A REAL VARIABLE CHARACTERIZATION OF GROMOV HYPERBOLICITY OF FLUTE SURFACES

ANA PORTILLA[(1)], JOSÉ M. RODRÍGUEZ [(1)] AND EVA TOURÍS[(1)]

Departamento de Matemáticas
Escuela Politécnica Superior
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés (Madrid), SPAIN
emails: apferrei@math.uc3m.es, jomaro@math.uc3m.es, etouris@math.uc3m.es
Corresponding author: José M. Rodríguez

Abstract. In this paper we give a characterization of the Gromov hyperbolicity of trains (a large class of Denjoy domains which contains the flute surfaces) in terms of the behavior of a real function. This function describes somehow the distances between some remarkable geodesics in the train. This theorem has several consequences; in particular, it allows to deduce a result about stability of hyperbolicity, even though the original surface and the modified one are not quasi-isometric.

Key words and phrases: Denjoy domain, flute surface, Gromov hyperbolicity, Riemann surface, train.
2000 AMS Subject Classification: 41A10, 46E35, 46G10.

1. Introduction.

To understand the connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [4], [10], [13], [21], [22], [23], [24], [29], [30], [34]), Gromov hyperbolic spaces are a useful tool. Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [15], [17], [18] and the references therein).

A geodesic metric space is called hyperbolic (in the Gromov sense) if there exists an upper bound of the distance of every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.3). The latter condition is known as Rips condition.

But, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. Some specific examples are showing that the Klein-Hilbert metric ([8], [25]) is Gromov hyperbolic (under particular conditions on the domain of definition), that the Gehring-Osgood metric ([20]) is Gromov hyperbolic, and that the Vuorinen metric ([20]) is not Gromov hyperbolic (except for a particular case). Recently, some interesting results by Balogh and Buckley [5] about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see also [9], [35] and the references therein). Another interesting instance is that of a Riemann surface endowed with the Poincaré metric. With such metric structure a Riemann surface is always negatively curved, but not every Riemann surface is Gromov hyperbolic, since topological obstacles may impede it: for instance, the two-dimensional jungle-gym (a $\mathbb{Z}^2$-covering of a torus with genus two) is not hyperbolic.

We are interested in studying when Riemann surfaces equipped with their Poincaré metric are Gromov hyperbolic (see e.g. [31], [32], [33], [26], [27], [28], [3]). To be more precise, in the current paper our main aim is to study the hyperbolicity of Denjoy domains, that is to say, plane domains Ω with $\partial\Omega \subset \mathbb{R}$. This kind of

Date: December 7, 2006.
(1) Research partially supported by three grants from M.E.C. (MTM 2006-11976, MTM 2006-13000-C03-02 and MTM 2006-26627-E), Spain.

surfaces are becoming more and more important in Geometric Theory of Functions, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, they are more manageable due to its symmetry. For instance, Garnett and Jones have proved the Corona Theorem for Denjoy domains ([14]), and in [2] the authors have got the characterization of Denjoy domains which satisfy a linear isoperimetric inequiality.

Denjoy domains are such a wide class of Riemann surfaces that characterization criteria are not straightforward to apply. That is the main reason that led us to focus on a particular type of Denjoy domain, which we have called *trains* (see Definition 2.5). Trains do include a especially important case of surfaces which are the flute surfaces (see, e.g. [6], [7]). These ones are the simplest examples of infinite ends, and besides, in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [19]). In [3] there are some partial results on hyperbolicity of trains.

The main result of this paper, Theorem 4.2, provides a characterization of the hyperbolicity of trains in terms of the behavior of a real function (with two integer parameters). This function describes somehow the distances between some remarkable geodesics in the train. This theorem is such manageable as to allow to deduce a result about stability of hyperbolicity, even though the original surface and the modified one are not quasi-isometric (see Theorem 4.8).

Theorem 4.2 allows to deduce also both sufficient and necessary conditions that either guarantee or discard hyperbolicity (see Theorems 4.14, 4.16 and 4.17). Besides, these three theorems give a much simpler characterization (see Theorem 4.18) than Theorem 4.2 for an interesting case of trains.

Theorem 4.22 gives some answers to the following question: how do some perturbations affect on the hyperbolicity of a flute surface?

Notations. We denote by X a geodesic metric space. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X.

We denote by Ω a train with its Poincaré metric.

Given a subset F of the complex plane, we define $F^+ = F \cap \{z \in \mathbb{C} : \Im z \geq 0\}$, where $\Im z$ is the imaginary part of z.

If E is either a function or constant related to a domain Ω, we will denote by E' or E^j the same function or constant related to a domain Ω' or Ω^j, respectively.

Finally, we denote by c and c_i, positive constants which can assume different values in different theorems.

Acknowledgements. We would like to thank Professor J. L. Fernández for some useful discussions.

2. Background in Gromov spaces and Riemann surfaces.

In our study of hyperbolic Gromov spaces we use the notations of [15]. We give now the basic facts about these spaces. We refer to [15] for more background and further results.

Definition 2.1. *Let us fix a point w in a metric space (X, d). We define the* Gromov product *of $x, y \in X$ with respect to the point w as*

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big) \geq 0 .$$

*We say that the metric space (X, d) is δ-*hyperbolic *($\delta \geq 0$) if*

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta ,$$

for every $x, y, z, w \in X$. We say that X is hyperbolic *(in the Gromov sense) if the value of δ is not important.*

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples:

(1) Every bounded metric space X is $(diamX)$-hyperbolic (see e.g. [15, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [15, p. 52]).

(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [15, p. 29]).

Definition 2.2. *If $\gamma : [a,b] \longrightarrow X$ is a continuous curve in a metric space (X,d), the* length *of γ is*

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that γ is a geodesic *if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s,t \in [a,b]$. We say that X is a* geodesic metric space *if for every $x,y \in X$ there exists a geodesic joining x and y; we denote by $[x,y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).*

Definition 2.3. *Consider a geodesic metric space X. If $x_1, x_2, x_3 \in X$, a* geodesic triangle $T = \{x_1, x_2, x_3\}$ *is the union of three geodesics $[x_1,x_2]$, $[x_2,x_3]$ and $[x_3,x_1]$. We say that T is* δ-thin *if for every $x \in [x_i, x_j]$ we have that $d(x, [x_j,x_k] \cup [x_k,x_i]) \le \delta$. The space X is* δ-thin (*or satisfies the* Rips condition *with constant δ) if every geodesic triangle in X is δ-thin.*

As the following basic result states, hyperbolicity is equivalent to Rips condition:

Theorem 2.4. *(*[15, p. 41]*) Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin.*
(2) *If X is δ-thin, then it is 4δ-hyperbolic.*

A *non-exceptional* Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1-|z|^2)$. Therefore, any simply connected subset of S is isometric to a subset of $\mathbb{D}$. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1, and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the *exceptional* Riemann surfaces, that is to say, the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases. The Poincaré metric is natural and useful in Complex Analysis: for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

A *Denjoy domain* is a domain Ω in the Riemann sphere with $\partial\Omega \subset \mathbb{R} \cup \{\infty\}$. As we mentioned in the introduction of this paper, Denjoy domains are becoming more and more interesting in Geometric Function Theory (see e.g. [1], [2], [14], [16]).

It is obvious that as we focus on more particular kind of surfaces, we can obtain more powerful results. That is the reason because we introduce now a new type of space.

We have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously can not be minimizing geodesics. We will continue using the word geodesic with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

Definition 2.5. *A* train *is a Denjoy domain $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty} (a_n, b_n)$, such that $-\infty \le a_0$ and $b_n \le a_{n+1}$ for every n. A* flute surface *is a train with $b_n = a_{n+1}$ for every n.*

We say that a curve in a train Ω is a fundamental geodesic *if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_0,b_0) and (a_n,b_n) for some $n>0$; we denote by γ_n the fundamental geodesic corresponding to n and $2l_n := L_\Omega(\gamma_n)$. A curve in a train Ω is a* second fundamental geodesic *if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_n,b_n) and (a_{n+1},b_{n+1}) for some $n \ge 0$; we denote by σ_n the second fundamental geodesic corresponding to n and $2r_n := L_\Omega(\sigma_n)$. If $b_n = a_{n+1}$, we define σ_n as the puncture at this point and $r_n = 0$. Given $z \in \Omega$, we define the* height *of z as $h(z) := d_\Omega(z, (a_0,b_0))$.*

A train is a flute surface if and only if every second fundamental geodesic is a puncture.

Flute surfaces are the simplest examples of infinite ends; furthermore, in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [19]).

3. Trigonometric lemmas.

In this section some technical lemmas are collected. All of them will be used in order to simplify the proof of Theorem 4.2.

Definition 3.1. *Given a surface M, a geodesic γ in M, and a continuous unit vector field ξ along γ, orthogonal to γ, we define the* Fermi coordinates *based on γ as the map $E(u,v) := \exp_{\gamma(u)} v\xi(u)$.*

It is well known that the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dv^2 + \eta^2(u,v)\,du^2$, where $\eta(u,v)$ is the solution of the scalar equation $\partial^2\eta/\partial v^2 + K\eta = 0$, $\eta(u,0) = 1$, $\partial\eta/\partial v(u,0) = 0$, and K is the curvature of M (see e.g. [11, p. 247]). Consequently, if M is a non-exceptional Riemann surface, the Poincaré metric in Fermi coordinates (based on any geodesic γ) is $ds^2 = dv^2 + \cosh^2 v\, du^2$, since $K = -1$ in the Poincaré metric. We always consider in a train the Fermi coordinates based on (a_0, b_0).

Definition 3.2. *Let us consider Fermi coordinates (u,v) in $\mathbb{D}$. We define the distances $d_1\big((u_1,v_1),(u_2,v_2)\big)$, $d_2\big((u_1,v_1),(u_2,v_2)\big)$ as follows: without loss of generality we can assume that $v_1 \ge v_2$; then*

$$d_1\big((u_1,v_1),(u_2,v_2)\big) := d\big((u_1,v_1),(u_1,v_2)\big) + d\big((u_1,v_2),(u_2,v_2)\big) = v_1 - v_2 + d\big((u_1,v_2),(u_2,v_2)\big),$$

$$d_2\big((u_1,v_1),(u_2,v_2)\big) := d\big((u_1,v_1),(u_2,v_1)\big) + d\big((u_2,v_1),(u_2,v_2)\big) = d\big((u_1,v_1),(u_2,v_1)\big) + v_1 - v_2\,.$$

The following lemma shows that the "cartesian distances" d_1 and d_2 are comparable to d.

Lemma 3.3. *Let us consider Fermi coordinates (u,v) in $\mathbb{D}$ and the distances d_1 and d_2. Then*

$$\frac{1}{2}\, d_1 \le d \le d_1\,, \qquad \frac{1}{3}\, d_2 \le d \le d_2\,.$$

Proof. Triangle inequality gives directly $d \le d_1$ and $d \le d_2$. Let us consider $v_1 \ge v_2$. It is easy to check that

$$d\big((u_1,v_1),(u_1,v_2)\big) \le d\big((u_1,v_1),(u_2,v_2)\big), \qquad d\big((u_1,v_2),(u_2,v_2)\big) \le d\big((u_1,v_1),(u_2,v_2)\big)$$

and this implies $d_1 \le 2d$.

