3]$; now it is direct that this inequality also holds for $x \in [-\delta_0, -\delta_3]$. Therefore $|V(x) - U(0-)| = |V_1(x) - U(0-)| \le |U(x) - U(0-)| + |u(0)x|$ if $x \in [-\delta_0, -\delta_3 - \varepsilon_1]$.

Let us consider $x \in [-\delta_3 - \varepsilon_1, -\delta_3]$; on the one hand, if x satisfies $V(x) \le U(0-)$, we have that $|V(x) - U(0-)| \le |V_1(x) - U(0-)| \le |U(x) - U(0-)| + |u(0)x|$, since $V_1(x) \le V(x)$; on the other hand, if x satisfies $V(x) > U(0-)$, then

$$-u(0)x \ge u(0)\delta_3/2 \ge \int_{-\delta_3-\varepsilon_1}^{-\delta_3} (v_2 - v_1) \ge \int_{-\delta_3-\varepsilon_1}^{x} (v_2 - v_1) = V(x) - V_1(x)\,,$$
$$V(x) - U(0-) \le V_1(x) - U(0-) - u(0)x \le U(x) - U(0-) - u(0)x \le |U(x) - U(0-)| + |u(0)x|\,;$$

it follows, in any case, that $|V(x) - U(0-)| \le |U(x) - U(0-)| + |u(0)x|$ if $x \in [-\delta_3 - \varepsilon_1, -\delta_3]$.

If $x \in [-\delta_3, -\delta_3 + \varepsilon_2]$, then $V(x) \ge V_1(x)$; it is clear that

$$-u(0)x \ge u(0)(\delta_3-\varepsilon_2) \ge u(0)\delta_3/2 \ge \int_{-\delta_3-\varepsilon_1}^{-\delta_3} (v_2-v_1) = V(-\delta_3)-V_1(-\delta_3) = V(-\delta_3)-r(-\delta_3) \ge V(x)-r(x)\,,$$

if $x \in [-\delta_3, -\delta_3 + \varepsilon_2]$ (since $v_2 \le u(0)$), and hence $V(x) - U(0-) \le 0$; we also have

$$\begin{aligned}-u(0)x &\ge \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) \ge \int_{-\delta_3}^{x} (u(0) - v_2) = r(x) - r(-\delta_3) - V(x) + V(-\delta_3)\\ &\ge r(x) - r(-\delta_3) - V(x) + V_1(-\delta_3) = r(x) - V(x)\,,\end{aligned}$$

if $x \in [-\delta_3, -\delta_3 + \varepsilon_2]$, and hence $V(x) - U(0-) \ge r(x) - U(0-) + u(0)x = 2u(0)x$ in this interval; it follows that $|V(x) - U(0-)| \le 2|u(0)x|$ if $x \in [-\delta_3, -\delta_3 + \varepsilon_2]$.

If $x \in [-\delta_3 + \varepsilon_2, 0)$, then $V(x) = r(x)$, since $V'(x) = v(x) = u(0) = r'(x)$ in this interval, and

$$\begin{aligned}r(-\delta_3 + \varepsilon_2) - V(-\delta_3 + \varepsilon_2) &= \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) + r(-\delta_3) - V(-\delta_3)\\ &= \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) - \big(V(-\delta_3) - V_1(-\delta_3)\big)\\ &= \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) - \int_{-\delta_3-\varepsilon_1}^{-\delta_3} (v_2 - v_1) = 0\,.\end{aligned}$$

Hence $V(x) - U(0-) = u(0)x$ and $|V(x) - U(0-)| = |u(0)x|$ if $x \in [-\delta_3 + \varepsilon_2, 0)$.

(1.2) If $v_1(-\delta_3) = u(0)$, we define $v(x) := v_1(x)$ if $x \le -\delta_3$ and $v(x) := u(0)$ if $x > -\delta_3$. We can argument as in the case $v_1(-\delta_3) < u(0)$.

(2) If $U(x) < r(x) := U(0-) + u(0)x$, for every point in a left neighborhood of 0, we can use a similar construction of v (taking now $v_1 \ge u$).

(3) If $U(x_n) = r(x_n)$, for a sequence $x_n \nearrow 0$, it is also possible to use a similar construction of v (taking $v_1 = u$ and $-\delta_3 = x_n$ for some n large enough).

(*ii*) Let us assume now that $\lim_{x\to 0^-} U(x)$ does not exist; then $u \notin L^1([-\delta_0, 0])$.

(1) Consider first the case $U(x) > r(x) := \lambda + u(0)x$, for every point in a left neighborhood of 0. The function $u_0 := u(0) - |u - u(0)|$ verifies $|u_0 - u(0)| = |u - u(0)|$ and $\lim_{x\to 0^-} \int_{-\delta_0}^{x} u_0 = -\infty$. It is possible to choose $0 < \delta_2 < \delta_1 < \delta$ and a function $v_1 \in C([-\delta_0, 0])$ with $v_1(x) = u(x)$ if $x \le -\delta_1$, $v_1(x) = u_0(x)$ if $x \ge -\delta_2$, and $|v_1(x) - u(0)| \le 2|v(x) - u(0)|$ for every x. If $V_1(x) := \int_{-\delta_0}^{x} v_1$, it is clear that $\lim_{x\to 0^-} V_1(x) = -\infty$, and consequently there exists a minimum $-\delta_3 \in (-\delta_1, 0)$ with $V_1(-\delta_3) = r(-\delta_3)$.

Now it is sufficient to choose the functions v_2 and v as in the case (*i*), and do the same computations.

(2) If $U(x) < r(x) := \lambda + u(0)x$, for every point in a left neighborhood of 0, we can repeat the argument with $u_1 := u(0) + |u - u(0)|$ instead of u_0.

(3) If $U(x_n) = r(x_n)$, for a sequence $x_n \nearrow 0$, it is also possible to use a similar construction of v (taking $v_1 = u$ and $-\delta_3 = x_n$ for some n large enough). ♯

Definition 4.1. Let us consider a weight w_1 such that $S(w_1) \cap [a-\delta, a+\delta] = \{a\}$ for some $\delta > 0$. We say that w_1 is *left-dominated at* a if there exists a constant c such that any function $F \in C([a-\delta, a])$ with $0 \le F \le 1/w_1$ a.e. verifies $\int_{a-\delta}^a F \le c$. We say that w_1 is *right-dominated at* a if there exists a constant c such that any function $F \in C([a, a+\delta])$ with $0 \le F \le 1/w_1$ a.e. verifies $\int_a^{a+\delta} F \le c$. We denote by $D^-(w_1)$ (respectively, $D^+(w_1)$) the set of left-dominated (respectively, right-dominated) points of w_1.

Remarks.

1. Every weight w_1 with $1/w_1 \in L^1([a, a+\delta])$ is right-dominated at a.

2. There exists weights w_1 right-dominated at a, with $1/w_1 \notin L^1([a, a+\delta])$: Let us consider a Borel set $E \subset [0,1]$ with $0 < |E \cap I| < |I|$ for every interval $I \subset [0,1]$ (see e.g. [Ru, Chapter 2]). Since $\int_E dx/x + \int_{[0,1]\setminus E} dx/x = \int_0^1 dx/x = \infty$, without loss of generality we can assume that $\int_E dx/x = \infty$ (in other case we can take $[0,1] \setminus E$ instead of E). Then, $w_1(x) := x\chi_E(x) + \chi_{[0,1]\setminus E}(x)$ is right-dominated at 0 and $1/w_1 \notin L^1([0,1])$.

Lemma 4.2. *Let us consider a weight w_1 in $[a-\delta, a]$ with $S(w_1) = \{a\}$. Then $a \notin D^-(w_1)$ if and only if there exists a function $F \in C([a-\delta, a))$ with $0 \le F \le 1/w_1$ a.e. and $\int_{a-\delta}^a F = \infty$.*

Proof. Let us assume that there exists a function $F \in C([a-\delta, a))$ with $0 \le F \le 1/w_1$ a.e. and $\int_{a-\delta}^a F = \infty$. For each n we can consider a function $F_n \in C([a-\delta, a])$ with $0 \le F_n \le F \le 1/w_1$ a.e. and $F = F_n$ in $[a-\delta, a-1/n]$. Then $\lim_{n\to\infty} \int_{a-\delta}^a F_n = \int_{a-\delta}^a F = \infty$ and $a \notin D^-(w_1)$.

Let us assume now that $a \notin D^-(w_1)$. Then, for each n there exists a function $F_n \in C([a-\delta, a])$ with $0 \le F_n \le 1/w_1$ a.e. and $\int_{a-\delta}^a F_n > n$. Let us choose $a_n \in (a-1/n, a)$ with $\int_{a-\delta}^{a_n} F_n > n$. Since $S(w_1) = \{a\}$, then $1/w_1 \in L^1_{loc}([a-\delta, a))$, and consequently $\int_{a-\delta}^x F_n \le \int_{a-\delta}^x 1/w_1 \in C([a-\delta, a))$. Therefore, there exists a subsequence $\{a_{n_k}\}_k$ with $\int_{a_{n_{k-1}}}^{a_{n_k}} F_{n_k} > 1$, and hence we can construct a function $F \in C([a-\delta, a))$ with $0 \le F \le F_{n_k} \le 1/w_1$ a.e. in $[a_{n_{k-1}}, a_{n_k}]$ and $\int_{a_{n_{k-1}}}^{a_{n_k}} F > 1$. Then $\int_{a-\delta}^a F = \infty$. $\sharp$

Lemma 4.3. *Let us consider two weights w_0, w_1, in $[a-\delta_0, a]$ with $\operatorname{ess\,lim}_{x\to a^-} w_0(x) = 0$ and $a \notin D^-(w_1)$. Then for each $f \in W^{1,\infty}(w_0, w_1) \cap C^1([a-\delta_0, a])$, each $\delta, \varepsilon > 0$, and each $s \in \mathbf{R}$, there exists $g \in C^1([a-\delta_0, a])$ with $\|f - g\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon$, $g(x) = f(x)$ if $x \notin (a-\delta, a]$, $g' = f'$ in some neighborhood of a, and $g(a) = s$.*

Remark. A similar result is true for $f \in W^{1,\infty}(w_0, w_1) \cap C^1([a, a+\delta_0])$.

Proof. By Lemma 4.2, there exists a function $F \in C([a-\delta_0, a))$ with $0 \le F \le 1/w_1$ a.e. and $\int_{a-\delta_0}^a F = \infty$. Without loss of generality, we can assume that $a = 0$ and $s > f(0)$: the case $s < f(0)$ is similar, and the case $s = f(0)$ is trivial (it is sufficient to take $g = f$). Since $\operatorname{ess\,lim}_{x\to 0^-} w_0(x) = 0$, then there exists $0 < \delta_1 < \delta$ with $(s - f(0))w_0(x) < \varepsilon/3$ for almost every $x \in (-\delta_1, 0)$.

Since $F \in C([-\delta_1, 0))$, $F \ge 0$ and $\int_{-\delta_1}^0 F = \infty$, it is clear that we can find a function $J \in C_c([-\delta_1, 0))$ with $0 \le J \le \varepsilon F/2$ and $\int_{-\delta_1}^0 J = s - f(0)$. Let us define $h(x) := \int_{-\delta_1}^x J$ and $g := f + h$. Then we have $0 \le h(x) \le s - f(0)$. It is clear that $g(x) = f(x)$ if $x \notin (-\delta, 0]$, $g' = f'$ in some neighborhood of 0, and $g(0) = s$. We only need to check that $\|h\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon$, and this fact is a consequence of

$$\|h\|_{L^\infty(w_0)} = \operatorname*{ess\,sup}_{x\in[-\delta_1,0]} h(x)w_0(x) \le \operatorname*{ess\,sup}_{x\in[-\delta_1,0]} (s - f(0))w_0(x) \le \frac{\varepsilon}{3} < \frac{\varepsilon}{2}\,,$$

$$\|h'\|_{L^\infty(w_1)} = \operatorname*{ess\,sup}_{x\in[-\delta_1,0]} J(x)w_1(x) \le \frac{\varepsilon}{2} \operatorname*{ess\,sup}_{x\in[-\delta_1,0]} F(x)w_1(x) \le \frac{\varepsilon}{2}\,. \qquad \sharp$$

Lemma 4.4. *Let us consider two weights w_0, w_1, in $[a-\delta_0, a]$ with $S(w_1) = \{a\}$ and $a \in D^-(w_1)$. Let us assume that there exists $f \in W^{1,\infty}(w_0, w_1)$ and $\{g_n\}_n \in W^{1,\infty}(w_0, w_1) \cap C^1([a-\delta_0, a])$ converging to f in $W^{1,\infty}(w_0, w_1)$. Then $\{g_n'\}_n$ converges to f' in $L^1([a-\delta_0, a])$ and f is continuous to the left in a.*

Remark. A similar result is true if we change $[a-\delta_0, a]$ by $[a, a+\delta_0]$ everywhere.