We also have $d\big((u_2,v_1),(u_2,v_2)\big) \le d\big((u_1,v_1),(u_2,v_2)\big)$, and then

$$d\big((u_1,v_1),(u_2,v_1)\big) \le d\big((u_1,v_1),(u_2,v_2)\big) + d\big((u_2,v_1),(u_2,v_2)\big) \le 2\, d\big((u_1,v_1),(u_2,v_2)\big),$$

$$d_2\big((u_1,v_1),(u_2,v_2)\big) = d\big((u_1,v_1),(u_2,v_1)\big) + d\big((u_2,v_1),(u_2,v_2)\big) \le 3\, d\big((u_1,v_1),(u_2,v_2)\big).$$

□

Lemma 3.4. *Let Ω be a train and l_0 any positive constant. We have*

$$d_1(z, \gamma_n \cap (a_n,b_n)) \le 2\, d_\Omega(z,(a_n,b_n)) + 2 \operatorname{Arcsinh} \frac{1}{\sqrt{2\tanh l_0}}\,,$$

for every $n > 0$ and $z \in \Omega$ with $l_0 \le h(z) \le l_n$.

Proof. Let w be the nearest point in (a_n,b_n) to z, and define $v := \gamma_n \cap (a_n,b_n)$, let v_0 be the nearest point in (a_0,b_0) to v and w_0 the nearest point in (a_0,b_0) to w. Consider the geodesic quadrilateral in Ω^+ with vertices v, w, w_0 and v_0. Standard hyperbolic trigonometry gives that

$$\tanh d_\Omega(w,w_0) = \tanh d_\Omega(v,v_0) \cosh d_\Omega(v_0,w_0) = \tanh l_n \cosh d_\Omega(v_0,w_0)\,.$$

Denote by v' (respectively w') the point in $\gamma_n^+ = [v,v_0] \subset \Omega^+$ (respectively in $[w,w_0] \subset \Omega^+$) with $h(v') = h(z)$ (respectively $h(w') = h(z)$). Consider the geodesic quadrilateral in Ω with vertices v', w', w_0 and v_0. Standard hyperbolic trigonometry (see e.g. [12, p. 88]) gives that

$$\begin{aligned}\sinh \frac{d_\Omega(v',w')}{2} &= \sinh \frac{d_\Omega(v_0,w_0)}{2}\, \cosh h(z) = \cosh h(z) \sqrt{\frac{\cosh d_\Omega(v_0,w_0) - 1}{2}} \\ &= \frac{1}{\sqrt{2}}\, \cosh h(z) \sqrt{\frac{\tanh d_\Omega(w,w_0)}{\tanh l_n} - 1} \;\le\; \frac{1}{\sqrt{2}}\, \cosh h(z) \sqrt{\frac{1}{\tanh h(z)} - 1} \\ &= \frac{1}{\sqrt{2}}\, \cosh h(z) \sqrt{\frac{1 - \tanh^2 h(z)}{\tanh h(z)}} = \frac{1}{\sqrt{2\tanh h(z)}} \le \frac{1}{\sqrt{2\tanh l_0}}\,.\end{aligned}$$

This fact and Lemma 3.3 imply

$$\begin{aligned} d_1(z,v) &= d_\Omega(z,v') + d_\Omega(v',v) \le d_\Omega(v',w') + d_\Omega(z,w') + d_\Omega(w',w) \\ &\le 2\,\text{Arcsinh}\,\frac{1}{\sqrt{2\tanh l_0}} + d_1(z,w) \le 2\,d_\Omega(z,w) + 2\,\text{Arcsinh}\,\frac{1}{\sqrt{2\tanh l_0}}\,. \end{aligned}$$

□

Lemma 3.5. *Let us consider Fermi coordinates (u,v) in $\mathbb{D}$. Fix $u_1 < u_4$, $g_1 := \{(u,v) : u = u_1,\ 0 \le v \le x\}$, $g_4 := \{(u,v) : u = u_4,\ v \ge 0\}$, and g_2 the (infinite) geodesic orthogonal to g_1 in (u_1,x). We assume that g_2 does not intersects g_4. Consider $(u_4,h) \in g_4$, with $h \ge x$, and $(u_2,v_2) \in g_2$, with $d\big((u_2,v_2),(u_4,h)\big) = d\big(g_2,(u_4,h)\big)$. Then*

$$d\big(g_2,(u_4,h)\big) \le d\big(g_2,(u_3,h)\big) + d\big((u_3,h),(u_4,h)\big) \le 6\,d\big(g_2,(u_4,h)\big)\,,$$

for every $u_2 \le u_3 \le u_4$.

Proof. We only need to prove the second inequality. Fix $u_3 \in [u_2,u_4]$.

Let us assume that $v_2 \le h$. Then Lemma 3.3 implies

$$\begin{aligned} d\big(g_2,(u_3,h)\big) + d\big((u_3,h),(u_4,h)\big) &\le d\big((u_2,v_2),(u_2,h)\big) + d\big((u_2,h),(u_3,h)\big) + d\big((u_3,h),(u_4,h)\big) \\ &\le d\big((u_2,v_2),(u_2,h)\big) + 2\,d\big((u_2,h),(u_4,h)\big) \\ &\le 2\,d_2\big((u_2,v_2),(u_4,h)\big) \le 6\,d\big((u_2,v_2),(u_4,h)\big) = 6\,d\big(g_2,(u_4,h)\big)\,. \end{aligned}$$

Let us assume now that $v_2 \ge h$. Lemma 3.3 also implies

$$\begin{aligned} d\big(g_2,(u_3,h)\big) + d\big((u_3,h),(u_4,h)\big) &\le d\big((u_2,v_2),(u_2,h)\big) + d\big((u_2,h),(u_3,h)\big) + d\big((u_3,h),(u_4,h)\big) \\ &\le d\big((u_2,v_2),(u_2,h)\big) + 2\,d\big((u_2,h),(u_4,h)\big) \\ &\le 2\,d_1\big((u_2,v_2),(u_4,h)\big) \le 4\,d\big((u_2,v_2),(u_4,h)\big) = 4\,d\big(g_2,(u_4,h)\big)\,. \end{aligned}$$

□

Lemma 3.6. *Let us define F as*

$$F(a,x) := \begin{cases} \dfrac{1}{\sinh 1}\,\sinh a \cosh x\,, & \text{if } 0 \le a \le 1\,, \\[2ex] \log\big(\sinh a \cosh x\big)\,, & \text{if } a \ge 1\,. \end{cases}$$

Then

$$F(a,x) \le a\,e^x \le 2\sinh a \cosh x\,,$$

for every $a,x \ge 0$.

Proof. The last inequality is a direct consequence of $a \le \sinh a$ and $e^x \le 2\cosh x$.

If $a \ge 1$, the function $h(x) := a\,e^x - a - x$ satisfies $h'(x) = a\,e^x - 1 \ge a - 1 \ge 0$ for every $x \ge 0$. Hence, $h(x) \ge h(0) = 0$ for every $x \ge 0$, and we conclude

$$a\,e^x \ge a + x = \log\big(e^a e^x\big) \ge \log\big(\sinh a \cosh x\big),$$

for $a \ge 1$ and $x \ge 0$.

Since the function $H(a) := \sinh a - a\sinh 1$ is convex in $[0,1]$, it satisfies $H(a) \le \max\{H(0),\, H(1)\} = 0$ for every $0 \le a \le 1$. Hence,

$$a\,e^x \ge \frac{1}{\sinh 1}\,\sinh a\,e^x \ge \frac{1}{\sinh 1}\,\sinh a \cosh x\,,$$

for $0 \le a \le 1$ and $x \ge 0$. □

This result has the following direct corollary.

Corollary 3.7. *For a set* $E \subset \{(a,x) :\ a,x \ge 0\}$, *we have* $\operatorname{Arcsinh}\big(\sinh a \cosh x\big) \le c_1$, *for every* $(a,x) \in E$ *and some constant* c_1, *if and only if* $a\, e^x \le c_2$, *for every* $(a,x) \in E$ *and some constant* c_2.

Furthermore, if one of the inequalities holds, the constant in the other inequality only depends on the first constant.

As usual, we denote by x_+ the positive part of x: $x_+ := x$ if $x \ge 0$ and $x_+ := 0$ if $x < 0$.

Proposition 3.8.

(1) *There exists a universal constant* c_1 *such that*

$$f(x,y,t) := \operatorname{Arccosh} \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \ge c_1\big(e^{-x} + e^{-y} + e^{-\frac12 (x+y-t)_+} + (t-x-y)_+\big)\,,$$

for every $x,y,t \ge 0$.

(2) *For each* $l_0 > 0$, *there exists a constant* c_2, *which only depends on* l_0, *such that*

$$\operatorname{Arccosh} \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \le c_2\big(e^{-x} + e^{-y} + e^{-\frac12 (x+y-t)_+} + (t-x-y)_+\big)\,,$$

for every $t \ge 0$ *and* $x,y \ge l_0$.

Remark. This result is interesting by itself: if H is a right-angled hexagon in the unit disk for which three pairwise non-adjacent sides X, Y, T are given (with respective lengths x, y, t), then the opposite side of T in H has length $f(x,y,t)$ (see e.g. [12, p. 86], or the proof of Theorem 4.2).

Proof. First, we remark that if $x \ge l_0$, then $e^{-2l_0} e^{2x} \ge 1$ and $e^{2x} - 1 \ge (1 - e^{-2l_0}) e^{2x}$. Therefore, if we define $c_3^{-1} := (1 - e^{-2l_0})/2$, we have

$$e^{2x} - 1 \ge 2\, c_3^{-1} e^{2x}\,, \qquad \sinh x \ge c_3^{-1} e^{x}\,, \qquad \coth x = 1 + \frac{2}{e^{2x}-1} \le 1 + c_3\, e^{-2x}\,, \qquad \text{for every } x \ge l_0\,.$$

We also have

$$\coth x = 1 + \frac{2}{e^{2x}-1} \ge 1 + 2\, e^{-2x}\,, \qquad \text{for every } x \ge 0\,.$$

Let us start with the proof of item (1).

If $f \ge 3$, then $f \ge e^{-x} + e^{-y} + e^{-\frac12 (x+y-t)_+}$. If $f \le 3$, then $1 + \frac23 c_4^{-2} f^2 \ge \cosh f$, for some universal constant $c_4 \le 1$, and

$$\begin{aligned}
1 + \frac23 c_4^{-2} f^2 &\ge \cosh f \ge 2\, e^{t-x-y} + \coth x \coth y \ge 2\, e^{-(x+y-t)} + \big(1 + 2\, e^{-2x}\big)\big(1 + 2\, e^{-2y}\big)\,,\\
1 + \frac23 c_4^{-2} f^2 &\ge 1 + 2\big(e^{-2x} + e^{-2y} + e^{-(x+y-t)_+}\big)\,,\\
c_4^{-1} f &\ge \sqrt{3}\sqrt{e^{-2x} + e^{-2y} + e^{-(x+y-t)_+}} \ge e^{-x} + e^{-y} + e^{-\frac12 (x+y-t)_+}\,,\\
f &\ge c_4\big(e^{-x} + e^{-y} + e^{-\frac12 (x+y-t)_+}\big)\,,
\end{aligned}$$

where we have used the inequality $\sqrt{3}\sqrt{a+b+c} \ge \sqrt{a} + \sqrt{b} + \sqrt{c}$, for every $a,b,c \ge 0$. This inequality is (1) if $t \le x + y$. If $t \ge x + y$, then

$$\cosh f > \frac{\cosh t}{\sinh x \sinh y} + 1 \ge 2\, e^{t-x-y} + 1 > \frac42\, e^{t-x-y} + \frac{1}{4 \cdot 2}\, e^{-(t-x-y)} = \cosh\big(t - x - y + \log 4\big)$$

and $f > t - x - y + \log 4 > (t-x-y)_+ + e^{-\frac12 (x+y-t)_+}$.

Consequently we have

$$f \ge c_1\big(e^{-x} + e^{-y} + e^{-\frac12 (x+y-t)_+} + (t-x-y)_+\big)\,,$$

for every $x,y,t \ge 0$, with $c_1 := c_4/2$, since $c_4 \le 1$.

Next, let us prove item (2). Fix $l_0 > 0$. We have seen that $\sinh x \ge c_3^{-1} e^x$ and $\coth x \le 1 + c_3\, e^{-2x}$, for every $x \ge l_0$.

Let us assume $t \ge x+y$. If $x, y \ge l_0$, then

$$\frac{1}{2}\, e^f \le \cosh f = \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \le c_3^2\, e^{t-x-y} + \operatorname{cotanh}^2 l_0 \,.$$

Consequently,

$$e^f \le 2\, c_3^2\, e^{t-x-y} + 2 \operatorname{cotanh}^2 l_0 \le e^{t-x-y+c_5},$$

with $c_5 := \log\big(2\, c_3^2 + 2 \operatorname{cotanh}^2 l_0\big)$, since $t-x-y \ge 0$. Hence, $f \le t-x-y+c_5 = (t-x-y)_+ + c_5 e^{-\frac{1}{2}(x+y-t)_+}$, for every $t \ge 0$ and $x, y \ge l_0$ with $t \ge x+y$.

Let us assume $t \le x+y$. If $x, y \ge l_0$, then

$$\begin{aligned}
1 + \frac{1}{2}\, f^2 &\le \cosh f \le c_3^2\, e^{t-x-y} + \operatorname{cotanh} x \operatorname{cotanh} y \le c_3^2\, e^{t-x-y} + \big(1 + c_3\, e^{-2x}\big)\big(1 + c_3\, e^{-2y}\big) \,, \\
\frac{1}{2}\, f^2 &\le c_3^2\, e^{t-x-y} + c_3\, e^{-2x} + c_3\, e^{-2y} + c_3^2\, e^{-2x-2y}, \\
\frac{1}{2}\, f^2 &\le c_3^2\, e^{t-x-y} + c_3\, e^{-2x} + c_3\, e^{-2y} + \frac{1}{2}\, c_3^2 \left(e^{-2x} + e^{-2y}\right) , \\
f^2 &\le 2\, c_3^2\, e^{-(x-y-t)} + \big(2\, c_3 + c_3^2\big) e^{-2x} + \big(2\, c_3 + c_3^2\big) e^{-2y}, \\
f^2 &\le c_6^2 \big(e^{-2x} + e^{-2y} + e^{-(x+y-t)_+}\big) , \\
f &\le c_6 \big(e^{-x} + e^{-y} + e^{-(x+y-t)_+} + (t-x-y)_+\big) ,
\end{aligned}$$

where $c_6^2 := \max\big\{2\, c_3^2,\, 2\, c_3 + c_3^2\big\}$, for every $t \ge 0$ and $x, y \ge l_0$ with $t \le x+y$. Then we have (2) with $c_2 := \max\{1,\, c_5,\, c_6\}$. ☐

Proposition 3.9. *For each $l_0 > 0$, we have*

$$F(x,y,t,h) := \operatorname{Arcsinh} \frac{\cosh x \cosh(y-h) + \cosh t \cosh h}{\sinh y} \asymp e^{-h+x} + e^{-(y-h-t)_+} + (t+h-y)_+ \,,$$

for every $x, y, t, h \ge 0$, verifying $y \ge h \ge x$ and $y \ge l_0$. Furthermore, the constants in the inequalities only depend on l_0.

Remark. This result is interesting by itself: if H is a right-angled hexagon in the unit disk for which three pairwise non-adjacent sides X, Y, T are given (with respective lengths x, y, t), P is the nearest point to X in Y, and P_h is the point in Y with $d(P_h, P) = h$, then $F(x,y,t,h)$ is the distance between P_h and the opposite side of Y in H (see the proof of Theorem 4.2).

Proof. We have seen that if $y \ge l_0$, and $c_3^{-1} := (1 - e^{-2l_0})/2$, we have $c_3^{-1} e^y \le \sinh y \le e^y/2$. We also have $e^z/2 \le \cosh z \le e^z$, for every $z \ge 0$.

Then $\sinh F \asymp e^{-h+x} + e^{-y+h+t}$, since $y \ge l_0$ and $y \ge h$, and the constants in the inequalities only depend on l_0.

If $h+t \le y$, then $e^{-h+x} + e^{-y+h+t} \le 2$, and

$$F \asymp \sinh F \asymp e^{-h+x} + e^{-(y-h-t)} = e^{-h+x} + e^{-(y-h-t)_+} + (t+h-y)_+ \,.$$

If $h+t \ge y$, then $e^{-h+x} + e^{-y+h+t} \ge 1$, and

$$e^F \asymp \sinh F \asymp e^{-h+x} + e^{-y+h+t} \asymp e^{t+h-y} = e^{-1} e^{1+(t+h-y)_+} \,.$$

Since

$$F \ge \operatorname{Arcsinh} \frac{\big(e^x e^{y-h} + e^t e^h\big)/4}{e^y/2} \ge \operatorname{Arcsinh} \frac{1}{2}\big(e^{-h+x} + e^{-y+h+t}\big) \ge \operatorname{Arcsinh} \frac{1}{2} > 0 \,,$$

and $1 + (t+h-y)_+ \ge 1 > 0$ for every $x, y, t, h \ge 0$, and $e^F \asymp e^{1+(t+h-y)_+}$ for every $x, y, t, h \ge 0$, verifying $h+t \ge y \ge h \ge x$ and $y \ge l_0$, we obtain that $F \asymp 1 + (t+h-y)_+$. Since $1 \le e^{-h+x} + 1 = e^{-h+x} + e^{-(y-h-t)_+} \le 2$, we also conclude that $F \asymp e^{-h+x} + e^{-(y-h-t)_+} + (t+h-y)_+$, if $h+t \ge y$. ☐

The following corollary can be directly deduced from this result.

Corollary 3.10. *For each $l_0 > 0$, let us consider a set $E \subset \{(x, y, t, h) : x, y, t, h \ge 0,\ y \ge h \ge x,\ y \ge l_0\}$. We have $F(x, y, t, h) \le c_1$, for every $(x, y, t, h) \in E$ and some constant c_1, if and only if $(t + h - y)_+ \le c_2$, for every $(x, y, t, h) \in E$ and some constant c_2.*

Furthermore, if one of the inequalities holds, the constant in the other inequality only depends on the first constant and l_0.

Obviously, we can replace condition $(t + h - y)_+ \le c_2$ by $t + h - y \le c_2$. We prefer the first one since F will be a distance and $(t + h - y)_+ \ge 0$.

4. The main results.

It is not difficult to see that the values of $\{l_n\}$ and $\{r_n\}$ determine a train. Then there must be a characterization of hyperbolicity in terms of the lengths of the fundamental geodesics.

In order to obtain this characterization, we need to introduce the following functions.

Definition 4.1. *Let us consider a sequence of positive numbers $\{l_n\}_{n=1}^\infty$ and a sequence of non-negative numbers $\{r_n\}_{n=1}^\infty$. Consider $n \ge 1$ and $0 \le h \le l_n$. We define $A_n(h) := \max\{m < n : l_m \le h\}$ if this set is non-empty and $A_n(h) := 1$ in other case, $B_n(h) := \min\{m > n : l_m \le h\}$ if this set is non-empty and $B_n(h) := \infty$ in other case,*

$$\Delta(k) := e^{-l_k} + e^{-l_{k+1}} + e^{-\frac{1}{2}(l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+ \,,$$

and

$$\Gamma_{nm}(h) := \begin{cases} \big(r_m + h - l_{m+1}\big)_+ + e^h \displaystyle\sum_{k=m+1}^{n-1} \Delta(k)\,, & \text{if } m < n \text{ and } l_m \le h\,, \\ l_m - h + e^h \displaystyle\sum_{k=m}^{n-1} \Delta(k)\,, & \text{if } m < n \text{ and } l_m > h\,, \\ \min\big\{h,\, l_n - h\big\}\,, & \text{if } m = n\,, \\ l_m - h + e^h \displaystyle\sum_{k=n}^{m-1} \Delta(k)\,, & \text{if } m > n \text{ and } l_m > h\,, \\ \big(r_{m-1} + h - l_{m-1}\big)_+ + e^h \displaystyle\sum_{k=n}^{m-2} \Delta(k)\,, & \text{if } m > n \text{ and } l_m \le h\,. \end{cases}$$

The functions $\Gamma_{nm}(h)$ are naturally associated to trains by taking $\{l_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ as the half-lengths of their fundamental geodesics.

Theorem 4.2. *A train Ω is hyperbolic if and only if*

$$K := \sup_{n \ge 1} \sup_{h \in [0, l_n]} \min_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}(h) < \infty \,.$$

Furthermore, if Ω is δ-hyperbolic, then K is bounded by a constant which only depends on δ; if $K < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K.

Remarks.

(1) Notice that this is a real variable characterization of the hyperbolicity, although the hyperbolicity is a concept of complex geometry, since we consider the Poincaré metric in each train.

(2) Theorem 4.2 clearly improves [3, Theorem 5.3]: we need to know the lengths of the fundamental geodesics instead of the precise location of these geodesics and the distances to $\mathbb{R}$ from their points.

(3) The proof of Theorem 4.2 gives that its conclusion also holds if we replace K by

$$K(l_0) := \sup_{n \ge 1} \sup_{h \in [l_0, l_n]} \min_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}(h) < \infty \,,$$

for any fixed $l_0 > 0$. In this case, the constant δ depends on $K(l_0)$ and l_0.

Proof. By [3, Theorem 5.3], Ω is δ-hyperbolic if and only if

$$K_1 := \sup_{n\geq 1} \sup_{z\in\gamma_n} \inf_{m\geq 0} d_\Omega\big(z,(a_m,b_m)\big) < \infty\,,$$

with the appropriate dependence of the constants (if Ω is δ-hyperbolic, then K_1 is bounded by a constant which only depends on δ; if $K_1<\infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K_1).

Fix any constant $l_0>0$. Notice that:

(1) $d_\Omega\big(z,(a_0,b_0)\big)=h(z)$ and $d_\Omega\big(z,(a_n,b_n)\big)=l_n-h(z)$. Since any z with $h(z)<l_0$ verifies

$$\inf_{m\geq 0} d_\Omega\big(z,(a_m,b_m)\big) \leq d_\Omega\big(z,(a_0,b_0)\big) = h(z) < l_0\,,$$

we only need to consider z with $l_0\leq h(z)\leq l_n$.

From now on, let us fix $n\geq 1$ and $z\in\gamma_n$ with $l_0\leq h(z)\leq l_n$.

(2) If $k<m<n$, with $l_m\leq h(z)$, let us consider the geodesic σ which gives the minimum distance between z and (a_k,b_k). Define the point $w:=\sigma\cap\gamma_m$; hence $d_\Omega\big(z,w\big)<d_\Omega\big(z,(a_k,b_k)\big)$ and Lemma 3.3 gives

$$d_\Omega\big(z,(a_m,b_m)\big) \leq d_2\big(z,(a_m,b_m)\cap\gamma_m\big) \leq d_2\big(z,w\big) \leq 3\,d_\Omega\big(z,w\big) < 3\,d_\Omega\big(z,(a_k,b_k)\big)\,.$$

In a similar way, if $k>m>n$, with $l_m\leq h(z)$, then $d_\Omega\big(z,(a_m,b_m)\big)<3\,d_\Omega\big(z,(a_k,b_k)\big)$. Hence we only need to consider $d_\Omega\big(z,(a_m,b_m)\big)$ with $m\in\{0\}\cup[A_n(h(z)),B_n(h(z))]$, in order to study if K_1 is finite.