Proof. Since $S(w_1) = \{a\}$, then $1/w_1 \in L^1_{loc}([a-\delta_0, a))$. For any $0 < \delta < \delta_0$, we obtain

$$\|f' - g_n'\|_{L^1([a-\delta_0,a-\delta])} = \int_{a-\delta_0}^{a-\delta} |f' - g_n'| \frac{w_1}{w_1} \le \|f' - g_n'\|_{L^\infty(w_1)} \int_{a-\delta_0}^{a-\delta} \frac{1}{w_1}\,.$$

Then, $\{g_n'\}_n$ converges to f' in $L^1([a-\delta_0, a-\delta])$, for any $0<\delta<\delta_0$. Furthermore, $\{g_n'\}_n$ is a Cauchy sequence in $L^1([a-\delta_0, a])$: Since $a \in D^-(w_1)$, there exists a constant c such that any function $F \in C([a-\delta_0, a])$ with $0 \le F \le 1/w_1$ a.e. verifies $\int_{a-\delta_0}^a F \le c$. We have $|g_n' - g_m'|/\|g_n' - g_m'\|_{L^\infty(w_1)} \le 1/w_1$ a.e., and hence $\int_{a-\delta_0}^a |g_n' - g_m'| \le c\|g_n' - g_m'\|_{L^\infty(w_1)}$. Therefore $\{g_n'\}_n$ converges to f' in $L^1([a-\delta_0, a])$.

Let us consider $\overline{g_n}(x) := g_n(x) - g_n(a-\delta_0) + f(a-\delta_0) \in C^1([a-\delta_0, a])$. Then $|f(x) - \overline{g_n}(x)| = \left|\int_{a-\delta_0}^x (f' - g_n')\right| \le \|f' - g_n'\|_{L^1([a-\delta_0,a])}$, for every $x \in [a-\delta_0, a]$. Consequently $\{\overline{g_n}\}_n$ converges uniformly to f in $[a-\delta_0, a]$ and f is continuous to the left in a. $\sharp$

The following definition makes sense because of Lemma A.

Definition 4.2. Let us consider a weight w_1. For each f with $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$, let us define $u_f(a) := 0$ if $a \in S_1(w_1)$, and $u_f(a) := \operatorname{ess\,lim}_{x\to a,\, w_1(x)\ge\eta} f'(x)$ for any $\eta > 0$ small enough if $a \notin S_1(w_1)$.

Let us remark that $u_f(a)$ is finite by Lemma A. We can state now our first theorem in this section.

Theorem 4.1. *Let us consider two weights w_0, w_1, in $[\alpha,\beta]$ such that $S(w_1) = \{a\}$, and $d > 0$. Then every function in*

$$\begin{aligned} H_1 := \Big\{ f \in W^{1,\infty}(w_0,w_1) : \;& f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)},\; f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}, \\ & f \text{ is continuous to the right if } a \in D^+(w_1), \\ & f \text{ is continuous to the left if } a \in D^-(w_1), \\ & \operatorname*{ess\,lim}_{x\to a} |f(x) - f(a)| w_0(x) = 0,\; \operatorname*{ess\,lim}_{x\to a} u_f(a)(x-a) w_0(x) = 0 \Big\}, \end{aligned}$$

can be approximated by functions $\{g_n\}_n$ in $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0,w_1)$ with the norm of $W^{1,\infty}(w_0,w_1)$ and with $g_n(x) = f(x)$ if $x \notin (a-d, a+d)$. Furthermore, if f also satisfies $\operatorname{ess\,lim}_{x\to a} |f'(x) - u_f(a)| w_1(x) = 0$, each function g_n is a polynomial of degree at most 1 in a neighborhood of a.

Remarks.

1. Notice that the hypothesis $\operatorname{ess\,lim}_{x\to a} u_f(a)(x-a)w_0(x) = 0$ for every function f with $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$, is a consequence of any of the following conditions:

(*a*) $\operatorname{ess\,lim}_{x\to a} (x-a) w_0(x) = 0$,

(*b*) $a \notin S_2(w_1)$, i.e. $\operatorname{ess\,lim}_{x\to a} w_1(x) = 0$ or $\operatorname{ess\,lim\,sup}_{x\to a} w_1(x) = \infty$ (in both cases, $u_f(a) = 0$).

2. Either of the following conditions guarantees $\operatorname{ess\,lim}_{x\to a} |f(x) - f(a)| w_0(x) = 0$ for every function $f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}$:

(*a*) $a \in S^+(w_0) \cap S^-(w_0)$, i.e., $\operatorname{ess\,lim\,inf}_{x\to a^+} w_0(x) = \operatorname{ess\,lim\,inf}_{x\to a^-} w_0(x) = 0$,

(*b*) $a \in S^+(w_0)$ and $w_0 \in L^\infty([a-\varepsilon, a])$, for some $\varepsilon > 0$,

(*c*) $a \in S^-(w_0)$ and $w_0 \in L^\infty([a, a+\varepsilon])$, for some $\varepsilon > 0$,

(*d*) $w_0 \in L^\infty([a-\varepsilon, a+\varepsilon])$, for some $\varepsilon > 0$.

3. Either of the following conditions guarantees $\operatorname{ess\,lim}_{x\to a} |f'(x) - u_f(a)| w_1(x) = 0$ for every function f with $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$:

(*a*) $a \in S^+(w_1) \cap S^-(w_1)$, i.e., $\operatorname{ess\,lim\,inf}_{x\to a^+} w_1(x) = \operatorname{ess\,lim\,inf}_{x\to a^-} w_1(x) = 0$,

(*b*) $a \in S^+(w_1)$ and $w_1 \in L^\infty([a-\varepsilon, a])$, for some $\varepsilon > 0$,

(*c*) $a \in S^-(w_1)$ and $w_1 \in L^\infty([a, a+\varepsilon])$, for some $\varepsilon > 0$,

(*d*) $a = \alpha$ or $a = \beta$ (since $a \in S(w_1)$).

4. Notice that we do not have any hypothesis about the singularities of w_0.

Proof. The heart of the proof is to use Lemma 4.1 in the approximation in $[\alpha, a]$ and the "right version" of Lemma 4.1 in the approximation in $[a, \beta]$. If these two approximations do not glue in a continuous way, we must use Lemma 4.3 in order to obtain a continuous function. Without loss of generality, we can assume that $a \in (\alpha,\beta)$, since the cases $a = \alpha$ and $a = \beta$ are easier (in these cases we do not use Lemma 4.3.

If $a \in S^-(w_1) \cap R^+(w_1)$, then every $f \in H_1$ belongs to $C^1([a,\beta])$, and we only need to apply Lemma 4.1; if $a \in S^+(w_1) \cap R^-(w_1)$, then every $f \in H_1$ belongs to $C^1([\alpha, a])$, and we only need to apply the "right version" of Lemma 4.1; then, without loss of generality, we can assume that $a \in S^+(w_1) \cap S^-(w_1)$, since the other cases are easier. In this case $a \in S^+(w_1) \cap S^-(w_1)$, every $f \in H_1$ satisfies $\operatorname{ess\,lim}_{x\to a} |f'(x) - u_f(a)| w_1(x) = 0$ (see Theorem 2.1 and Lemma A; in the case $a \in S_1(w_1)$ we have in fact $\operatorname{ess\,lim}_{x\to a} |f'(x) - \lambda| w_1(x) = 0$ for any $\lambda \in \mathbf{R}$, since $\operatorname{ess\,lim}_{x\to a} w_1(x) = 0$).

Let us consider any $f \in H_1$ and $\varepsilon > 0$. Let us define $u := f'$ in $[\alpha, \beta] \setminus \{a\}$ and $u(a) := u_f(a)$. Since $f \in H_1$, it is possible to choose $0 < \delta < d$ with

$$3\|f' - u(a)\|_{L^\infty([a-\delta,a+\delta],w_1)} < \frac{\varepsilon}{6}\,, \quad 4\|f - f(a)\|_{L^\infty([a-\delta,a+\delta],w_0)} < \frac{\varepsilon}{6}\,, \quad 4|u(a)|\,\|x - a\|_{L^\infty([a-\delta,a+\delta],w_0)} < \frac{\varepsilon}{6}\,.$$

We also require to δ that

$$\begin{aligned} &|f(x) - f(a-)| \le |f(x) - f(a)| \text{ for } x \in [a-\delta, a) \text{ if there exists } f(a-) \ne f(a), \text{ and} \\ &|f(x) - f(a+)| \le |f(x) - f(a)| \text{ for } x \in (a, a+\delta] \text{ if there exists } f(a+) \ne f(a). \end{aligned} \tag{4.1}$$

Let us define $U(x) := f(x) - f(\alpha) = \int_\alpha^x f'$ if $x \in [\alpha, a)$, and $U(x) := f(x) - f(\beta) = \int_\beta^x f'$ if $x \in (a, \beta]$. Consider the function $v \in C([\alpha, a])$ in Lemma 4.1 satisfying $v(x) = u(x)$ if $x \notin (a-\delta, a)$, $|v(x) - u(a)| \le 2|u(x) - u(a)|$ for every $x \in [\alpha, a)$,

$$V(a) = \begin{cases} f(a-) - f(\alpha)\,, & \text{if there exists } f(a-)\,, \\ f(a) - f(\alpha)\,, & \text{in other case}\,, \end{cases}$$

and $|V(x) - V(a)| \le |U(x) - V(a)| + 2|u(a)||x - a|$ for every $x \in [\alpha, a)$, if $V(x) := \int_\alpha^x v$. Consider also the function $\tilde v \in C([a, \beta])$ in the "right version" of Lemma 4.1 satisfying $\tilde v(x) = u(x)$ if $x \notin (a, a+\delta)$, $|\tilde v(x) - u(a)| \le 2|u(x) - u(a)|$ for every $x \in (a, \beta]$,

$$\tilde V(a) = \begin{cases} f(a+) - f(\beta)\,, & \text{if there exists } f(a+)\,, \\ f(a) - f(\beta)\,, & \text{in other case}\,, \end{cases}$$

and $|\tilde V(x) - \tilde V(a)| \le |U(x) - \tilde V(a)| + 2|u(a)||x - a|$ for every $x \in (a, \beta]$, if $\tilde V(x) := \int_\beta^x \tilde v$.

Let us consider the function g_0 given by $g_0(x) := V(x) + f(\alpha)$ if $x \in [\alpha, a]$, and $g_0(x) := \tilde V(x) + f(\beta)$ if $x \in (a, \beta]$. Notice that $g_0 \in C^1([\alpha, \beta] \setminus \{a\})$ and $g_0'(a-) = g_0'(a+) = u(a)$. In fact, g_0 is a polynomial of degree at most 1 in a left neighborhood (respectively right) of a, since $g_0'(x) = u(a)$ there (by Lemma 4.1).

This function also satisfies $g_0(x) = f(x)$ if $x \notin (a-\delta, a+\delta)$, and $|g_0'(x) - u(a)| \le 2|f'(x) - u(a)|$ for every $x \in [\alpha, \beta] \setminus \{a\}$. It follows that g_0 verifies

$$\begin{aligned} \|f - g_0\|_{W^{1,\infty}(w_0,w_1)} &= \|f - g_0\|_{L^\infty(w_0)} + \|f' - g_0'\|_{L^\infty(w_1)} \\ &= \max\big\{\|U - V\|_{L^\infty([a-\delta,a],w_0)}, \|U - \tilde V\|_{L^\infty([a,a+\delta],w_0)}\big\} + \|f' - g_0'\|_{L^\infty([a-\delta,a+\delta],w_1)} \\ &\le \|U - V(a)\|_{L^\infty([a-\delta,a],w_0)} + \|V - V(a)\|_{L^\infty([a-\delta,a],w_0)} \\ &\quad + \|U - \tilde V(a)\|_{L^\infty([a,a+\delta],w_0)} + \|\tilde V - \tilde V(a)\|_{L^\infty([a,a+\delta],w_0)} \\ &\quad + \|f' - u(a)\|_{L^\infty([a-\delta,a+\delta],w_1)} + \|g_0' - u(a)\|_{L^\infty([a-\delta,a+\delta],w_1)} \\ &\le 2\|U - V(a)\|_{L^\infty([a-\delta,a],w_0)} + 2|u(a)|\,\|x - a\|_{L^\infty([a-\delta,a],w_0)} \\ &\quad + 2\|U - \tilde V(a)\|_{L^\infty([a,a+\delta],w_0)} + 2|u(a)|\,\|x - a\|_{L^\infty([a,a+\delta],w_0)} \\ &\quad + 3\|f' - u(a)\|_{L^\infty([a-\delta,a+\delta],w_1)} \\ &\le 2\|f - f(a)\|_{L^\infty([a-\delta,a],w_0)} + 2|u(a)|\,\|x - a\|_{L^\infty([a-\delta,a],w_0)} \\ &\quad + 2\|f - f(a)\|_{L^\infty([a,a+\delta],w_0)} + 2|u(a)|\,\|x - a\|_{L^\infty([a,a+\delta],w_0)} \\ &\quad + 3\|f' - u(a)\|_{L^\infty([a-\delta,a+\delta],w_1)} \\ &\le 4\|f - f(a)\|_{L^\infty([a-\delta,a+\delta],w_0)} + 4|u(a)|\,\|x - a\|_{L^\infty([a-\delta,a+\delta],w_0)} \\ &\quad + 3\|f' - u(a)\|_{L^\infty([a-\delta,a+\delta],w_1)} \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}\,, \end{aligned}$$

where we have used (4.1) in the third inequality. In order to finish the proof we only need to construct a function $g \in C^1([\alpha,\beta])$ with $\|g-g_0\|_{W^{1,\infty}([\alpha,\beta],w_0,w_1)} < \varepsilon/2$, $g(x)=g_0(x)=f(x)$ if $x \notin (a-d,a+d)$ and $g'=g_0'=u(a)$ in a neighborhood of a.

Let us recall that $g_0(a-)=f(a-)$ if there exists $f(a-)$ and $g_0(a-)=f(a)$ in other case, $g_0(a+)=f(a+)$ if there exists $f(a+)$ and $g_0(a+)=f(a)$ in other case. We also have $g_0'(a-)=g_0'(a+)=u(a)$. Hence, $g_0 \in C^1([\alpha,\beta])$ if and only if $g_0(a-)=g_0(a+)$; in this case, it is sufficient to take $g := g_0$.

We analyze now the different cases:

(1) If $a \in D^-(w_1)\cap D^+(w_1)$, then $f \in C([\alpha,\beta])$. Therefore we can take $g := g_0$.

(2) Let us assume now that $a \notin D^-(w_1)\cap D^+(w_1)$.

(2.1) If there exist neither $f(a-)$ nor $f(a+)$, then we also have $g_0 \in C([\alpha,\beta])$.

(2.2) Let us assume that there exists $f(a-)$ and there not exists $f(a+)$ (the case in which there exists $f(a+)$ and there not exists $f(a-)$ is similar). If $f(a-)=f(a)$, it follows that $g_0 \in C([\alpha,\beta])$. If $f(a-)\neq f(a)$, it follows that $\operatorname{ess\,lim}_{x\to a^-} w_0(x)=0$ and $a \notin D^-(w_1)$: if $\operatorname{ess\,lim\,sup}_{x\to a^-} w_0(x)>0$, then Lemma A and its remark imply that $f(a)=\operatorname{ess\,lim}_{x\to a^-,\, w_0(x)\ge\eta} f(x)=f(a-)$, for any $\eta>0$ small enough, which is a contradiction; if $a\in D^-(w_1)$, then f is continuous to the left at a, which is a contradiction. Consequently we can apply Lemma 4.3 to $g_0|_{[\alpha,a]}$ in order to obtain a function $g \in C^1([\alpha,a])$ with $\|g-g_0\|_{W^{1,\infty}([\alpha,a],w_0,w_1)} < \varepsilon/2$, $g'(a-)=g_0'(a-)=g_0'(a+)$, $g(a)=g_0(a+)$ and $g(x)=g_0(x)=f(x)$ if $x\notin(a-d,a]$; if we define $g := g_0$ in $(a,\beta]$, this g is the required function.

Notice that lemmas 4.1 and 4.3 guarantee that g is a polynomial of degree at most 1 in a neighborhood of a, since g' is constant in a neighborhood of a.

(2.3) Finally, let us assume that there exist $f(a-)$ and $f(a+)$. If $f(a-)=f(a+)$, it follows that $g_0\in C([\alpha,\beta])$. If $f(a-)\neq f(a+)$, we consider two cases:

If $\operatorname{ess\,lim}_{x\to a} w_0(x)=0$, without loss of generality, we can assume that $a\notin D^-(w_1)$ (the case $a\notin D^+(w_1)$ is similar). Consequently we can apply Lemma 4.3 as in the case (2.2).

If $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x)>0$, without loss of generality, we can assume that $\operatorname{ess\,lim\,sup}_{x\to a^+} w_0(x)>0$ (the case $\operatorname{ess\,lim\,sup}_{x\to a^-} w_0(x)>0$ is similar). Then, Lemma A and its remark imply that $f(a)=\operatorname{ess\,lim}_{x\to a^+,\, w_0(x)\ge\eta} f(x)=f(a+)$. It follows that $\operatorname{ess\,lim}_{x\to a^-} w_0(x)=0$, since if this is not so, $f(a)=\operatorname{ess\,lim}_{x\to a^-,\, w_0(x)\ge\eta} f(x)=f(a-)$ and hence $f(a+)=f(a-)$, which is a contradiction. We also have $a\notin D^-(w_1)$, since if this is not so, f is continuous to the left at a, which is a contradiction. Consequently we can apply Lemma 4.3 as in the case (2.2).

This finishes the proof of the theorem. $\sharp$

Lemma 4.5. *Let us consider a weight w_0 with $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x)=\infty$ and $\operatorname{ess\,lim}_{x\to a}|x-a|w_0(x)=0$. If $f\in L^\infty(w_0)$ and $\|f\|_{L^\infty([a-\delta,a+\delta],w_0)}\ge c>0$ for every $\delta>0$, then $\operatorname{dist}_{L^\infty(w_0)}(f, C^1(\mathbf{R})\cap L^\infty(w_0))\ge c$.*

Proof. Without loss of generality, we can assume that $a=0$. If $g\in C^1(\mathbf{R})\cap L^\infty(w_0)$, then $g(0)=0$, since $\operatorname{ess\,lim\,sup}_{x\to 0} w_0(x)=\infty$, and consequently $\lim_{x\to0} g(x)/x=g'(0)$. It follows that

$$\operatorname*{ess\,lim}_{x\to0}|g(x)|w_0(x)=\Big(\operatorname*{ess\,lim}_{x\to0}\frac{|g(x)|}{|x|}\Big)\big(\operatorname*{ess\,lim}_{x\to0}|x|w_0(x)\big)=|g'(0)|\cdot 0=0\,.$$

Therefore, given any $\varepsilon>0$ there exists $\delta>0$ such that $\|g\|_{L^\infty([-\delta,\delta],w_0)}\le\varepsilon$. Hence

$$\|f-g\|_{L^\infty(w_0)}\ge\|f-g\|_{L^\infty([-\delta,\delta],w_0)}\ge\|f\|_{L^\infty([-\delta,\delta],w_0)}-\|g\|_{L^\infty([-\delta,\delta],w_0)}\ge c-\varepsilon\,,$$

for every $\varepsilon>0$, and consequently $\|f-g\|_{L^\infty(w_0)}\ge c$. $\sharp$

The three following theorems describe the set of functions which can be approximated by C^1 functions, when there is just one singular point of w_1.

Theorem 4.2. *Let us consider two weights w_0, w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$ and $\operatorname{ess\,lim}_{x\to a}|x-a|w_0(x)=0$. Then the closure of $C^1(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ is equal to*

$$\begin{aligned} H_2 := \big\{ f\in W^{1,\infty}(w_0,w_1) : {} & f\in\overline{C(\mathbf{R})\cap L^\infty(w_0)}^{L^\infty(w_0)},\ f'\in\overline{C(\mathbf{R})\cap L^\infty(w_1)}^{L^\infty(w_1)},\\ & f \ \textit{ is continuous to the right if } a\in D^+(w_1),\\ & f \ \textit{ is continuous to the left if } a\in D^-(w_1),\\ & \operatorname*{ess\,lim}_{x\to a}|f(x)-f(a)|w_0(x)=0\big\}\,. \end{aligned}$$

Furthermore, if $w_0, w_1 \in L^\infty([\alpha,\beta])$, then the closure of the space of polynomials in $W^{1,\infty}(w_0,w_1)$ is also H_2. In fact, for each $f \in H_2$ and $d > 0$ there exist $\{g_n\}_n$ in $C^1(\mathbf{R})$ with $\lim_{n\to\infty} \|f - g_n\|_{W^{1,\infty}(w_0,w_1)} = 0$ and $g_n(x) = f(x)$ if $x \notin (a-d, a+d)$.

Remarks.

1. It is a remarkable fact that the approximation method is constructive.

2. Notice that we require $\operatorname{ess\,lim}_{x\to a} |f(x) - f(a)| w_0(x) = 0$ in H_2, even if $a \notin S(w_0)$.

Proof. If f is in the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$, it follows that $f \in W^{1,\infty}(w_0,w_1)$, $f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}$, and $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$. Lemma 4.4 implies that f is continuous to the right if $a \in D^+(w_1)$ and f is continuous to the left if $a \in D^-(w_1)$. If $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x) < \infty$, we can deduce that $\operatorname{ess\,lim}_{x\to a} |f(x) - f(a)| w_0(x) = 0$: We see that $\operatorname{ess\,lim}_{x\to a^+} |f(x) - f(a)| w_0(x) = 0$ (the left limit is similar); it is a consequence of Theorem 2.1 if $a \in S^+(w_0)$, and if this is not so, f is continuous to the right at a, as a consequence of $f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}$ and Theorem 2.1. If $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x) = \infty$, we have $f(a) = 0$, and Lemma 4.5 implies that there not exists $c > 0$ with $\|f\|_{L^\infty([a-\delta,a+\delta],w_0)} \ge c$ for every $\delta > 0$; therefore we obtain $\operatorname{ess\,lim}_{x\to a} |f(x) - f(a)| w_0(x) = 0$ also in this case. Then $f \in H_2$.

It is clear that H_2 is contained in the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$, since $f \in H_1$: $u_f(a)$ is finite and we have the hypothesis $\operatorname{ess\,lim}_{x\to a} |x-a| w_0(x) = 0$, and consequently $\operatorname{ess\,lim}_{x\to a} u_f(a)|x-a| w_0(x) = 0$. Then it is possible to apply Theorem 4.1, which allows to choose $\{g_n\}_n$ in $C^1(\mathbf{R})$ with $\lim_{n\to\infty} \|f - g_n\|_{W^{1,\infty}(w_0,w_1)} = 0$ and $g_n(x) = f(x)$ if $x \notin (a-d, a+d)$.