(3) If $m\in(A_n(h(z)),n)$, then $l_0\leq h(z)<l_m$. By Lemma 3.4, we can replace $d_\Omega\big(z,(a_m,b_m)\big)$ by $d_1\big(z,\gamma_m\cap(a_m,b_m)\big)$. If z_m is the point in γ_m with $h(z_m)=h(z)$, then $d_1\big(z,\gamma_m\cap(a_m,b_m)\big):=d_\Omega(z,z_m)+l_m-h(z)$. Standard hyperbolic trigonometry in quadrilaterals (see e.g. [12, p. 88]) gives that

$$d_\Omega(z,z_m) = 2\,\mathrm{Arcsinh}\,\Big(\sinh\frac{1}{2}\,d_\Omega(\gamma_m,\gamma_n)\cosh h(z)\Big)\,.$$

Recall that (a_0,b_0) contains the shortest geodesic joining γ_m and γ_n. By Corollary 3.7 we can replace $d_\Omega(z,z_m)$ by $d_\Omega(\gamma_m,\gamma_n)\,e^{h(z)}$, and therefore $d_1\big(z,\gamma_m\cap(a_m,b_m)\big)$ by $d_\Omega(\gamma_m,\gamma_n)\,e^{h(z)}+l_m-h(z)$. Standard hyperbolic trigonometry in right-angled hexagons (see e.g. [12, p. 86]) gives that

$$d_\Omega(\gamma_k,\gamma_{k+1}) = \mathrm{Arccosh}\,\frac{\cosh r_k+\cosh l_k\cosh l_{k+1}}{\sinh l_k\sinh l_{k+1}}$$

for every $k\geq 1$. Proposition 3.8 gives

$$d_\Omega(\gamma_k,\gamma_{k+1}) = f(l_k,l_{k+1},r_k) \asymp e^{-l_k}+e^{-l_{k+1}}+e^{-\frac{1}{2}(l_k+l_{k+1}-r_k)_+}+(r_k-l_k-l_{k+1})_+ = \Delta(k)\,,$$

for every $k\in(A_n(h(z)),n)$, since then $l_k,l_{k+1}\geq h(z)\geq l_0$. Therefore we can replace $d_\Omega\big(z,(a_m,b_m)\big)$ by

$$l_m-h(z)+e^{h(z)}\sum_{k=m}^{n-1}\Delta(k)\,.$$

A symmetric argument gives that if $m\in(n,B_n(h(z)))$, then we can replace $d_\Omega\big(z,(a_m,b_m)\big)$ by

$$l_m-h(z)+e^{h(z)}\sum_{k=n}^{m-1}\Delta(k)\,.$$

(4) If $m=A_n(h(z))$, then $h(z)\geq l_m$. If z_{m+1} is the point in γ_{m+1} with $h(z_{m+1})=h(z)$, by Lemma 3.5, we can replace $d_\Omega\big(z,(a_m,b_m)\big)$ by $d_\Omega\big(z,z_{m+1}\big)+d_\Omega\big(z_{m+1},(a_m,b_m)\big)$. We have seen in (3) that we can replace $d_\Omega\big(z,z_{m+1}\big)$ by

$$e^{h(z)}\sum_{k=m+1}^{n-1}\Delta(k)\,.$$

Standard hyperbolic trigonometry in pentagons (see e.g. [12, p. 87]) gives that

$$\sinh d_\Omega\big(z_{m+1},(a_m,b_m)\big) = -\cosh l_m\sinh h(z)+\sinh l_m\cosh h(z)\cosh d_\Omega(\gamma_m,\gamma_{m+1})\,.$$

Standard hyperbolic trigonometry in right-angled hexagons (see e.g. [12, p. 86]) gives that

$$\cosh d_\Omega(\gamma_m,\gamma_{m+1}) = \frac{\cosh r_m + \cosh l_m \cosh l_{m+1}}{\sinh l_m \sinh l_{m+1}}\,,$$

and hence

$$\begin{aligned}\sinh d_\Omega\big(z_{m+1},(a_m,b_m)\big) &= -\cosh l_m \sinh h(z) + \cosh h(z)\,\frac{\cosh r_m + \cosh l_m \cosh l_{m+1}}{\sinh l_{m+1}}\\ &= \frac{\cosh l_m\big(\cosh l_{m+1}\cosh h(z) - \sinh l_{m+1}\sinh h(z)\big) + \cosh r_m \cosh h(z)}{\sinh l_{m+1}}\\ &= \frac{\cosh l_m \cosh\big(l_{m+1}-h(z)\big) + \cosh r_m \cosh h(z)}{\sinh l_{m+1}} = \sinh F\big(l_m,l_{m+1},r_m,h(z)\big)\,,\end{aligned}$$

where F is the function in Proposition 3.9. Therefore, Corollary 3.10 gives that we can replace $d_\Omega\big(z_{m+1},(a_m,b_m)\big)$ by $\big(r_m+h(z)-l_{m+1}\big)_+$. Consequently, we can substitute $d_\Omega\big(z,(a_m,b_m)\big)$ by

$$\big(r_m+h(z)-l_{m+1}\big)_+ + e^{h(z)}\sum_{k=m+1}^{n-1}\Delta(k)\,.$$

A symmetric argument gives that if $m=B_n(h(z))$, then we can replace $d_\Omega\big(z,(a_m,b_m)\big)$ by

$$\big(r_{m-1}+h(z)-l_{m-1}\big)_+ + e^{h(z)}\sum_{k=n}^{m-2}\Delta(k)\,.$$

Notice that each time that we replace a quantity by other in this proof, the constants are under control. Let us remark that (1), (2), (3) and (4) give the result, with $\inf_{m\in[A_n(h),B_n(h)]}\Gamma_{nm}(h)$ instead of $\min_{m\in[A_n(h),B_n(h)]}\Gamma_{nm}(h)$.

Let us see now that this infimum is attained. Seeking for a contradiction, suppose that the latest statement is not true. Therefore, $B_n(h)=\infty$ and $l_m>h$ for every $m>n$. Then, there exists an increasing sequence of integer numbers $\{m_j\}$ with $\lim_{j\to\infty}\Gamma_{nm_j}(h)=\inf_{m\in[A_n(h),\infty)}\Gamma_{nm}(h)$. By choosing a subsequence if it is necessary, we can assume that $\{\Gamma_{nm_j}(h)\}_j$ is a decreasing sequence. Hence,

$$\Gamma_{nm_{j+1}}(h) = l_{m_{j+1}} - h + e^h\sum_{k=n}^{m_{j+1}-1}\Delta(k) < \Gamma_{nm_j}(h) = l_{m_j} - h + e^h\sum_{k=n}^{m_j-1}\Delta(k)\,.$$

Consequently, we have that $l_{m_{j+1}} < l_{m_j} < l_{m_1}$ for every j, and

$$\Gamma_{nm_j}(h) = l_{m_j} - h + e^h\sum_{k=n}^{m_j-1}\Delta(k) \ge e^h\sum_{k=n}^{m_j} e^{-l_k} \ge e^h\sum_{k=1}^{j} e^{-l_{m_k}} \ge e^h\, j\, e^{-l_{m_1}}\,.$$

Hence, $\lim_{j\to\infty}\Gamma_{nm_j}(h)=\lim_{j\to\infty} e^h\, j\, e^{-l_{m_1}}=\infty$, which is a contradiction. This finishes the proof. □

Lemma 4.3. *For every $r_k\ge 0$ and $0<l_k\le h\le l_{k+1}$, we have*

$$\big(r_k+h-l_{k+1}\big)_+ < e^h\Delta(k)\,.$$

Proof. Let us remark that it is sufficient to prove

$$r_k+h-l_{k+1} < e^h\big(e^{-\frac12(l_k+l_{k+1}-r_k)_+} + (r_k-l_k-l_{k+1})_+\big)\,,$$

for every $r_k\ge 0$ and $0<l_k\le h\le l_{k+1}$.

Since the left hand side of the inequality does not depend on l_k and the right hand side is a decreasing function on l_k, it is enough to prove

$$r_k+h-l_{k+1} < e^h\big(e^{-\frac12(h+l_{k+1}-r_k)_+} + (r_k-h-l_{k+1})_+\big)\,,$$

for every $r_k\ge 0$ and $0<h\le l_{k+1}$.

If $r_k\le h+l_{k+1}$, then the inequality is

$$r_k+h-l_{k+1} < e^h e^{-\frac12(h+l_{k+1}-r_k)} = e^{\frac12(r_k+h-l_{k+1})}\,,$$

which trivially holds since $t < e^{t/2}$ for every real number t.

If $r_k \ge h + l_{k+1}$, then the inequality is

$$r_k + h - l_{k+1} < e^h (1 + r_k - h - l_{k+1}) .$$

Since $h > 1$, it is clear that the function

$$U(r_k) := e^h (1 + r_k - h - l_{k+1}) - r_k - h + l_{k+1}$$

is increasing in $r_k \in [h + l_{k+1}, \infty)$. Then $U(r_k) \ge U(h + l_{k+1}) = e^h - 2h > 0$, and the inequality holds. □

Proposition 4.4. *In any train Ω we have*

$$\min_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}(h) = \min_{m \ge 1} \Gamma_{nm}(h) ,$$

for every $n \ge 1$ and $0 \le h \le l_n$.

Proof. Fix $n \ge 1$ and $0 \le h \le l_n$. If $m < A_n(h)$, then Lemma 4.3 gives $\Gamma_{nm}(h) > \Gamma_{nA_n(h)}(h)$:

$$\begin{aligned}\Gamma_{nm}(h) &\ge e^h \sum_{k=m+1}^{n-1} \Delta(k) \ge e^h \sum_{k=A_n(h)}^{n-1} \Delta(k) = e^h \Delta(A_n(h)) + e^h \sum_{k=A_n(h)+1}^{n-1} \Delta(k) \\ &> \big(r_{A_n(h)} + h - l_{A_n(h)+1}\big)_+ + e^h \sum_{k=A_n(h)+1}^{n-1} \Delta(k) = \Gamma_{nA_n(h)}(h) .\end{aligned}$$

The case $m > B_n(h)$ is similar. □

Proposition 4.5. *If for some n we have $l_m \ge l_n$ for every $m \ge n$, then the conclusion of Theorem 4.2 also holds if we replace $[A_n(h), B_n(h)]$ by $[A_n(h), n]$ for this n.*

Proof. It is enough to remark that for every $z \in \gamma_n$ and $m > n$, we have $d_\Omega(z, (a_n, b_n)) = l_n - h(z) \le l_m - h(z) < d_\Omega(z, (a_m, b_m))$. □

Although to compute the minimum and the supremum in Theorem 4.2 can be difficult in the general case, Theorem 4.2 is the main tool in order to obtain the remaining results of this paper. We start with an elementary corollary.