If $w_0, w_1 \in L^\infty([\alpha,\beta])$, the closure of the polynomials is H_2 as well, as a consequence of Bernstein's proof of Weierstrass' Theorem (see e.g. [D, p.113]), which gives a sequence of polynomials converging uniformly up to the k-th derivative for any function in $C^k([\alpha,\beta])$. ♯

Proposition 4.1. *Let us consider two weights w_0, w_1, in $[\alpha,\beta]$, with $\operatorname{ess\,lim\,sup}_{x\to a} |x-a| w_0(x) > 0$ and $a \in S(w_1)$.*

(a) If f belongs to the closure of $C^1(\mathbf{R}) \cap L^\infty(w_0)$ in $L^\infty(w_0)$, then for each $\eta > 0$ small enough there exists $l := \operatorname{ess\,lim}_{x\to a,\, |x-a| w_0(x) \ge \eta} f(x)/(x-a)$. We also have $\lim_{n\to\infty} g_n'(a) = l$, for any sequence $\{g_n\} \subset C^1(\mathbf{R}) \cap L^\infty(w_0)$ converging to f in $L^\infty(w_0)$.

(b) If f belongs to the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ and $a \notin S_1(w_1)$, then $u_f(a) = l$. Furthermore, if there exists $f'(a)$, then $u_f(a) = f'(a)$.

(c) If f' belongs to the closure of $C(\mathbf{R}) \cap L^\infty(w_1)$ in $L^\infty(w_1)$ and $a \notin S_1(w_1)$, then $u_f(a) = \lim_{n\to\infty} h_n(a)$, if $\{h_n\} \subset C(\mathbf{R}) \cap L^\infty(w_1)$ converges to f' in $L^\infty(w_1)$.

Proof. Let us fix $0 < \eta < \operatorname{ess\,lim\,sup}_{x\to a} |x-a| w_0(x)$. Seeking a contradiction, suppose that

$$\operatorname*{ess\,lim\,inf}_{x\to a,\, |x-a| w_0(x) \ge \eta} \frac{f(x)}{x-a} = c_1 < c_2 = \operatorname*{ess\,lim\,sup}_{x\to a,\, |x-a| w_0(x) \ge \eta} \frac{f(x)}{x-a} .$$

If g is any function in $C^1(\mathbf{R}) \cap L^\infty(w_0)$, it follows that $g(a) = 0$ (by $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x) = \infty$) and

$$\|f - g\|_{L^\infty(w_0)} \ge \eta \left\| \frac{f(x) - g(x)}{x-a} \right\|_{L^\infty(\{|x-a| w_0(x) \ge \eta\})} \ge \eta \max\left\{ |c_1 - g'(a)|, |c_2 - g'(a)| \right\} \ge \eta \, \frac{c_2 - c_1}{2} .$$

This is a contradiction with f belonging to the closure of $C^1(\mathbf{R}) \cap L^\infty(w_0)$ in $L^\infty(w_0)$.

Let us choose $g_n \in C^1(\mathbf{R}) \cap L^\infty(w_0)$ with $\|f - g_n\|_{L^\infty(w_0)} \le 1/n$. Hence

$$\eta \left| \frac{f(x) - g_n(x)}{x-a} \right| \le |f(x) - g_n(x)| w_0(x) \le \|f - g_n\|_{L^\infty(w_0)} \le \frac{1}{n} ,$$

for almost every x with $|x-a| w_0(x) \ge \eta$. Therefore, it follows that $\eta\, |l - g_n'(a)| \le 1/n$, for every n, since $g_n(a) = 0$ (by $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x) = \infty$). Hence l is finite and $\lim_{n\to\infty} g_n'(a) = l$.

Let us assume now that f belongs to the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ and $a \notin S_1(w_1)$. Notice that Lemma A gives that there exists $u_f(a) := \operatorname{ess\,lim}_{x\to a,\, w_1(x) \ge \eta} f'(x)$, for each $\eta > 0$ small enough, since $a \notin S_1(w_1)$. We have that there exists $g_n \in C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ with $\|f - g_n\|_{W^{1,\infty}(w_0,w_1)} \le 1/n$. Hence

$$\eta\,|f'(x) - g_n'(x)| \le |f'(x) - g_n'(x)| w_1(x) \le \|f' - g_n'\|_{L^\infty(w_1)} \le \frac{1}{n}\,,$$

for almost every x with $w_1(x) \ge \eta$. Consequently, it follows that $\eta|u_f(a) - g_n'(a)| \le 1/n$, for every n, and we deduce that $l = \lim_{n\to\infty} g_n'(a) = u_f(a)$. (The same argument allows to deduce that $\lim_{n\to\infty} h_n(a) = u_f(a)$, for any sequence $\{h_n\} \subset C(\mathbf{R}) \cap L^\infty(w_1)$ converging to f' in $L^\infty(w_1)$. This proves (c).)

Let us assume now that there exists $f'(a)$. Then it follows that $f'(a) = l$ and consequently $f'(a) = l = u_f(a)$. ♯

Proposition 4.2. *Let us consider two weights w_0, w_1, in $[\alpha, \beta]$, with $\operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x) = \infty$ and $a \in S(w_1)$. If f belongs to the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$, then $u_f(a) = 0$.*

Proof. We only need to consider the case $a \in S(w_1) \setminus S_1(w_1)$, since $u_f(a) = 0$ if $a \in S_1(w_1)$ (recall Definition 4.2).

If we take $g_n \in C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ with $\|f - g_n\|_{W^{1,\infty}(w_0,w_1)} \le 1/n$, then parts (a) and (b) of Proposition 4.1 imply that $\lim_{n\to\infty} g_n'(a) = u_f(a)$.

Since $\operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x) = \infty$, for each m

$$m \left| \frac{g_n(x)}{x - a} \right| \le |g_n(x)| w_0(x) \le \|g_n\|_{L^\infty(w_0)} \le \|f\|_{L^\infty(w_0)} + \frac{1}{n}\,,$$

for almost every x with $|x - a| w_0(x) \ge m$. Then $m|g_n'(a)| \le \|f\|_{L^\infty(w_0)} + 1/n$ for every m, since $g_n(a) = 0$. Consequently, it follows that $g_n'(a) = 0$ and $u_f(a) = 0$. ♯

Definition 4.3. Let us consider a weight w_0 in $[\alpha, \beta]$, with $\operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x) > 0$ and $a \in S(w_1)$, and a function f in the closure of $C^1(\mathbf{R}) \cap L^\infty(w_0)$ in $L^\infty(w_0)$. We define the *derivative of f in a through* $\{|x - a| w_0(x) \ge \eta\}$ as $l(f, a) := \operatorname{ess\,lim}_{x\to a,\, |x-a|w_0(x)\ge\eta} f(x)/(x - a)$, for any $0 < \eta < \operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x)$.

Theorem 4.3. *Let us consider two weights w_0, w_1, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$ and $0 < \operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x) < \infty$. Then the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$ is equal to*

$$\begin{aligned} H_3 := \Big\{ f \in W^{1,\infty}(w_0, w_1) : \; & f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}, \; f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}, \\ & f \;\; \textit{is continuous to the right if } a \in D^+(w_1), \\ & f \;\; \textit{is continuous to the left if } a \in D^-(w_1), \\ & \exists\, l(f, a) \;\; \textit{and} \;\; \operatorname*{ess\,lim}_{x\to a} |f(x) - l(f, a)(x - a)| w_0(x) = 0, \\ & \textit{and if } a \notin S_1(w_1), \textit{ then } \; u_f(a) = l(f, a) \Big\}\,. \end{aligned}$$

In fact, for each $f \in H_3$ and $d > 0$ there exist $\{g_n\}_n$ in $C^1(\mathbf{R})$ with $\lim_{n\to\infty} \|f - g_n\|_{W^{1,\infty}(w_0,w_1)} = 0$ and $g_n(x) = f(x)$ if $x \notin (a - d, a + d)$.

Remark. Condition "if $a \notin S_1(w_1)$, then $u_f(a) = l(f, a)$" shows the interaction that must exist between f, w_0 and w_1 in order to approximate f by smooth functions (compare with Theorem 4.2). The example after the proof of Theorem 4.3 shows that this condition is independent of the other hypotheses in the definition of H_3.

Proof. If f is in the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$, we will see that it belongs to H_3. It is clear that $f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}$, and $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$. Lemma 4.4 allows to deduce that f is continuous to the right if $a \in D^+(w_1)$ and f is continuous to the left if $a \in D^-(w_1)$. Proposition 4.1 implies that if $a \notin S_1(w_1)$, then $u_f(a) = l(f, a)$. Let us choose a sequence $\{g_n\} \subset C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ converging to f in $W^{1,\infty}(w_0, w_1)$. By Proposition 4.1 it follows that $l(f, a) = \operatorname{ess\,lim}_{x\to a,\, |x-a|w_0(x)\ge\eta} f(x)/(x - a) = \lim_{n\to\infty} g_n'(a)$, for $\eta > 0$ small enough.

Let us fix $\varepsilon > 0$. It is clear that

$$\operatorname*{ess\,lim}_{x\to a,\,|x-a|w_0(x)\ge\eta} |f(x)-l(f,a)(x-a)|w_0(x) = \operatorname*{ess\,lim}_{x\to a,\,|x-a|w_0(x)\ge\eta} \Big|\frac{f(x)}{x-a}-l(f,a)\Big||x-a|w_0(x) = 0\,,$$

since $\operatorname{ess\,lim\,sup}_{x\to a} |x-a|w_0(x) < \infty$; then there exists $\delta_1 > 0$ with

$$\|f(x)-l(f,a)(x-a)\|_{L^\infty([a-\delta_1,a+\delta_1]\cap\{|x-a|w_0(x)\ge\eta\},w_0)} < \varepsilon\,.$$

Now, it is sufficient to prove that $\|f(x)-l(f,a)(x-a)\|_{L^\infty([a-\delta,a+\delta]\cap\{|x-a|w_0(x)<\eta\},w_0)} < \varepsilon$, for some $0<\delta\le\delta_1$. Proposition 4.1 allows to choose n with $\|f-g_n\|_{L^\infty(w_0)} < \varepsilon/2$ and $|g_n'(a)-l(f,a)|\eta < \varepsilon/2$; hence, there exists $0<\delta\le\delta_1$ with $|g_n(x)/(x-a)-l(f,a)|\eta<\varepsilon/2$ for every $0<|x-a|<\delta$. Consequently

$$\begin{aligned}\|g_n(x)-&l(f,a)(x-a)\|_{L^\infty([a-\delta,a+\delta]\cap\{|x-a|w_0(x)<\eta\},w_0)}\\ &= \Big\|\frac{g_n(x)}{x-a}-l(f,a)\Big\|_{L^\infty([a-\delta,a+\delta]\cap\{|x-a|w_0(x)<\eta\},|x-a|w_0)} \le \frac{\varepsilon}{2}\,.\end{aligned}$$

We also have $\|f-g_n\|_{L^\infty(w_0)} < \varepsilon/2$; therefore $\|f(x)-l(f,a)(x-a)\|_{L^\infty([a-\delta,a+\delta]\cap\{|x-a|w_0(x)<\eta\},w_0)} < \varepsilon$, and $\|f(x)-l(f,a)(x-a)\|_{L^\infty([a-\delta,a+\delta],w_0)} < \varepsilon$. Then $f\in H_3$.

Let us fix now $f\in H_3$. The hypothesis $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|w_0(x)<\infty$ implies that there exists $0<\delta_0<d/2$ such that $x-a\in L^\infty([a-2\delta_0,a+2\delta_0],w_0)$; if $\operatorname{ess\,lim\,sup}_{x\to a}w_1(x)<\infty$, we also require $w_1\in L^\infty([a-2\delta_0,a+2\delta_0])$. Let us fix $\phi\in C_c^\infty([a-2\delta_0,a+2\delta_0])$ with $0\le\phi\le1$ and $\phi=1$ in $[a-\delta_0,a+\delta_0]$. We see now that $l(f,a)(x-a)\phi(x)\in C_c^\infty([a-2\delta_0,a+2\delta_0])\cap W^{1,\infty}(w_0,w_1)$: it is clear that it belongs to $L^\infty(w_0)$; its derivative is in $L^\infty(w_1)$ if $\operatorname{ess\,lim\,sup}_{x\to a}w_1(x)<\infty$; if this is not so, $a\notin S_1(w_1)$, and it follows that $l(f,a)=0$: if $\{h_n\}\subset C(\mathbf{R})\cap L^\infty(w_1)$ converges to f' in $L^\infty(w_1)$, part (c) of Proposition 4.1 implies that $u_f(a)=\lim_{n\to\infty}h_n(a)$; the fact $\operatorname{ess\,lim\,sup}_{x\to a}w_1(x)=\infty$ implies $h_n(a)=0$, and we have $0=u_f(a)=l(f,a)$, since $f\in H_3$.

We consider the function $g(x):=f(x)-l(f,a)(x-a)\phi(x)$. Since $l(f,a)(x-a)\phi(x)$ is a smooth function in $W^{1,\infty}(w_0,w_1)$, it is sufficient to show that g can be approximated by C^1 functions in $W^{1,\infty}(w_0,w_1)$. We have that $f(a)=g(a)=0$ since $\operatorname{ess\,lim\,sup}_{x\to a}w_0(x)=\infty$; then $\operatorname{ess\,lim}_{x\to a}|g(x)-g(a)|w_0(x)=0$, since $f\in H_3$. Notice that $u_g(a)=0$ if $a\in S_1(w_1)$; if $a\notin S_1(w_1)$, it follows that $u_g(a)=\operatorname{ess\,lim}_{x\to a,\,w_1(x)\ge\eta}f'(x)-l(f,a)=u_f(a)-l(f,a)=0$. Then Theorem 4.1 implies that g can be approximated by functions $\{g_n\}_n$ in $C^1\cap W^{1,\infty}(w_0,w_1)$, with $g_n(x)=g(x)=f(x)$ if $x\notin(a-d,a+d)$. $\sharp$

Example. There exist weights w_0,w_1, and a function f such that $a\notin S_1(w_1)$, $u_f(a)\ne l(f,a)$, and verifying the other hypotheses in the definition of H_3.

Let us consider the function $f(x)=x^2\sin(1/x)$ and the weights in $[0,1]$,

$$w_0(x)=\frac1x\,,\qquad w_1(x)=\begin{cases}1\,, & \text{if } x\in\Big(\dfrac{1}{2\pi n+1/(n+1)},\dfrac{1}{2\pi n-1/n}\Big]\,,\\[2mm] \dfrac1n\,, & \text{if } x\in\Big(\dfrac{1}{2\pi n-1/n},\dfrac{1}{2\pi(n-1)+1/n}\Big]\,.\end{cases}$$

It is clear that $a=0$, $a\notin S_1(w_1)$, $f\in C([0,1])$, $f'\in C((0,1])$, $l(f,0)=f'(0)=0$ and $\operatorname{ess\,lim}_{x\to0}f(x)w_0(x)=0$. A direct computation shows that $u_f(0)=-1$ and $\operatorname{ess\,lim}_{x\to0}|f'(x)+1|w_1(x)=0$ (then f' belongs to the closure of $C(\mathbf{R})\cap L^\infty(w_1)$ in $L^\infty(w_1)$).

We can deduce the following result from Theorem 4.3. We say that two functions u,v are comparable in the set A if there are positive constants c_1,c_2 such that $c_1v(x)\le u(x)\le c_2v(x)$ for almost every $x\in A$.

Corollary 4.1. *Let us consider two weights w_0,w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$ and w_0 is comparable to $1/|x-a|$ in a neighborhood of a. Then the closure of $C^1(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ is equal to*

$$\begin{aligned}\big\{f\in W^{1,\infty}(w_0,w_1): &\ f\in\overline{C(\mathbf{R})\cap L^\infty(w_0)}^{L^\infty(w_0)},\ f'\in\overline{C(\mathbf{R})\cap L^\infty(w_1)}^{L^\infty(w_1)},\\ &f \text{ is continuous to the right if } a\in D^+(w_1),\\ &f \text{ is continuous to the left if } a\in D^-(w_1),\\ &\exists\, f'(a) \text{ and if } a\notin S_1(w_1), \text{ then } u_f(a)=f'(a)\big\}\,.\end{aligned}$$

Proof. It is clear that $l(f,a)=f'(a)$, since w_0 is comparable to $1/|x-a|$, and it follows that $\operatorname{ess\,lim}_{x\to a}|f(x)-f'(a)(x-a)|w_0(x)=0$, since f is differentiable in a. $\sharp$

We introduce now the following condition which will be essential in the characterization of the functions f which can be approximated by smooth functions in $W^{1,\infty}(w_0, w_1)$ in the last case:

Let us consider two weights w_0, w_1, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$ and $\operatorname{ess\,lim\,sup}_{x\to a} |x-a| w_0(x) = \infty$, and $f \in W^{1,\infty}(w_0, w_1)$.

$$\begin{aligned} &\text{For some } d_0 > 0 \text{ and each } n \in \mathbf{N}, \\ &\text{there exists } \phi_n \in C^1([a-d_0, a+d_0]) \cap W^{1,\infty}([a-d_0, a+d_0], w_0, w_1) \\ &\text{such that } \operatorname*{ess\,lim\,sup}_{x\to a} |f(x) - \phi_n(x)|\, w_0(x) < 1/n\,. \end{aligned} \tag{4.2}$$

Lemma 4.6. *Let us consider two weights w_0, w_1, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$ and $\operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x) = \infty$. If f verifies condition (4.2), then for each $0 < d \le d_0$ we can choose the functions ϕ_n with the additional property $\phi_n \in C_c^1((a-d, a+d))$.*

Proof. Let us fix $0 < d \le d_0$. We prove that we can choose ϕ_n with the additional property $\phi_n = 0$ in a neighborhood of $a - d$. The argument in a neighborhood of $a + d$ is similar.

Let us assume first that $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x) = \infty$ for every $t \in [a-d, a]$. Then $\phi_n' = 0$ in $[a-d, a]$, and $\phi_n(a) = 0$ since $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x) = \infty$. Hence, $\phi_n = 0$ in $[a-d, a]$.

In other case, there exists $t \in [a-d, a]$ with $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x) < \infty$. Then, there exists a closed interval $A = [a_1, a_2] \subset (a-d, a)$ with $w_1 \in L^\infty(A)$. Let us fix $\varphi \in C^1(\mathbf{R})$ with $\varphi = 0$ in $(-\infty, a_1]$ and $\phi = 1$ in $[a_2, \infty)$. It is clear that $\varphi\phi_n \in W^{1,\infty}([a-d, a+d], w_0, w_1)$ since $w_1 \in L^\infty(A)$. Hence, we can substitute ϕ_n by $\varphi\phi_n$. ♯

Theorem 4.4. *Let us consider two weights w_0, w_1, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$ and $\operatorname{ess\,lim\,sup}_{x\to a} |x - a| w_0(x) = \infty$. Then the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$ is equal to*

$$\begin{aligned} H_4 := \Big\{ f \in W^{1,\infty}(w_0, w_1) : \; & f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)},\; f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}, \\ & f \text{ is continuous to the right if } a \in D^+(w_1), \\ & f \text{ is continuous to the left if } a \in D^-(w_1), \\ & f \text{ satisfies (4.2) and } u_f(a) = 0 \Big\}. \end{aligned}$$

In fact, for each $f \in H_4$ and $d > 0$ there exist $\{g_n\}_n$ in $C^1(\mathbf{R})$ with $\lim_{n\to\infty} \|f - g_n\|_{W^{1,\infty}(w_0,w_1)} = 0$ and $g_n(x) = f(x)$ if $x \notin (a-d, a+d)$.

Remarks.

1. Although (4.2) is not a condition so clean than those in H_2 or H_3, it simplifies notably the approximation problem, since it is a local condition and there is no reference to f' (we do not need to approximate simultaneously f and f'). Condition (5.1) below implies (4.2), and Proposition 5.2 characterizes (5.1) in many situations.

2. Condition (4.2) shows the interaction that must exist between f, w_0 and w_1 in order to approximate f by smooth functions (notice that $\phi_n \in C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$).

3. If $f(a) \in W^{1,\infty}(w_0, w_1)$ for any $f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}$ with $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$ (in particular, if $w_0 \in L^\infty([\alpha, \beta])$), then condition (4.2) can be removed (since $\operatorname{ess\,lim}_{x\to a} |f(x) - f(a)|\, w_0(x) = 0$) if we are in some of the following situations (see Remark 2 to Theorem 4.1):

(*a*) $a \in S^+(w_0) \cap S^-(w_0)$, i.e., $\operatorname{ess\,lim\,inf}_{x\to a^+} w_0(x) = \operatorname{ess\,lim\,inf}_{x\to a^-} w_0(x) = 0$,

(*b*) $a \in S^+(w_0)$ and $w_0 \in L^\infty([a-\varepsilon, a])$, for some $\varepsilon > 0$,

(*c*) $a \in S^-(w_0)$ and $w_0 \in L^\infty([a, a+\varepsilon])$, for some $\varepsilon > 0$,

(*d*) $w_0 \in L^\infty([a-\varepsilon, a+\varepsilon])$, for some $\varepsilon > 0$.

Therefore, in this situation, the statement of Theorem 4.4 is nicer.

Proof. It is clear that if f belongs to the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$, then $f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)}$ and $f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}$. Lemma 4.4 implies that f is continuous to the right if $a \in D^+(w_1)$ and f is continuous to the left if $a \in D^-(w_1)$. Proposition 4.2 implies that $u_f(a) = 0$. We prove now that f satisfies (4.2): Seeking a contradiction, suppose that f does not satisfy (4.2); then there exist positive constants c, d such that $\operatorname{ess\,lim\,sup}_{x\to a} |f(x) - \phi(x)|\, w_0(x) \ge c$ for every $\phi \in C^1([a-d, a+d]) \cap W^{1,\infty}([a-d, a+d], w_0, w_1)$. This means that $\|f - \phi\|_{L^\infty([a-\delta,a+\delta],w_0)} \ge c$ for every $\delta > 0$. Hence, $\|f - \phi\|_{L^\infty(w_0)} \ge \|f - \phi\|_{L^\infty([a-\delta,a+\delta],w_0)} \ge c$ for every $\phi \in C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$, which provides the expected contradiction. Then $f \in H_4$.

Let us see now that H_4 is contained in the closure of $C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$. By Lemma 4.6, given $f_0 \in H_4$, $d > 0$ and $\varepsilon > 0$ we can choose $\phi \in C^1_c((a-d, a+d)) \cap W^{1,\infty}(w_0, w_1)$ such that the function defined by $f := f_0 - \phi$ verifies $\operatorname{ess\,lim\,sup}_{x\to a} |f(x)|\, w_0(x) < \varepsilon/24$; besides, $f(x) = f_0(x)$ if $x \notin (a-d, a+d)$. Then there exists $\delta > 0$ with $4\|f - f(a)\|_{L^\infty([a-\delta,a+\delta],w_0)} < \varepsilon/6$ (recall that $f(a) = 0$ since $\operatorname{ess\,lim\,sup}_{x\to a} w_0(x) = \infty$).

Since $u_f(a) = 0$, then applying the argument in the proof of Theorem 4.1 it is possible to find $g \in C^1(\mathbf{R}) \cap W^{1,\infty}(w_0, w_1)$ with $\|f - g\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon$ and $g(x) = f(x)$ if $x \notin (a-d, a+d)$. Hence, if $g_0 := g + \phi$, it follows that $\|f_0 - g_0\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon$ and $g_0(x) = f_0(x)$ if $x \notin (a-d, a+d)$. $\sharp$

The following result allows to reduce the global approximation problem in $W^{1,\infty}(I, w_0, w_1)$ by smooth functions to a local approximation problem, under some technical conditions.

Theorem B. [R1, Theorem 5.2] *Let us consider strictly increasing sequences of real numbers $\{\alpha_n\}_{n\in J}$, $\{\beta_n\}_{n\in J}$ (J is either a finite set, $\mathbf{Z}$, $\mathbf{Z}^+$ or $\mathbf{Z}^-$) with $\alpha_{n+1} < \beta_n < \alpha_{n+2}$ for every n. Let w_0, w_1 be weights in the interval $I := \cup_n[\alpha_n, \beta_n]$. Assume that for each n there exists an interval $I_n \subset [\alpha_{n+1}, \beta_n]$ with $w_1 \in L^\infty(I_n)$ and $\int_{I_n} w_0 > 0$. Then f can be approximated by functions of $C^1(I)$ in $W^{1,\infty}(I, w_0, w_1)$ if and only if it can be approximated by functions of $C^1([\alpha_n, \beta_n])$ in $W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ for each n. The same result is true if we replace C^1 by C^∞ in both cases.*

Remarks.