Proposition 4.6. *Let us consider a train Ω with $l_n \le c$ for every n. Then Ω is δ-hyperbolic, where δ is a constant which only depends on c.*

Proof. For each positive integer n, we have $\Gamma_{nn}(h) := \min \big\{h,\, l_n - h\big\} \le l_n \le c$ for every $h \in [0, l_n]$. Hence, $K \le c$ and Theorem 4.2 finishes the proof. □

One of the important problems in the study of any property is to obtain its stability under appropriate deformations. Theorem 4.2 allows to prove a result which shows that hyperbolicity is stable under bounded perturbations of the lengths of the fundamental geodesics. Theorem 4.8 is particularly remarkable since there are very few results on hyperbolic stability which do not involve quasi-isometries. We need a previous lemma; it deals with some kind of reverse inequality to the one in Lemma 4.3.

Lemma 4.7. *For every $r_k, l_{k+1} \ge 0$ and $0 \le h \le l_k$, we have*

$$e^h \big(e^{-\frac12 (l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+\big) \le \big(1 + (r_k + h - l_{k+1})_+\big)\, e^{\frac12 (r_k + h - l_{k+1})_+} .$$

Proof. Since the right hand side of the inequality does not depend on l_k and the left hand side is a decreasing function on l_k, it is sufficient to prove

$$e^h \big(e^{-\frac12 (h + l_{k+1} - r_k)_+} + (r_k - h - l_{k+1})_+\big) \le (1 + \big(r_k + h - l_{k+1})_+\big)\, e^{\frac12 (r_k + h - l_{k+1})_+} .$$

for every $r_k, l_{k+1}, h \ge 0$.

If $h + l_{k+1} - r_k \ge 0$, the inequality is direct since

$$e^h \big(e^{-\frac12 (h + l_{k+1} - r_k)_+} + (r_k - h - l_{k+1})_+\big) = e^h e^{-\frac12 (h + l_{k+1} - r_k)} = e^{\frac12 (r_k + h - l_{k+1})} .$$

If $h + l_{k+1} - r_k < 0$, then $r_k - l_{k+1} > h$ and $(r_k + h - l_{k+1})_+ > 2h$; consequently,

$$e^h\big(e^{-\frac{1}{2}(h+l_{k+1}-r_k)_+} + (r_k - h - l_{k+1})_+\big) = e^h\big(1 + r_k - h - l_{k+1}\big) < \big(1 + \big(r_k + h - l_{k+1})_+\big)\, e^{\frac{1}{2}(r_k+h-l_{k+1})_+}.$$

□

Next, the result about stability that we have talked about before Lemma 4.7. Theorem 4.8 is both, a qualitative and a quantitative result.

Theorem 4.8. *Let us consider two trains Ω, Ω' and a constant c such that $|r'_n - r_n| \le c$, and $|l'_n - l_n| \le c$ for every $n \ge 1$. Then Ω is hyperbolic if and only if Ω' is hyperbolic.*
Furthermore, if Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ and c.

Remarks.
(1) Notice that in many cases Ω and Ω' are not quasi-isometric (for example, if there exists a subsequence $\{n_k\}_k$ with $\lim_{k\to\infty} l_{n_k} = 0$ and $l'_{n_k} \ge c_0 > 0$).
(2) We have examples which show that Theorem 4.8 is sharp: if we change the constants in Theorem 4.8 by any function growing slowly to infinity, then the conclusion of Theorem 4.8 does not hold. For instance, if $\{r_n\}$ is bounded and $\{r'_n\}$ is not bounded, then there exists $\{l_n\} = \{l'_n\}$ with Ω hyperbolic and Ω' not hyperbolic.

Proof. By symmetry, it is sufficient to prove that if Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ and c. Therefore, let us assume that Ω is δ-hyperbolic.

Notice that $e^{-l_k} + e^{-l_{k+1}} \le e^c\big(e^{-l'_k} + e^{-l'_{k+1}}\big)$.

If $l_k + l_{k+1} \le r_k$, then $e^{-\frac{1}{2}(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+ = 1 + r_k - l_k - l_{k+1}$ and

$$e^{-\frac{1}{2}(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+ \le 1 + 3c + r_k - l_k - l_{k+1} \le (1+3c)\big(e^{-\frac{1}{2}(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+\big).$$

If $l'_k + l'_{k+1} \ge r'_k$, then

$$e^{-\frac{1}{2}(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+ = e^{-\frac{1}{2}(l'_k+l'_{k+1}-r'_k)_+} \le e^{3c/2}\big(e^{-\frac{1}{2}(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+\big).$$

If $l_k + l_{k+1} > r_k$ and $l'_k + l'_{k+1} < r'_k$, then

$$l_k + l_{k+1} - r_k \le l'_k + l'_{k+1} - r'_k + 3c < 3c\,,$$
$$r'_k - l'_k - l'_{k+1} \le r_k - l_k - l_{k+1} + 3c < 3c\,,$$

and consequently

$$\begin{aligned} e^{-\frac{1}{2}(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+ &= 1 + r'_k - l'_k - l'_{k+1} < (1+3c)\, e^{3c/2} e^{-3c/2} \\ &< (1+3c)\, e^{3c/2}\big(e^{-\frac{1}{2}(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+\big). \end{aligned}$$

Therefore

$$e^{-l'_k} + e^{-l'_{k+1}} + e^{-\frac{1}{2}(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+ \le (1+3c)\, e^{3c/2}\big(e^{-l_k} + e^{-l_{k+1}} + e^{-\frac{1}{2}(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+\big),$$

i.e. $\Delta'(k) \le (1+3c)\, e^{3c/2}\Delta(k)$. We also have

$$(r'_m + h - l'_{m+1})_+ \le 2c + (r_m + h - l_{m+1})_+\,,$$
$$l'_m - h \le c + l_m - h\,,$$
$$\min\big\{h,\, l'_n - h\big\} \le c + \min\big\{h,\, l_n - h\big\}\,.$$

Hence, we conclude

$$\big(\Gamma_{nm}\big)'(h) \le (1+3c)\, e^{3c/2}\Gamma_{nm}(h) + 2c\,,$$

for every $n, m \ge 1$ and $h \ge 0$ with either $m = n$ or $l_m, l'_m \le h$ or $l_m, l'_m > h$.

We deal now with the other cases. Let us assume that $m \in [A'_n(h), n)$. The case $m \in (n, B'_n(h)]$ is similar.

If $l'_m \le h < l_m$, then $m = A'_n(h)$ and $l'_m \le h < l'_{m+1}$. Applying Lemma 4.3 we obtain

$$\begin{aligned}\big(\Gamma_{nm}\big)'(h) &= \big(r'_m + h - l'_{m+1}\big)_+ + e^h \sum_{k=m+1}^{n-1} \Delta'(k) < e^h \sum_{k=m}^{n-1} \Delta'(k) \\ &\le l_m - h + (1+3c)\, e^{3c/2} e^h \sum_{k=m}^{n-1} \Delta(k) \le (1+3c)\, e^{3c/2} \Gamma_{nm}(h)\,.\end{aligned}$$

If $l_m \le h < l'_m$, then $m > A'_n(h)$ and $h < l'_{m+1}$. We also have $l'_m - h \le l'_m - l_m \le c$. Applying Lemma 4.7 we obtain

$$\begin{aligned}\big(\Gamma_{nm}\big)'(h) &= l'_m - h + e^{h-l'_m} + e^{h-l'_{m+1}} + e^h \big(e^{-\frac12 (l'_m + l'_{m+1} - r'_m)_+} + (r'_m - l'_m - l'_{m+1})_+\big) + e^h \sum_{k=m+1}^{n-1} \Delta'(k) \\ &\le c + 2 + \big(1 + (r'_m + h - l'_{m+1})_+\big)\, e^{\frac12 (r'_m + h - l'_{m+1})_+} + (1+3c)\, e^{3c/2} e^h \sum_{k=m+1}^{n-1} \Delta(k) \\ &\le c + 2 + \big(1 + 2c + (r_m + h - l_{m+1})_+\big)\, e^c e^{\frac12 (r_m + h - l_{m+1})_+} + (1+3c)\, e^{3c/2} e^h \sum_{k=m+1}^{n-1} \Delta(k) \\ &\le c + 2 + \big(1 + 2c + \Gamma_{nm}(h)\big)\, e^c e^{\frac12 \Gamma_{nm}(h)} + (1+3c)\, e^{3c/2} \Gamma_{nm}(h)\,.\end{aligned}$$

We can conclude in any case

$$\begin{aligned}\sup_{h\in[0,\min\{l_n,l'_n\}]} \min_{m\in[A'_n(h),B'_n(h)]} \big(\Gamma_{nm}\big)'(h) &= \sup_{h\in[0,\min\{l_n,l'_n\}]} \min_{m\ge1} \big(\Gamma_{nm}\big)'(h) \\ &\le \sup_{h\in[0,l_n]} \min_{m\ge1} \Big(c + 2 + \big(1 + 2c + \Gamma_{nm}(h)\big)\, e^c e^{\frac12 \Gamma_{nm}(h)} + (1+3c)\, e^{3c/2} \Gamma_{nm}(h)\Big) \\ &\le c + 2 + \big(1 + 2c + K\big)\, e^c e^{\frac12 K} + (1+3c)\, e^{3c/2} K,\end{aligned}$$

for every $n \ge 1$, where K only depends on δ, by Theorem 4.2 and Proposition 4.4.

If for some n we have $l_n < l'_n$ and $h \in [l_n, l'_n]$, then $\big(\Gamma_{nn}\big)'(h) \le l'_n - h \le l'_n - l_n \le c$ and

$$\sup_{h\in[l_n,l'_n]} \min_{m\in[A'_n(h),B'_n(h)]} \big(\Gamma_{nm}\big)'(h) \le c\,.$$

Therefore, $K' \le c + 2 + \big(1 + 2c + K\big)\, e^c e^{\frac12 K} + (1+3c)\, e^{3c/2} K$, and the conclusion holds by Theorem 4.2. □

Theorem 4.8 has the following direct consequence.

Corollary 4.9. *Let us consider two trains Ω, Ω' such that $r'_n = r_n$, and $l'_n = l_n$ for every $n \ge N$. Then Ω is hyperbolic if and only if Ω' is hyperbolic.*

Theorems 4.11 and 4.12 are simpler versions of Theorem 4.2, which can be applied in many occasions, and are obtained by replacing $\Gamma_{nm}(h)$ for $\Gamma^*_{nm}(h)$ and $\Gamma^0_{nm}(h)$, respectively. We define now these functions.