1. The proof of this theorem in [R1] is constructive and the main idea is natural: it suffices to consider functions g_n which aproximate f in $[\alpha_n, \beta_n]$ and to obtain a function g which aproximate f in I by "pasting" $\{g_n\}_n$ with an appropriate partition of unity. Since the pasting process occurs in $\cup_n I_n$, we have $g = g_n$ in $[\beta_{n-1}, \alpha_{n+1}]$; furthermore, if there exists a first index n_1 in J, then $g = g_{n_1}$ in $[\alpha_{n_1}, \alpha_{n_1+1}]$, and if there exists a last index n_2 in J, then $g = g_{n_2}$ in $[\beta_{n_2-1}, \beta_{n_2}]$; in particular, $g(\alpha_{n_1}) = g_{n_1}(\alpha_{n_1})$ and $g(\beta_{n_2}) = g_{n_2}(\beta_{n_2})$.

2. Condition $\alpha_{n+1} < \beta_n$ means that (α_n, β_n) and $(\alpha_{n+1}, \beta_{n+1})$ overlap; $(\alpha_n, \beta_n) \cap (\alpha_{n+2}, \beta_{n+2}) = \emptyset$ since $\beta_n < \alpha_{n+2}$.

In fact, Theorem 5.2 in [R1] is a more general result, but the statement we present here is good enough for our purposes.

Definition 4.4. The weights w_0, w_1 are *jointly admissible* on the interval I, if there exist strictly increasing sequences of real numbers $\{\alpha_n\}_{n\in J}$, $\{\beta_n\}_{n\in J}$ (J is either a finite set, $\mathbf{Z}$, $\mathbf{Z}^+$ or $\mathbf{Z}^-$) with $\alpha_{n+1} < \beta_n < \alpha_{n+2}$ for every n and $I := \cup_n[\alpha_n, \beta_n]$, and verifying the following conditions:

There exists a partition J_1, J_2, J_3 of J, such that

($a1$) if $n \in J_1$, then $w_0 \in L^\infty([\alpha_n, \beta_n])$ and $1/w_1 \in L^1([\alpha_n, \beta_n])$,

($a2$) if $n \in J_2$, then $S(w_1) \cap [\alpha_n, \beta_n] = \{a_n\}$,

($a3$) if $n \in J_3$, then $S(w_1) \cap [\alpha_n, \beta_n] = \emptyset$.

Remark. Without loss of generality we can assume that $a_n \in (\beta_{n-1}, \alpha_{n+1})$ if $n \in J_2$: if $a_n \in (\alpha_n, \beta_n)$ and $a_n \le \beta_{n-1}$, it suffices to take β_{n-1} smaller; if $a_n \in (\alpha_n, \beta_n)$ and $\alpha_{n+1} \le a_n$, it suffices to take α_{n+1} bigger; if $a_n = \alpha_n$, it suffices to take α_n bigger (and then $n \in J_3$); if $a_n = \beta_n$, it suffices to take β_n smaller (and then we also have $n \in J_3$). We always assume this property.

Now, we can state the main result of this section. Notice that we do not have any hypothesis about the singularities of w_0, that the weights w_0, w_1 have a great deal of independence among them, and that the interval I is not required to be bounded.

Theorem 4.5. *Let us consider two weights w_0, w_1 which are jointly admissible on the interval I. Then the* *closure of* $C^1(I)\cap W^{1,\infty}(w_0,w_1)$ *in* $W^{1,\infty}(w_0,w_1)$ *is equal to*

$$
\begin{aligned}
H := \big\{ f\in W^{1,\infty}(w_0,w_1):\ & f\in \overline{C(I)\cap L^\infty(w_0)}^{L^\infty(w_0)},\ f'\in \overline{C(I)\cap L^\infty(w_1)}^{L^\infty(w_1)},\\
& \text{for each } \{a_n\}=S(w_1)\cap[\alpha_n,\beta_n],\ \text{with } n\in J_2,\ \text{we have}\\
& f \ \text{ is continuous to the right if } a_n\in D^+(w_1),\\
& f \ \text{ is continuous to the left if } a_n\in D^-(w_1),\\
& \text{if } \operatorname*{ess\,lim}_{x\to a_n}|x-a_n|w_0(x)=0,\quad \operatorname*{ess\,lim}_{x\to a_n}|f(x)-f(a_n)|w_0(x)=0,\\
& \text{if } 0<\operatorname*{ess\,lim\,sup}_{x\to a_n}|x-a_n|w_0(x)<\infty,\\
& \qquad \exists\, l(f,a_n)\ \text{ and }\ \operatorname*{ess\,lim}_{x\to a_n}|f(x)-l(f,a_n)(x-a_n)|w_0(x)=0,\\
& \qquad \text{and if } a_n\notin S_1(w_1),\ \text{then } u_f(a_n)=l(f,a_n),\\
& \text{if } \operatorname*{ess\,lim\,sup}_{x\to a_n}|x-a_n|w_0(x)=\infty,\quad f \ \text{satisfies (4.2) and } u_f(a_n)=0 \big\}\,.
\end{aligned}
$$

Remarks.

1. Notice that this theorem has a wide range of application. Let us consider the particular case of Jacobi weights: $w_0(x)=(1+x)^{s_1}(1-x)^{s_2}$, $w_1(x)=(1+x)^{t_1}(1-x)^{t_2}$, in $[-1,1]$. Theorem 4.5 describes the closure of $C^1([-1,1])\cap W^{1,\infty}([-1,1],w_0,w_1)$ in $W^{1,\infty}([-1,1],w_0,w_1)$ for every possible value of the exponents; if $t_1\le 0$ (respectively $t_2\le 0$), then -1 (respectively 1) is a regular point of w_1.

It is obvious that Theorem 4.5 also describes the closure of C^1 functions with weights with many singular points, as $w_0(x)=|x-a_1|^{s_1}|x-a_2|^{s_2}\cdots|x-a_m|^{s_m}$, $w_1(x)=|x-b_1|^{t_1}|x-b_2|^{t_2}\cdots|x-b_n|^{t_n}$. The same is true if we change each power $|x-\alpha|^\beta$ by any function with a singularity in α, and even if we consider weights defined in some interval I such that $S(w_1)$ has no accumulation point in the interior of I.

2. Let us observe that in Theorem 4.5 we do not have as hypotheses the technical conditions which appear in the statement of Theorem B.

In order to prove Theorem 4.5, we need two previous results.

Proposition 4.3. *Let us consider two weights* w_0,w_1, *in* $A=[\alpha,\beta]$ $(-\infty\le\alpha<\beta\le\infty)$, *with* $w_0\in L^\infty(A)$ *and* $1/w_1\in L^1(A)$. *Then*

$$
\overline{C^1(A)\cap W^{1,\infty}(A,w_0,w_1)}^{W^{1,\infty}(A,w_0,w_1)} = \big\{ f\in W^{1,\infty}(A,w_0,w_1):\ f'\in \overline{C(A)\cap L^\infty(A,w_1)}^{L^\infty(A,w_1)} \big\}\,.
$$

Furthermore, if $f\in \overline{C^1(A)\cap W^{1,\infty}(A,w_0,w_1)}^{W^{1,\infty}(A,w_0,w_1)}$, *we can obtain a sequence of functions* $\{F_n\}\subset C^1(A)\cap W^{1,\infty}(A,w_0,w_1)$ *converging to* f *in* $W^{1,\infty}(A,w_0,w_1)$ *with* $F_n(\alpha)=f(\alpha)$ *and* $F_n(\beta)=f(\beta)$. *The same result is true if we replace* $C^1(A)$ *and* $C(A)$ *by* $C^\infty(A)$ *everywhere.*

Proof. We prove the non-trivial inclusion. If $f'\in \overline{C(A)\cap L^\infty(A,w_1)}^{L^\infty(A,w_1)}$, let us consider a sequence $\{g_n\}\subset C(A)\cap L^\infty(A,w_1)$ which converges to f' in $L^\infty(A,w_1)$. Notice that $f'\in L^\infty(A,w_1)$ and $1/w_1\in L^1(A)$ imply that $f'\in L^1(A)$ and hence f is an absolutely continuous function on A. Then the functions $G_n(x):=f(\alpha)+\int_\alpha^x g_n$ belongs to $C^1(A)$, satisfy $G_n(\alpha)=f(\alpha)$ and

$$
\big|f(x)-G_n(x)\big| = \Big|\int_\alpha^x (f'-g_n)\Big| \le \int_A |f'-g_n|\,\frac{w_1}{w_1} \le \|f'-g_n\|_{L^\infty(A,w_1)}\int_A \frac{1}{w_1}\,.
$$

Then, $\|f-G_n\|_{L^\infty(A,w_0)}\le \|f'-g_n\|_{L^\infty(A,w_1)}\|w_0\|_{L^\infty(A)}\|1/w_1\|_{L^1(A)}$, and we have proved the inclusion. Let us remark that $\lim_{n\to\infty}G_n(\beta)=f(\beta)$.

If $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x)=\infty$ for every $t\in A$, then any $g\in C^1(A)\cap W^{1,\infty}(A,w_0,w_1)$ verifies $g'=0$ in A, and therefore is constant. Hence the closure of $C^1(A)\cap W^{1,\infty}(A,w_0,w_1)$ is the space of constants, and then the last conclusion of the proposition is direct.

In we do not have $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x)=\infty$ for every $t\in A$, then there exists an interval $B\subset A$ with $w_1\in L^\infty(B)$. Let us consider a function $h\in C(A)$ with $\operatorname{supp} h\subset B$ and $\int h=1$. In this case we can define the functions $F_n(x) := G_n(x) + (f(\beta) - G_n(\beta)) \int_\alpha^x h \in C^1(A) \cap W^{1,\infty}(A, w_0, w_1)$, which verify $F_n(\alpha) = f(\alpha)$ and $F_n(\beta) = f(\beta)$. Since $\lim_{n\to\infty}(f(\beta) - G_n(\beta)) = 0$, we also have that $\{F_n\}$ converges to f in $W^{1,\infty}(A, w_0, w_1)$.

If we replace $C^1(A)$ and $C(A)$ by $C^\infty(A)$ everywhere in this proof, we obtain that

$$\overline{C^\infty(A) \cap W^{1,\infty}(A, w_0, w_1)}^{W^{1,\infty}(A,w_0,w_1)} = \big\{ f \in W^{1,\infty}(A, w_0, w_1) : \, f' \in \overline{C^\infty(A) \cap L^\infty(A, w_1)}^{L^\infty(A,w_1)} \big\}. \ \sharp$$

Proposition 4.4. *Let us consider strictly increasing sequences of real numbers $\{\alpha_n\}_{n\in J}$, $\{\beta_n\}_{n\in J}$ (J is either a finite set,* **Z**, $\mathbf{Z}^+$ *or* $\mathbf{Z}^-$*) with $\alpha_{n+1} < \beta_n < \alpha_{n+2}$ for every n. Let w_0, w_1 be weights in the interval $I := \cup_n [\alpha_n, \beta_n]$. Let us fix $f \in W^{1,\infty}(I, w_0, w_1)$. Assume that for each n $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x) = \infty$ for every $t \in [\alpha_{n+1}, \beta_n]$, and that there exist $\{g_n^k\}_k$ in $C^1([\alpha_n, \beta_n]) \cap W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ with $\lim_{k\to\infty} \|f - g_n^k\|_{W^{1,\infty}([\alpha_n,\beta_n],w_0,w_1)} = 0$, $g_n^k(\alpha_n) = f(\alpha_n)$ and $g_n^k(\beta_n) = f(\beta_n)$. Then f belongs to the closure of $C^1(I) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$. The same result is true if we replace C^1 by C^∞ in both cases.*

Proof. For each n, let us consider $\{g_n^k\}_k$ in $C^1([\alpha_n, \beta_n])$ with $\|f - g_n^k\|_{W^{1,\infty}([\alpha_n,\beta_n],w_0,w_1)} < 1/k$, $g_n^k(\alpha_n) = f(\alpha_n)$ and $g_n^k(\beta_n) = f(\beta_n)$. Since $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x) = \infty$ for every $t \in [\alpha_{n+1}, \beta_n]$, we have that $(g_n^k)' = (g_{n+1}^k)' = f' = 0$ in $[\alpha_{n+1}, \beta_n]$. Consequently, $g_n^k(x) = f(x) = f(\beta_n)$ for every $x \in [\alpha_{n+1}, \beta_n]$, and $g_{n+1}^k(x) = f(x) = f(\alpha_{n+1})$ for every $x \in [\alpha_{n+1}, \beta_n]$. Since $g_{n+1}^k = g_n^k$ in $[\alpha_{n+1}, \beta_n]$, for each k we can define a function $g^k \in C^1(I)$ as $g^k = g_n^k$ in $[\alpha_n, \beta_n]$ for each n, and then $\|f - g^k\|_{W^{1,\infty}(w_0,w_1)} < 1/k$. It is clear now that the same result is true if we replace C^1 by C^∞ in both cases. $\sharp$

Proof of Theorem 4.5. Theorems 4.2, 4.3 and 4.4, and Proposition 4.3 allow to deduce that any function in the closure of $C^1(I)$ in $W^{1,\infty}(w_0, w_1)$ belongs to H. Let us observe that the closure of $C^1([\alpha_n, \beta_n]) \cap W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ in $W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ is $C^1([\alpha_n, \beta_n]) \cap W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ if $n \in J_3$, since the closure of $C([\alpha_n, \beta_n]) \cap L^\infty([\alpha_n, \beta_n], w_1)$ in $L^\infty([\alpha_n, \beta_n], w_1)$ is $C([\alpha_n, \beta_n]) \cap L^\infty([\alpha_n, \beta_n], w_1)$, by Theorem 2.1.

We prove now the other inclusion. Let us consider the sequences $\{\alpha_n\}_{n\in J}$ and $\{\beta_n\}_{n\in J}$ in the definition of jointly admissible weights. Recall that $a_n \in (\beta_{n-1}, \alpha_{n+1})$ if $n \in J_2$. This fact allows to take the approximations in theorems 4.2, 4.3 and 4.4 with the same values of the approximated function in α_n and β_n.

We show that each function $f \in H$ can be approximated by functions of $C^1(I)$ in $W^{1,\infty}(I, w_0, w_1)$ if it can be approximated by functions of $C^1([\alpha_n, \beta_n])$ in $W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ for each n; then we can apply theorems 4.2, 4.3 and 4.4, and Proposition 4.3, which show that any function in H belongs to the closure of $C^1([\alpha_n, \beta_n])$ in $W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ for every n. We use an argument with two steps, using Theorem B and Proposition 4.4.

Let us assume first that for each n there exists an interval $I_n \subset [\alpha_{n+1}, \beta_n]$ with $w_1 \in L^\infty(I_n)$.

Let us remark that $a_n \notin I_n$ if $n \in J_2$, since $a_n < \alpha_{n+1}$. Then every function f in H belongs to $C(I_n)$: if $n \in J_2 \cup J_3$, then $S(w_1) \cap I_n = \emptyset$ and $f \in C^1(I_n)$; if $n \in J_1$, then $f' \in L^1(I_n)$ and $f \in AC(I_n)$. For each $f \in H$, let us define $c_n := \|f\|_{L^\infty(I_n)}^{-1}$ if $\|f\|_{L^\infty(I_n)} > 0$ and $c_n := 1$ in other case. Then $f \in L^\infty(w_0^*)$, where $w_0^* := w_0 + \sum_n c_n \chi_{I_n}$, since $\|f\|_{L^\infty(w_0^*)} \le \|f\|_{L^\infty(w_0)} + 1$. We also have $\int_{I_n} w_0^* > 0$ for each $n \in J$. Hence, theorems B, 4.2, 4.3 and 4.4, and Proposition 4.3 finish the proof of Theorem 4.5 in this case, since the closures of $C^1([\alpha_n, \beta_n])$ in $W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ and in $W^{1,\infty}([\alpha_n, \beta_n], w_0^*, w_1)$ are the same (recall that any f in the closure of $C^1([\alpha_n, \beta_n])$ in $W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)$ belongs to $C(I_n)$).

In the general case, there are some n's with $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x) = \infty$ for every $t \in [\alpha_{n+1}, \beta_n]$. The simplified version of Theorem 4.5 which we have proved allows to joint some intervals in a single interval (recall the first remark to Theorem B); therefore, we can assume that $\operatorname{ess\,lim\,sup}_{x\to t} w_1(x) = \infty$ for every $t \in [\alpha_{n+1}, \beta_n]$ and every n. Then, Proposition 4.4, theorems 4.2, 4.3 and 4.4, and Proposition 4.3 finish the proof. $\sharp$

5. APPROXIMATION BY C^∞ FUNCTIONS IN $W^{1,\infty}(I, w_0, w_1)$

We are also interested in approximation by more regular functions. With some additional hypothesis we can use Theorem 4.1 in order to approximate by C^∞ functions.

Theorem 5.1. *Let us consider two weights w_0, w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$ and $w_0, w_1 \in L^\infty_{loc}([\alpha,\beta]\setminus\{a\})$. Then every function in*

$$\begin{aligned} H_5 := \big\{ f \in W^{1,\infty}(w_0,w_1) : \,& f \in \overline{C(\mathbf{R}) \cap L^\infty(w_0)}^{L^\infty(w_0)},\ f' \in \overline{C(\mathbf{R}) \cap L^\infty(w_1)}^{L^\infty(w_1)}, \\ & f \ \text{ is continuous to the right if } a \in D^+(w_1), \\ & f \ \text{ is continuous to the left if } a \in D^-(w_1), \\ & \operatorname*{ess\,lim}_{x\to a} |f(x)-f(a)|w_0(x) = 0,\ \operatorname*{ess\,lim}_{x\to a} u_f(a)(x-a)w_0(x)=0, \\ & \text{and } \operatorname*{ess\,lim}_{x\to a} |f'(x)-u_f(a)|w_1(x)=0 \big\}\,, \end{aligned}$$

can be approximated by functions $\{g_n\}_n$ in $C^\infty(\mathbf{R}) \cap W^{1,\infty}(w_0,w_1)$ with the norm of $W^{1,\infty}(w_0,w_1)$, with $g_n(\alpha)=f(\alpha)$ if $a\neq\alpha$ and with $g_n(\beta)=f(\beta)$ if $a\neq\beta$.

Remark. In the remark after Theorem 4.1 appear simple conditions which guarantee the properties which define H_5.

Proof. Let us consider $f \in H_5$ and $\varepsilon > 0$. Theorem 4.1 implies that there exists $g_0 \in C^1(\mathbf{R})$ with $\|f-g_0\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon/2$, such that g_0 is a polynomial of degree at most 1 in $[a-2\delta, a+2\delta]$ for some $\delta>0$.

Let us choose an even function $\phi \in C_c^\infty([-1,1])$ with $\phi \ge 0$ and $\int \phi = 1$. For each $t>0$, we define $\phi_t(x) := t^{-1}\phi(x/t)$ and $g_t := g_0 * \phi_t$; these functions satisfy $\phi_t \in C_c^\infty([-t,t])$, $\phi_t \ge 0$ and $\int \phi_t = 1$.

It is well known that $g_t \in C^\infty(\mathbf{R})$, and that g_t (respectively g_t') converges uniformly in $[\alpha,\beta]$ to g_0 (respectively g_0') when $t \to 0$.

Notice that if h is a polynomial of degree at most 1, then $h * \phi_t = h$, since $1 * \phi_t = \int \phi_t = 1$ and $x * \phi_t = x$: it is sufficient to notice that $(x*\phi_t)(0) = \int y\phi_t(y)\,dy = 0$ and $(x*\phi_t)' = 1*\phi_t = 1$. Consequently, $g_t = g_0$ in $[a-\delta, a+\delta]$, for $0<t<\delta$, since under this hypothesis, the integral defining g_t only takes into account the values of g_0 in which it is a polynomial of degree at most 1.

Since $w_0, w_1 \in L^\infty_{loc}([\alpha,\beta]\setminus\{a\})$, there exists a constant M with $w_0, w_1 \le M$ in $[\alpha,\beta]\setminus(a-\delta,a+\delta)$. Therefore

$$\|g_t-g_0\|_{W^{1,\infty}(w_0,w_1)} = \|g_t-g_0\|_{W^{1,\infty}([\alpha,\beta]\setminus(a-\delta,a+\delta),w_0,w_1)} \le M\|g_t-g_0\|_{W^{1,\infty}([\alpha,\beta]\setminus(a-\delta,a+\delta))} < \frac{\varepsilon}{2}\,,$$

if t is small enough, since g_t and g_t' converge uniformly in $[\alpha,\beta]$ to g_0 and g_0' respectively.

Then $\|f-g_t\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon$ if t is small enough.

Let us assume that $a \neq \alpha$. Fix $\varphi \in C^\infty(\mathbf{R})$ with $\varphi = 1$ in $(-\infty,\alpha]$ and $\varphi = 0$ in $[a-\delta,\infty)$. Since g_t converges uniformly to g_0 in $[\alpha,\beta]$, $g_0(\alpha)=f(\alpha)$ and $w_0,w_1 \le M$ in $[\alpha, a-\delta]$, we can choose t with the additional condition $|f(\alpha)-g_t(\alpha)|\,\|\varphi\|_{W^{1,\infty}(w_0,w_1)} < \varepsilon$. Therefore, $\overline{g_t} := g_t + (f(\alpha)-g_t(\alpha))\varphi$ verifies $\overline{g_t}(\alpha) = f(\alpha)$ and $\|f-\overline{g_t}\|_{W^{1,\infty}(w_0,w_1)} \le \|f-g_t\|_{W^{1,\infty}(w_0,w_1)} + |f(\alpha)-g_t(\alpha)|\,\|\varphi\|_{W^{1,\infty}(w_0,w_1)} < 2\varepsilon$. If $a\neq\beta$, we use a similar argument in a neighborhood of β. ♯

Definition 5.1. We say that a weight w_1 in $[\alpha,\beta]$ is *balanced at* $a \in [\alpha,\beta]$, if it verifies some of the following conditions:

(*a*) $a \in S^+(w_1)\cap S^-(w_1)$, i.e., $\operatorname{ess\,lim\,inf}_{x\to a^+} w_1(x) = \operatorname{ess\,lim\,inf}_{x\to a^-} w_1(x) = 0$,
(*b*) $a \in S^+(w_1)$ and $w_1 \in L^\infty([a-\varepsilon,a])$, for some $\varepsilon>0$,
(*c*) $a \in S^-(w_1)$ and $w_1 \in L^\infty([a,a+\varepsilon])$, for some $\varepsilon>0$,
(*d*) $a=\alpha$ or $a=\beta$.

Theorem 5.1 and Remark 3 to Theorem 4.1, give the following result.

Corollary 5.1. *Let us consider two weights w_0,w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$, w_1 is balanced at a, and $w_0,w_1 \in L^\infty_{loc}([\alpha,\beta]\setminus\{a\})$. Then every function in H_1 can be approximated by functions $\{g_n\}_n$ in $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ with the norm of $W^{1,\infty}(w_0,w_1)$, with $g_n(\alpha)=f(\alpha)$ if $a\neq\alpha$ and with $g_n(\beta)=f(\beta)$ if $a\neq\beta$.*

We introduce now the following condition which plays the same role that (4.2) in the approximation by functions in C^∞:

Let us consider two weights w_0, w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$ and $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|w_0(x)=\infty$, and $f\in W^{1,\infty}(w_0,w_1)$.

$$
\text{(5.1)}\qquad
\begin{gathered}
\text{For some } d_0>0 \text{ and each } n\in\mathbf{N}\,,\\
\text{there exists }\ \phi_n\in C^\infty([a-d_0,a+d_0])\cap W^{1,\infty}([a-d_0,a+d_0],w_0,w_1)\\
\text{such that }\quad \operatorname*{ess\,lim\,sup}_{x\to a}|f(x)-\phi_n(x)|\,w_0(x)<1/n\,.
\end{gathered}
$$

Remarks.