Definition 4.10. *Let us consider a sequence of positive numbers* $\{l_n\}_{n=1}^{\infty}$ *and a sequence of non-negative numbers* $\{r_n\}_{n=1}^{\infty}$. *Consider* $n \ge 1$ *and* $0 \le h \le l_n$. *We define*

$$\Gamma^*_{nm}(h) := \begin{cases} \big(r_m + h - l_{m+1}\big)_+ + e^h \displaystyle\sum_{k=m+1}^{n} e^{-l_k}\,, & \text{if } m < n \text{ and } l_m \le h\,,\\ l_m - h + e^h \displaystyle\sum_{k=m}^{n} e^{-l_k}\,, & \text{if } m < n \text{ and } l_m > h\,,\\ \min\big\{h,\, l_n - h\big\}\,, & \text{if } m = n\,,\\ l_m - h + e^h \displaystyle\sum_{k=n}^{m} e^{-l_k}\,, & \text{if } m > n \text{ and } l_m > h\,,\\ \big(r_{m-1} + h - l_{m-1}\big)_+ + e^h \displaystyle\sum_{k=n}^{m-1} e^{-l_k}\,, & \text{if } m > n \text{ and } l_m \le h\,, \end{cases}$$

and

$$\Gamma^0_{nm}(h) := \begin{cases} e^h \displaystyle\sum_{k=m+1}^{n} e^{-l_k}\,, & \text{if } m < n \text{ and } l_m \le h\,,\\ e^h \displaystyle\sum_{k=n}^{m-1} e^{-l_k}\,, & \text{if } m > n \text{ and } l_m \le h\,,\\ \Gamma^*_{nm}(h)\,, & \text{if } m > n \text{ in other case.} \end{cases}$$

The functions $\Gamma^*_{nm}(h)$ *and* $\Gamma^0_{nm}(h)$ *are naturally associated to trains by taking* $\{l_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *as the half-lengths of their fundamental geodesics.*

Theorem 4.11. *Let us consider a train* Ω *such that there exists a constant* $c > 0$ *with* $r_n \le 2c + |l_n - l_{n+1}|$ *for every* $n \ge 1$. *Then* Ω *is hyperbolic if and only if*

$$K^* := \sup_{n\ge 1}\ \sup_{h\in[0,l_n]}\ \min_{m\in[A_n(h),B_n(h)]} \Gamma^*_{nm}(h) < \infty\,.$$

Furthermore, if Ω *is* δ*-hyperbolic, then* K^* *is bounded by a constant which only depends on* δ *and* c*; if* $K^* < \infty$*, then* Ω *is* δ*-hyperbolic, with* δ *a constant which only depends on* K^* *and* c*.*

Proof. First, let us consider the integer numbers k with $l_k + l_{k+1} \ge r_k$. The inequality $r_k - l_k - l_{k+1} \le 2c - 2\min\{l_k, l_{k+1}\}$ (which is equivalent to $r_k \le 2c + |l_k - l_{k+1}|$) gives

$$e^{-\frac12 (l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+ = e^{\frac12 (r_k - l_k - l_{k+1})} \le e^{c - \min\{l_k, l_{k+1}\}} \le e^c\big(e^{-l_k} + e^{-l_{k+1}}\big)\,.$$

And now, consider the integer numbers k with $l_k + l_{k+1} \le r_k$. The inequality $0 \le r_k - l_k - l_{k+1} \le 2c - 2\min\{l_k, l_{k+1}\}$ gives $\min\{l_k, l_{k+1}\} \le c$, and consequently

$$e^{-c} \le e^{-\min\{l_k, l_{k+1}\}}, \qquad 1 \le e^c\big(e^{-l_k} + e^{-l_{k+1}}\big)\,.$$

Hence

$$e^{-\frac12 (l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+ = 1 + r_k - l_k - l_{k+1} \le 1 + 2c \le (1+2c)\, e^c\big(e^{-l_k} + e^{-l_{k+1}}\big)\,.$$

Then

$$e^{-\frac12 (l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+ \le (1+2c)\, e^c\big(e^{-l_k} + e^{-l_{k+1}}\big)\,,$$
$$e^{-l_k} + e^{-l_{k+1}} \le \Delta(k) \le \big(1 + (1+2c)\, e^c\big)\big(e^{-l_k} + e^{-l_{k+1}}\big)\,,$$

for every $k \ge 1$. Hence, if we apply Theorem 4.2 we obtain the conclusion, with $\inf_{m\in[A_n(h),B_n(h)]} \Gamma^*_{nm}(h)$ instead of $\min_{m\in[A_n(h),B_n(h)]} \Gamma^*_{nm}(h)$. In order to see that the infimum is attained we can follow an argument similar to the one at the end of the proof of Theorem 4.2. □

Theorem 4.12. *Let us consider a train Ω such that there exists a constant $c > 0$ with $r_n \le c$ for every $n \ge 1$. Then Ω is hyperbolic if and only if*

$$K^0 := \sup_{n\ge 1} \sup_{h\in[0,l_n]} \min_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h) < \infty\,.$$

Furthermore, if Ω is δ-hyperbolic, then K^0 is bounded by a constant which only depends on δ and c; if $K^0 < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K^0 and c.

Remark. Notice that Γ^0_{nm} is much simpler than Γ_{nm}:

Firstly, the four terms in the definition of $\Delta(k)$ are replaced by its first term.

Furthermore, in the first and fifth cases in the definition of Γ^0_{nm} we remove the first term in the corresponding definition of Γ_{nm}.

In order to obtain these simplifications, we must pay with the hypothesis $r_n \le c$, but this is a usual hypothesis: for instance, every flute surface satisfies it.

Proof. Notice that $\big(r_m + h - l_{m+1}\big)_+ \le r_m \le c$ if $m = A_n(h)$ (since $l_{m+1} > h$) and $\big(r_{m-1} + h - l_{m-1}\big)_+ \le r_{m-1} \le c$ if $m = B_n(h)$.

Hence, if we apply Theorem 4.11 we obtain the conclusion, with $\inf_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h)$ instead of $\min_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h)$.

In order to see that the infimum is attained we can follow an argument similar to the one at the end of the proof of Theorem 4.2. □

Proposition 4.13. *In any train Ω we have*

$$\min_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h) = \min_{m\ge 1} \Gamma^0_{nm}(h)\,,$$

for every $n \ge 1$ and $0 \le h \le l_n$.

Proof. Fix $n \ge 1$ and $0 \le h \le l_n$. If $m < A_n(h)$, then $\Gamma^0_{nm}(h) > \Gamma^0_{nA_n(h)}(h)$:

$$\Gamma^0_{nm}(h) \ge e^h \sum_{k=m+1}^{n} e^{-l_k} > e^h \sum_{k=A_n(h)+1}^{n} e^{-l_k} = \Gamma^0_{nA_n(h)}(h)\,.$$

The case $m > B_n(h)$ is similar. □

We can deduce a simple sufficient condition for the hyperbolicity.

Theorem 4.14. *Let us consider a train Ω with $l_1 \le l^0$, $r_n \le c_1$ for every n and*

$$\sum_{k=n}^{\infty} e^{-l_k} \le c_2\, e^{-l_n}, \qquad \text{for every } n > 1\,. \tag{4.1}$$

Then Ω is δ-hyperbolic, where δ is a constant which only depends on c_1, c_2 and l^0.

Remark. Let us consider an increasing C^1 function f with $\lim_{x\to\infty} f(x) = \infty$, and define $l_n := f(n)$ for every n. A direct computation gives that $\{l_n\}$ satisfies (4.1) if and only if there exist constants c, M with $f'(x) \ge c > 0$ for every $x \ge M$.

Consequently, for $a, b > 0$ and $c \in \mathbb{R}$, the sequence $l_n := an^b + c$ satisfies (4.1) if and only if $b \ge 1$.

Proof. Let us consider $n \ge 1$ and $h \in [l^0, l_n]$. Since $l_1 \le l^0 \le h$, we have that $m = A_n(h)$ satisfies $l_m \le h < l_{m+1}$ and

$$\Gamma^0_{nm}(h) = e^h \sum_{k=m+1}^{n} e^{-l_k} \le e^h\, c_2\, e^{-l_{m+1}} < c_2\,.$$

If $h \in [0, l^0]$, then $\Gamma^0_{nn}(h) \le h \le l^0$. Hence, $K^0 \le \max\{c_2, l^0\}$, and Theorem 4.12 gives the result. □

Lemma 4.15.

(1) *Let us consider a sequence $\{l_n\}$ such that $l_m \le l_n + c$ for every positive integer numbers $m \le n$. Then there exists a non-decreasing sequence $\{l'_n\}$, such that $|l_n - l'_n| \le c$ for every n.*

(2) *Let us consider a non-decreasing sequence* $\{l'_n\}$. *If* $\{l_n\}$ *is a sequence with* $|l_n - l'_n| \le c$ *for every* n, *then* $l_m \le l_n + 2\,c$ *for every positive integer numbers* $m \le n$.

Proof. We prove now the first part of the lemma. We define a sequence $\{l'_n\}$ in the following way: $l'_n := \max\{l_1, l_2, \dots, l_n\}$. It is clear that $\{l'_n\}$ is a non-decreasing sequence. Since $l_m \le l_n + c$ for every $m = 1, 2, \dots, n$, we have $l_n \le l'_n \le l_n + c$. Consequently, $|l_n - l'_n| \le c$ for every n.

In order to prove the second part, notice that if $m \le n$, then $l_m \le l'_m + c \le l'_n + c \le l_n + 2\,c$. □

The two following theorems provide necessary conditions for hyperbolicity.

Theorem 4.16. *Let us consider an hyperbolic train* Ω *with* $l_m \le l_n + c_1$ *for every positive integer numbers* $m \le n$. *If* K *is the constant defined in Theorem 4.2, then*

$$r_n \le 2\max\{K,1\} + 2\log\max\{K,1\} + 3\,c_1\,, \qquad \text{for every } n \text{ with } l_{n+1} > 4(K + c_1)\,.$$

Proof. Let us define $M := \max\{K,1\}$ and fix n with $l_{n+1} > 4(K + c_1)$.

Let us assume that $r_n \le l_{n+1}$. Consider $\varepsilon \in (0, 1/2)$ and $h_{n+1} := l_{n+1} - \varepsilon r_n$. Then

$$\begin{aligned}
\Gamma_{n+1,n+1}(h_{n+1}) &= \min\{l_{n+1} - \varepsilon r_n,\, \varepsilon r_n\} = \varepsilon r_n\,,\\
\Gamma_{n+1,m}(h_{n+1}) &\ge l_m - h_{n+1} \ge l_{n+1} - c_1 - h_{n+1} = \varepsilon r_n - c_1\,, \qquad \text{if } m > n+1\,,\\
\Gamma_{n+1,n}(h_{n+1}) &\ge (r_n + h_{n+1} - l_{n+1})_+ = (1-\varepsilon) r_n\,, \qquad \text{if } l_n \le h_{n+1}\,,\\
\Gamma_{n+1,m}(h_{n+1}) &\ge e^{h_{n+1}}\Delta(n) \ge e^{l_{n+1} - \varepsilon r_n} e^{-\frac12 (l_n + l_{n+1} - r_n)} \ge e^{l_{n+1} - \varepsilon r_n} e^{-\frac12 (l_{n+1} + l_{n+1} + c_1 - r_n)}\\
&= e^{-\frac12 c_1 + (\frac12 - \varepsilon) r_n}, \qquad \text{if either } m < n \text{ or } m = n \text{ and } l_n > h_{n+1}\,.
\end{aligned}$$

Since $\varepsilon \in (0, 1/2)$

$$M \ge \min\left\{\varepsilon r_n,\, \varepsilon r_n - c_1,\, (1-\varepsilon) r_n,\, e^{-\frac12 c_1 + (\frac12 - \varepsilon) r_n}\right\} = \min\left\{\varepsilon r_n - c_1,\, e^{-\frac12 c_1 + (\frac12 - \varepsilon) r_n}\right\},$$

and we deduce

$$r_n \le \max\left\{\frac{M + c_1}{\varepsilon}\,,\, \frac{\log M + c_1/2}{1/2 - \varepsilon}\right\}.$$

Taking $\varepsilon = (M + c_1)/(2M + 2\log M + 3c_1)$ (notice that $\varepsilon \in (0, 1/2)$, since $\log M \ge 0$), we obtain the equality of the two terms inside the maximum, and therefore $r_n \le 2M + 2\log M + 3c_1$.

We prove now that $r_n \le l_{n+1}$. Seeking for a contradiction, assume that $r_n > l_{n+1}$, and consider $h^{n+1} := \frac34\, l_{n+1}$. A similar argument, with h^{n+1} instead of h_{n+1}, gives:
If $l_n + l_{n+1} < r_n$, since $l_{n+1} > 4(K + c_1)$,

$$K \ge \min\left\{\frac14\, l_{n+1},\, \frac14\, l_{n+1} - c_1,\, \frac34\, l_{n+1},\, e^{\frac34 l_{n+1}}\right\} = \frac14\, l_{n+1} - c_1 > K,$$

since $l_{n+1} > 4(K + c_1)$, and this is a contradiction. If $l_n + l_{n+1} \ge r_n$, we obtain with a similar argument

$$K \ge \min\left\{\frac14\, l_{n+1},\, \frac14\, l_{n+1} - c_1,\, \frac34\, l_{n+1},\, e^{\frac14 l_{n+1} - \frac12 c_1}\right\} = \min\left\{\frac14\, l_{n+1} - c_1,\, e^{\frac14 l_{n+1} - \frac12 c_1}\right\} > K,$$

since $l_{n+1} > 4(K + c_1)$, and this is the contradiction we are looking for. □

Condition $l_m \le l_n + c_1$ for every positive integer numbers $m \le n$ in Theorem 4.16 can seem superfluous, but we have examples which prove that, in fact, if it is removed, then the conclusion of the theorem is not true.

Theorem 4.17. *Let us consider an hyperbolic train* Ω *with* $l_m \le l_n + c_1$ *for every positive integer numbers* $m \le n$. *If* K *is the constant defined in Theorem 4.2, then*

$$\sum_{k=n}^{\infty} e^{-l_k} \le K\, e^{K + c_1}\, e^{-l_n}, \qquad \text{for every } n \text{ with } l_n > 2K + c_1\,.$$

Proof. Theorem 4.2 and Proposition 4.4 give that

$$\min_{m\ge 1} \Gamma_{nm}(h) \le K\,, \qquad \text{for every } n \ge 1 \text{ and } h \in [0, l_n]\,.$$

Let us fix n with $l_n > 2K + c_1$ and $n_0 \ge n$. Consider $\varepsilon > 0$ with $l_n \ge 2K + c_1 + \varepsilon$. If we define $h := l_n - K - c_1 - \varepsilon/2 \ge K + \varepsilon/2 > K$, then for any $m \ge n$ we have $l_m - h \ge l_n - h - c_1 = K + \varepsilon/2 > K$ and

$$\Gamma_{n_0m}(h) \ge \Gamma^0_{n_0m}(h) \ge K + \varepsilon/2 > K\,.$$

If $m < n$, we obtain

$$\Gamma_{n_0m}(h) \ge \Gamma^0_{n_0m}(h) \ge e^h \sum_{k=n}^{n_0} e^{-l_k}.$$

Consequently,

$$K \ge \min_{m\ge 1} \Gamma_{n_0m}(h) = \min_{1\le m<n} \Gamma_{n_0m}(h) \ge e^{l_n-K-c_1-\varepsilon/2} \sum_{k=n}^{n_0} e^{-l_k},$$

for every $n_0 \ge n$ and ε small enough. Therefore

$$K \ge e^{l_n-K-c_1} \sum_{k=n}^{\infty} e^{-l_k},$$

which finishes the proof. □

The last three theorems, Theorem 4.2 and Proposition 4.6 give the following characterization.

Theorem 4.18. *Let us consider a train Ω with $l_m \le l_n + c_1$ for every positive integer numbers $m \le n$.*

(1) *If $\{l_n\}$ is a bounded sequence, then Ω is hyperbolic.*
(2) *If $\lim_{n\to\infty} l_n = \infty$, then Ω is hyperbolic if and only if $\{r_n\}$ is a bounded sequence and* (4.1) *holds for some constant c_2.*

If we have an hyperbolic train, we want to study what kind of transformations in $\{l_n\}$ and $\{r_n\}$ allows to obtain another hyperbolic train.

Theorem 4.19. *Consider two trains Ω and Ω'. Let us assume that Ω is δ-hyperbolic. Then, Ω' is δ'-hyperbolic if we have either:*

(1) *$l'_n = l_n$ and $r'_n \le r_n$ for every n (and then $K' \le K$), or*
(2) *$l'_n = \lambda l_n$ and $r'_n = \lambda r_n$ for every n ($\lambda \ge 1$) (and then $K' \le \lambda K + (1+\lambda)K^\lambda$), or*
(3) *$l'_n = \lambda l_n$ and $r'_n = \mu r_n$ for every n ($\lambda \ge 1$, $0 \le \mu \le \lambda$) (and then $K' \le \lambda K + (1+\lambda)K^\lambda$).*

Proof. In case (1), $\big(\Gamma_{nm}\big)'(h) \le \Gamma_{nm}(h)$ for every $n, m \ge 1$, since $\Gamma_{nm}(h)$ is a non-decreasing function in each variable r_k. This allows to deduce (1).

In order to prove the second part, notice that (since $\lambda \ge 1$)

$$e^{\lambda h} \sum_k \big(e^{-\lambda l_k} + e^{-\lambda l_{k+1}} + e^{-\frac12(\lambda l_k+\lambda l_{k+1}-\lambda r_k)_+}\big) \le \Big(e^h \sum_k \big(e^{-l_k} + e^{-l_{k+1}} + e^{-\frac12(l_k+l_{k+1}-r_k)_+}\big)\Big)^\lambda .$$

Notice that $t \le (1+t)^\lambda$ for every $t \ge 0$ and $\lambda \ge 1$. Hence, if $r_k - l_k - l_{k+1} \ge 0$,

$$\begin{aligned} e^{\lambda h} \sum_k (\lambda r_k - \lambda l_k - \lambda l_{k+1})_+ &\le \lambda\, e^{\lambda h} \sum_k \big(1 + (r_k - l_k - l_{k+1})_+\big)^\lambda \\ &\le \lambda \Big(e^h \sum_k \big(e^{-\frac12(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+\big)\Big)^\lambda . \end{aligned}$$

We also have

$$\big(\lambda r_m + \lambda h - \lambda l_{m+1}\big)_+ = \lambda\big(r_m + h - l_{m+1}\big)_+\,, \quad \lambda l_m - \lambda h = \lambda(l_m - h)\,, \quad \min\big\{\lambda h,\, \lambda l_n - \lambda h\big\} = \lambda \min\big\{h,\, l_n - h\big\}\,.$$

Consequently, $\big(\Gamma_{nm}\big)'(\lambda h) \le \lambda\Gamma_{nm}(h) + \Gamma_{nm}(h)^\lambda + \lambda\Gamma_{nm}(h)^\lambda$ for every $n, m \ge 1$ and $0 \le h \le l_n$, and then $K' \le \lambda K + (1+\lambda)K^\lambda$.

Item (3) is a direct consequence of (1) and (2). □

We want to study now the following question: If we have an hyperbolic train with $\{r_n\} \in l^\infty$, what kind of perturbations are allowed on $\{l_n\}$ so that the train is still hyperbolic? Theorem 4.22 answers this question providing a great deal of hyperbolic flute surfaces.

We need the following definitions.

Definition 4.20. *We denote by H the following set of sequences:*

$$\begin{aligned} H := & \big\{\{x_n\}: \text{ the train with } l_n = x_n \text{ and } r_n = 0 \text{ for every } n \text{ is hyperbolic} \big\} \\ = & \big\{\{x_n\}: \text{ every train with } l_n = x_n \text{ for every } n \text{ and } \{r_n\} \in l^\infty \text{ is hyperbolic} \big\}. \end{aligned}$$

The second equality is a direct consequence of Theorem 4.8.

Definition 4.21. *We say that the sequence $\{y_n\}$ is a union of the sequences $\{x_n^1\}, \dots, \{x_n^N\}$, if $\{x_n^1\}, \dots, \{x_n^N\}$ are subsequences of $\{y_n\}$, and $\{x_n^1\}, \dots, \{x_n^N\}$ is a partition of $\{y_n\}$.*

Theorem 4.22. *Let us consider a sequence $\{l_n\} \in H$.*

1. *If $l'_n = l_n + x_n$ with $\{x_n\} \in l^\infty$, then $\{l'_n\} \in H$.*
2. *Fix a positive integer N. Let us assume that $\{l_n\}$ is a subsequence $\{l'_{n_k}\}$ of $\{l'_n\}$ such that $n_{k+1} - n_k \le N$ for every k, and $\max\{l'_{n_k}, l'_{n_{k+1}}\} \le l'_m + N$ for every $m \in (n_k, n_{k+1})$ and every k. Then $\{l'_n\} \in H$.*
3. *If $\{l'_n\}$ is any union of the sequences $\{l_n^1\}, \dots, \{l_n^N\} \in H$, then $\{l'_n\} \in H$.*
4. *If $\{l'_n\}$ is a union of $\{l_n\}$ and a sequence $\{x_n\} \in l^\infty$, then $\{l'_n\} \in H$.*
5. *Let us assume that $\{l'_n\}$ is any union of the sequences $\{l_n^1\}, \dots, \{l_n^N\}$ which verify*

$$\sum_{k=n}^{\infty} e^{-l_k^j} \le c\, e^{-l_n^j}, \qquad \text{for every } n > 1 \text{ and } j = 1, \dots, N.$$

Then $\{l'_n\} \in H$.

6. *Fix a positive integer N. Let us assume that $\{x_n\}$ is a subsequence $\{l'_{n_k}\}$ of $\{l'_n\}$ such that $\max\{l'_{n_k}, l'_{n_{k+1}}\} \le l'_m + N$ for every $m \in (n_k, n_{k+1})$ and every k. If $\{x_n\} \notin H$, then $\{l'_n\} \notin H$.*
7. *Fix a positive integer N. Let σ be a permutation of the positive integer numbers such that $|\sigma(n) - n| \le N$ for every n, and consider $l'_n := l_{\sigma(n)}$. Then $\{l'_n\} \in H$.*

Remarks.

(1) In fact, (7) gives the following stronger statement: If σ is a permutation of the positive integer numbers such that $|\sigma(n) - n| \le N$ for every n, then $\{l_{\sigma(n)}\} \in H$ if and only if $\{l_n\} \in H$ (since σ^{-1} also satisfies $|\sigma^{-1}(n) - n| \le N$ for every n).

(2) We have examples showing that the conclusions of Theorem 4.22 do not hold if we remove any of the hypothesis.

Proof. (1) is a direct consequence of Theorem 4.8.

(2) Fix $n \ge 1$ and $h \in [0, l'_n]$.

Let us consider the maximum integer k_0 such that $n_{k_0} \le n < n_{k_0+1}$.

If $l'_s \le h$ for some $s \in [n_{k_0}, n_{k_0+1}]$, by symmetry, without loss of generality we can assume that there exists some $s \in [n_{k_0}, n)$ with $l'_s \le h$ (the case $s = n$ is trivial: if $l'_n \le h$, then $h = l'_n$ and $\big(\Gamma^0_{nn}\big)'(h) = 0$). Hence $A'_n(h) \in [n_{k_0}, n)$ and then $l'_k \ge h$ for every $k \in (A'_n(h), n]$ and $n - A'_n(h) \le n - n_{k_0} \le N - 1$; consequently,

$$\big(\Gamma^0_{nA'_n(h)}\big)'(h) = \sum_{k=A'_n(h)+1}^{n} e^{h - l'_k} \le \sum_{k=A'_n(h)+1}^{n} 1 = n - A'_n(h) \le N - 1\,.$$

Let us assume now that $l'_s > h$ for every $s \in [n_{k_0}, n_{k_0+1}]$. There exists some integer m with $\Gamma^0_{k_0 m}(h) \le K^0$. By symmetry, without loss of generality we can assume that $m \le k_0$.