1. We will see in propositions 5.1 and 5.2 that condition (5.1) can be substituted in many cases by simpler conditions which only involve f.

2. The same argument that the one in the proof of Lemma 4.6 allows to deduce that if f verifies condition (5.1), then for each $0<d\le d_0$ we can choose the functions ϕ_n with the additional property $\phi_n\in C_c^\infty((a-d,a+d))$.

Let us assume that $w_0,w_1\in L^\infty_{loc}([\alpha,\beta]\setminus\{a\})$, $S(w_1)=\{a\}$, and w_1 is balanced at a. The argument in the proof of Theorem 4.2 (using Corollary 5.1) gives that if $\operatorname{ess\,lim}_{x\to a}|x-a|w_0(x)=0$, then the closure of $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ is H_2. In a similar way, if $0<\operatorname{ess\,lim\,sup}_{x\to a}|x-a|w_0(x)<\infty$, then the closure of $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ is H_3. We also have that, if $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|w_0(x)=\infty$, then the closure of $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ is H_4, if we change (4.2) by (5.1). We also obtain that if $f\in H_j$ $(2\le j\le 4)$, then it can be approximated by functions $\{g_n\}_n$ in $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ with the norm of $W^{1,\infty}(w_0,w_1)$, with $g_n(\alpha)=f(\alpha)$ if $a\ne\alpha$ and with $g_n(\beta)=f(\beta)$ if $a\ne\beta$.

Definition 5.2. The weights w_0,w_1 are *strongly jointly admissible* on the interval I, if they verify the conditions in the definition of jointly admissible (Definition 4.4), with $J_3=\emptyset$ and replacing $(a2)$ by

$(a2')$ if $n\in J_2$, then $S(w_1)\cap[\alpha_n,\beta_n]=\{a_n\}$, $w_0,w_1\in L^\infty_{loc}([\alpha_n,\beta_n]\setminus\{a_n\})$, and w_1 is balanced at a_n.

Remark. We choose $J_3=\emptyset$, since in this context we must require $w_0,w_1\in L^\infty([\alpha_n,\beta_n])$ additionally in $(a3)$, and these facts imply the hypothesis in $(a1)$. Hence, J_1 plays here the role of $J_1\cup J_3$ in Definition 4.4.

The following is the main result of this section.

Theorem 5.2. *Let us consider two weights w_0,w_1 which are strongly jointly admissible on the interval I. Then the closure of $C^\infty(I)\cap W^{1,\infty}(w_0,w_1)$ in $W^{1,\infty}(w_0,w_1)$ is equal to*

$$
\begin{aligned}
H_6:=\big\{f\in W^{1,\infty}(w_0,w_1):\ & f\in\overline{C(I)\cap L^\infty(w_0)}^{L^\infty(w_0)},\ f'\in\overline{C(I)\cap L^\infty(w_1)}^{L^\infty(w_1)},\\
& f'\in\overline{C^\infty([\alpha_n,\beta_n])\cap L^\infty([\alpha_n,\beta_n],w_1)}^{L^\infty([\alpha_n,\beta_n],w_1)},\ \textit{for any } n\in J_1,\\
& \textit{for each } \{a_n\}=S(w_1)\cap[\alpha_n,\beta_n],\ \textit{with } n\in J_2,\ \textit{we have}\\
& f\ \textit{is continuous to the right if } a_n\in D^+(w_1),\\
& f\ \textit{is continuous to the left if } a_n\in D^-(w_1),\\
& \textit{if } \operatorname*{ess\,lim}_{x\to a_n}|x-a_n|w_0(x)=0,\quad \operatorname*{ess\,lim}_{x\to a_n}|f(x)-f(a_n)|w_0(x)=0,\\
& \textit{if } 0<\operatorname*{ess\,lim\,sup}_{x\to a_n}|x-a_n|w_0(x)<\infty,\\
& \qquad \exists\ l(f,a_n)\ \textit{and}\ \operatorname*{ess\,lim}_{x\to a_n}|f(x)-l(f,a_n)(x-a_n)|w_0(x)=0,\\
& \qquad \textit{and if } a_n\notin S_1(w_1),\ \textit{then } u_f(a_n)=l(f,a_n),\\
& \textit{if } \operatorname*{ess\,lim\,sup}_{x\to a_n}|x-a_n|w_0(x)=\infty,\quad f\ \textit{satisfies (5.1) and } u_f(a_n)=0\big\}\,.
\end{aligned}
$$

Remark. In Theorem 2.1 and in [PQRT] we characterize $\overline{C^\infty\cap L^\infty(w)}^{L^\infty(w)}$ for a general kind of weights.

Proof. We only need to follow the argument in the proof of Theorem 4.5, replacing the functions in C or C^1, by functions in C^∞. This is the reason why we need to require that f' belongs to the closure of $C^\infty([\alpha_n,\beta_n]) \cap L^\infty([\alpha_n,\beta_n],w_1)$ in $L^\infty([\alpha_n,\beta_n],w_1)$ for any $n \in J_1$. ♯

In many situations we can simplify condition (5.1).

Proposition 5.1. *Let us consider two weights w_0, w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$, and $\operatorname{ess\,lim\,sup}_{x\to a} |x-a| w_0(x)=\infty$. Let us assume that for some function s verifying $0<m\le |s(x)|\le M<\infty$ a.e., there exists $\operatorname{ess\,lim}_{x\to a} \phi(x)s(x)w_0(x)$ for every $\phi \in C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$. Let us denote by $D(w_0,a)$ the set of values of these limits when we consider every $\phi\in C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ ($D(w_0,a)$ is either $\{0\}$ or $\mathbf{R}$). Then (5.1) is equivalent to the following: for any $f\in W^{1,\infty}(w_0,w_1)$ the limit $\operatorname{ess\,lim}_{x\to a} f(x)s(x)w_0(x)$ there exists and belongs to $D(w_0,a)$.*

Remarks.

1. By Remark 2 after (5.1), the functions in $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ can be substituted by $C^\infty([a-d,a+d])\cap W^{1,\infty}([a-d,a+d],w_0,w_1)$ everywhere in Proposition 5.1, for some (or for every) $d>0$.

2. The conclusion of Proposition 5.1 also holds if we substitute (5.1) by (4.2) and C^∞ by C^1 everywhere.

3. A natural choice for s is $s(x):=1$ or $s(x):=\operatorname{sgn}(x-a)$ (see the proof of Proposition 5.2).

Proof. Let us fix $f\in W^{1,\infty}(w_0,w_1)$. If the limit $d:=\operatorname{ess\,lim}_{x\to a} f(x)s(x)w_0(x)$ there exists and belongs to $D(w_0,a)$, we have (5.1) with $\phi_n:=\phi$, where ϕ is a function in $C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$ with $\operatorname{ess\,lim}_{x\to a}\phi(x)s(x)w_0(x)=d$, since then $\operatorname{ess\,lim}_{x\to a}|f(x)-\phi(x)|w_0(x)\le m^{-1}\operatorname{ess\,lim}_{x\to a}|f(x)s(x)w_0(x)-\phi(x)s(x)w_0(x)|=0$. If $d\notin D(w_0,a)$, then $D(w_0,a)$ is $\{0\}$, and consequently $d\ne 0$; hence, $\operatorname{ess\,lim\,sup}_{x\to a}|f(x)-\phi(x)|w_0(x)\ge |d|/M>0$, for every $\phi\in C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$. If the limit $\operatorname{ess\,lim}_{x\to a} f(x)s(x)w_0(x)$ does not exist, a similar argument implies that there exists a constant $c=c(f,M)>0$ such that $\operatorname{ess\,lim\,sup}_{x\to a}|f(x)-\phi(x)|w_0(x)\ge c>0$, for every $\phi\in C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$. ♯

Definition 5.3. We say that a weight w_0 has *potential growth* at a, if $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|^m w_0(x)<\infty$, for some natural number m. If w_0 has *potential growth* at a, we say that the *degree* of w_0 at a is m, if m is the minimum natural number with $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|^m w_0(x)<\infty$.

Proposition 5.2. *Let us consider two weights w_0,w_1, in $[\alpha,\beta]$ such that $S(w_1)=\{a\}$, $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|w_0(x)=\infty$ and w_0 has potential growth at a. Let us assume that m is the degree of w_0 at a.*

(1) *If $\operatorname{ess\,lim}_{x\to a}|x-a|^m w_0(x)=0$, then (5.1) is equivalent to $\operatorname{ess\,lim}_{x\to a}|f(x)|w_0(x)=0$.*

(2) *If $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|^m w_0(x)>0$ and $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|^{m-1}w_1(x)<\infty$, then we can substitute (5.1) by the following condition: there exists $l_m(f,a):=\operatorname{ess\,lim}_{x\to a,|x-a|^m w_0(x)\ge\eta} f(x)/(x-a)^m$ for η small enough, and $\operatorname{ess\,lim}_{x\to a}|f(x)-l_m(f,a)(x-a)^m|w_0(x)=0$.*

(3) *If $w_0(x)$ is comparable with $|x-a|^{-m}$ in a neighborhood of a, for some positive integer m, then (5.1) is equivalent to the existence of $\operatorname{ess\,lim}_{x\to a} f(x)/(x-a)^m$.*

Proof. (1) Let us fix $f\in W^{1,\infty}(w_0,w_1)$ with $\operatorname{ess\,lim}_{x\to a}|f(x)|w_0(x)=0$; then (5.1) holds with $\phi_n:=0$.

In order to see the other implication, let us fix $f\in W^{1,\infty}(w_0,w_1)$ satisfying (5.1). Let us consider $\phi\in C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$. Condition $\operatorname{ess\,lim\,sup}_{x\to a}|x-a|^{m-1}w_0(x)=\infty$ implies $\phi(a)=\phi'(a)=\cdots=\phi^{(m-1)}(a)=0$; then $\phi(x)\approx \phi^{(m)}(a)/m!(x-a)^m$, and condition $\operatorname{ess\,lim}_{x\to a}|x-a|^m w_0(x)=0$ gives $\operatorname{ess\,lim}_{x\to a}\phi(x)w_0(x)=0$ for every $\phi\in C^\infty(\mathbf{R})\cap W^{1,\infty}(w_0,w_1)$. Hence $\operatorname{ess\,lim\,sup}_{x\to a}|f(x)|w_0(x)=\operatorname{ess\,lim\,sup}_{x\to a}|f(x)-\phi_n(x)|w_0(x)<1/n$ for every n.

(2) Let us fix $f\in W^{1,\infty}(w_0,w_1)$ satisfying (5.1). An argument similar to the one in the proof of part (a) of Proposition 4.1 implies that there exists $l_m(f,a)$ for $0<\eta<\operatorname{ess\,lim\,sup}_{x\to a}|x-a|^m w_0(x)$, and that $\phi_n^{(m)}(a)/m!\longrightarrow l_m(f,a)$ as $n\to\infty$. In order to finish the proof of this implication, it is sufficient to follow the argument in the proof of the first part of Theorem 4.4, taking the function $l_m(f,a)(x-a)^m$ instead of $l(f,a)(x-a)$.

We deal with the other implication. Let us consider $f\in W^{1,\infty}(w_0,w_1)$ such that there exists $l_m(f,a)$ for η small enough, and $\operatorname{ess\,lim}_{x\to a}|f(x)-l_m(f,a)(x-a)^m|w_0(x)=0$. In order to verify (5.1), it is sufficient to take as $\phi_n=\phi$ the function $l_m(f,a)(x-a)^m$ multiplied by an appropriate meseta function (ϕ belongs to $W^{1,\infty}(w_0,w_1)$ by hypothesis).

(3) It is sufficient to apply Proposition 5.1 with $s(x):=(x-a)^{-m}/w_0(x)$. ♯

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