If $m = k_0$, then $\min\{h,\, l_{k_0} - h\} \le K^0$. If $\min\{h,\, l_{k_0} - h\} = h$, then $h \le K^0$ and we can deduce

$$\big(\Gamma^0_{nn}\big)'(h) = \min\{h,\, l'_n - h\} \le h \le K^0.$$

If $\min\{h,\, l_{k_0}-h\} = l_{k_0}-h$, then $l_{k_0}-h \le K^0$ and

$$\big(\Gamma^0_{nn_{k_0}}\big)'(h) = l'_{n_{k_0}} - h + \sum_{k=n_{k_0}}^{n} e^{h-l'_k} \le l_{k_0} - h + \sum_{k=n_{k_0}}^{n} 1 \le K^0 + N.$$

If $m < k_0$ and $l_m > h$, then $\Gamma^0_{k_0 m}(h) = l_m - h + e^h \sum_{k=m}^{k_0} e^{-l_k} \le K^0$. Hence

$$\begin{aligned}
\big(\Gamma^0_{nn_m}\big)'(h) &= l'_{n_m} - h + e^h \sum_{k=n_m}^{n_{k_0}} e^{-l'_k} + \sum_{k=n_{k_0}+1}^{n} e^{h-l'_k} \\
&\le l'_{n_m} - h + e^h\Big(e^{-l'_{n_m}} + \sum_{j=m+1}^{k_0} \sum_{k=n_{j-1}+1}^{n_j} e^{-l'_k}\Big) + \sum_{k=n_{k_0}+1}^{n} 1 \\
&\le l'_{n_m} - h + e^h\Big(e^{-l'_{n_m}} + \sum_{j=m+1}^{k_0} N\, e^{N-l'_{n_j}}\Big) + N - 1 \\
&\le N\, e^N \Big(l_m - h + e^h \sum_{j=m}^{k_0} e^{-l_j}\Big) + N - 1 \le N\, e^N K^0 + N - 1\,.
\end{aligned}$$

If $m < k_0$ and $l_m \le h$, a similar argument gives the same bound for $\big(\Gamma^0_{nn_m}\big)'(h)$.

Then, $\big(K^0\big)' \le N\, e^N K^0 + N$ and Theorem 4.12 implies (2).

(3) Assume first that $N = 2$; then $\{l'_n\}$ is the union of $\{l^1_n\}$ and $\{l^2_n\}$. We denote by $\{l'_{n^i_k}\}$ the subsequence $\{l^i_n\}$ in $\{l'_n\}$, for $i = 1, 2$. Fix $n \ge 1$ and $h \in [0, l'_n]$. By symmetry, without loss of generality we can assume that there exist k_1 with $n^1_{k_1} = n$ and $m_1 \le k_1$ with $\big(\Gamma^0_{k_1 m_1}\big)^1(h) \le (K^0)^1$.

We can assume that $l'_s > h$ for every $s \in (n^1_{m_1}, n^1_{k_1})$, since the other case is similar.

If there is no k with $n^2_k \in [n^1_{m_1}, n^1_{k_1}]$, then $\big(\Gamma^0_{n^1_{k_1} n^1_{m_1}}\big)'(h) = \big(\Gamma^0_{k_1 m_1}\big)^1(h) \le (K^0)^1$.

Assume now that there exists k with $n^2_k \in (n^1_{m_1}, n^1_{k_1})$. Let us define $k_2 := \max\{k :\, n^2_k \in (n^1_{m_1}, n^1_{k_1})\}$.

If there exists $m_2 \le k_2$ such that $\big(\Gamma^0_{k_2 m_2}\big)^2(h) \le (K^0)^2$, then

$$\big(\Gamma^0_{n^1_{k_1},\, \max\{n^1_{m_1}, n^2_{m_2}\}}\big)'(h) \le \big(\Gamma^0_{k_1 m_1}\big)^1(h) + \big(\Gamma^0_{k_2 m_2}\big)^2(h) \le (K^0)^1 + (K^0)^2.$$

If there exists k_3 verifying the next three conditions simultaneously:

(a) $n^2_{k_3} \in (n^1_{m_1}, n^1_{k_1})$,

(b) there exists $m_3 \le k_3$ such that $\big(\Gamma^0_{k_3 m_3}\big)^2(h) \le (K^0)^2$,

(c) for every $k \in (k_3, k_2]$ we have $\big(\Gamma^0_{km}\big)^2(h) > (K^0)^2$ for every $m \le k$,

then there exists $m_0 > k_2$ such that $\big(\Gamma^0_{k_3+1, m_0}\big)^2(h) \le (K^0)^2$: In fact, seeking for a contradiction, let us assume that there exists $m_0 \in (k_3+1, k_2]$ with $\big(\Gamma^0_{k_3+1, m_0}\big)^2(h) \le (K^0)^2$; then $\big(\Gamma^0_{m_0 m_0}\big)^2(h) \le \big(\Gamma^0_{k_3+1, m_0}\big)^2(h) \le (K^0)^2$ (recall that $l'_s > h$ for every $s \in (n^1_{m_1}, n^1_{k_1})$), which is actually a contradiction with (c). Hence,

$$\big(\Gamma^0_{n^1_{k_1},\, \max\{n^1_{m_1}, n^2_{m_3}\}}\big)'(h) \le \big(\Gamma^0_{k_1 m_1}\big)^1(h) + \big(\Gamma^0_{k_3 m_3}\big)^2(h) + \big(\Gamma^0_{k_3+1, m_0}\big)^2(h) \le (K^0)^1 + 2(K^0)^2.$$

If for any k with $n^2_k \in (n^1_{m_1} n^1_{k_1})$ we have $\big(\Gamma^0_{km}\big)^2(h) > (K^0)^2$ for every $m \le k$, let us define $k_4 := \min\{k :\, n^2_k \in (n^1_{m_1}, n^1_{k_1})\}$. As in the last case, then there exists $m_4 > k_2$ such that $\big(\Gamma^0_{k_4 m_4}\big)^2(h) \le (K^0)^2$, and hence

$$\big(\Gamma^0_{n^1_{k_1} n^1_{m_1}}\big)'(h) \le \big(\Gamma^0_{k_1 m_1}\big)^1(h) + \big(\Gamma^0_{k_4 m_4}\big)^2(h) \le (K^0)^1 + (K^0)^2.$$

Consequently, $\big(K^0\big)' \le 2(K^0)^1 + 2(K^0)^2$ and Theorem 4.12 implies (3) with $N = 2$. The result for N sequences is obtained by applying $N-1$ times this result for 2 sequences.

(4) is a direct consequence of (3) and Proposition 4.6.

(5) is a direct consequence of (3) and Theorem 4.14.

(6) Since $\{x_n\} \notin H$, by Theorem 4.12 and Proposition 4.13, for each $M > N$ there exist k_0 and $h \in (0, x_{k_0})$ with $\Gamma^0_{k_0 m}(h) \ge M$, for every $m \ge 1$.

Consider $m \ge 1$. By symmetry, without loss of generality we can assume that $m \le n_{k_0}$. If $m = n_{k_0}$, then

$$\big(\Gamma^0_{n_{k_0} n_{k_0}}\big)'(h) = \min\big\{h,\, l'_{n_{k_0}} - h\big\} = \min\big\{h,\, x_{k_0} - h\big\} = \Gamma^0_{k_0 k_0}(h) \ge M.$$

Notice that if $m \in (n_{k_0 - 1}, n_{k_0})$, then

$$l'_m - h \ge l'_{n_{k_0}} - h - N = x_{k_0} - h - N \ge \Gamma^0_{k_0 k_0}(h) - N \ge M - N > 0\,,$$

and $l'_m > h$. Hence $\big(\Gamma^0_{n_{k_0} m}\big)'(h) \ge l'_m - h \ge M - N$.

In the case $m \le n_{k_0 - 1}$, we have $n_{k_1 - 1} < m \le n_{k_1}$ for some $k_1 < k_0$.

If $x_{k_1} \le h$, then

$$\big(\Gamma^0_{n_{k_0} m}\big)'(h) \ge e^h \sum_{k=m+1}^{n_{k_0}} e^{-l'_k} \ge e^h \sum_{k=k_1+1}^{k_0} e^{-x_k} = \Gamma^0_{k_0 k_1}(h) \ge M.$$

If $x_{k_1} > h$ and $l'_m > h$, then

$$\begin{aligned}\big(\Gamma^0_{n_{k_0} m}\big)'(h) &= l'_m - h + e^h \sum_{k=m}^{n_{k_0}} e^{-l'_k} \ge l'_{n_{k_1}} - h - N + e^h \sum_{k=m}^{n_{k_0}} e^{-l'_k} \\ &\ge x_{k_1} - h - N + e^h \sum_{k=k_1}^{k_0} e^{-x_k} = \Gamma^0_{k_0 k_1}(h) - N \ge M - N.\end{aligned}$$

If $x_{k_1} > h$ and $l'_m \le h$, then $x_{k_1} - N = l'_{n_{k_1}} - N \le l'_m \le h$ and $0 \ge x_{k_1} - h - N$; therefore

$$\big(\Gamma^0_{n_{k_0} m}\big)'(h) = e^h \sum_{k=m+1}^{n_{k_0}} e^{-l'_k} \ge x_{k_1} - h - N + e^h e^{-x_{k_1}} - 1 + e^h \sum_{k=k_1+1}^{k_0} e^{-x_k} = \Gamma^0_{k_0 k_1}(h) - N - 1 \ge M - N - 1.$$

Consequently, $\big(K^0\big)' \ge M - N - 1$ for every $M > N$, and hence $\big(K^0\big)' = \infty$. Then $\{l'_n\} \notin H$ by Theorem 4.12.

(7) First, we want to remark the following elementary fact: If $i < j$ and $\sigma(i) > \sigma(j)$, then $|i - j| < 2N$: $|i - j| = j - i < j - \sigma(j) + \sigma(i) - i \le 2N$.

Fix $n \ge 1$ and $h \in [0, l'_n]$. There exists $\sigma(m)$ with $\Gamma^0_{\sigma(n)\sigma(m)}(h) \le K^0$. By symmetry, without loss of generality we can assume that $\sigma(m) \le \sigma(n)$.

If $m = n$, then $\sigma(m) = \sigma(n)$ and $\big(\Gamma^0_{nn}\big)'(h) = \Gamma^0_{\sigma(n)\sigma(n)}(h) \le K^0$.

We consider now the case $\sigma(m) < \sigma(n)$.

If $m > n$, then $m - n < 2N$.

If $B'_n(h) > m$, then $l'_k > h$ for every $k \in (n, m]$ and

$$\big(\Gamma^0_{nm}\big)'(h) = l'_m - h + \sum_{k=n}^{m} e^{h - l'_k} \le l_{\sigma(m)} - h + 2N \le \Gamma^0_{\sigma(n)\sigma(m)}(h) + 2N \le K^0 + 2N.$$

If $B'_n(h) \le m$, then $l'_k > h$ for every $k \in (n, B'_n(h))$ and

$$\big(\Gamma^0_{n B'_n(h)}\big)'(h) = \sum_{k=n}^{B'_n(h)-1} e^{h - l'_k} \le 2N.$$

We deal now with the case $m < n$. Notice first that $\sigma([m,n]) \subset [m - N, n + N]$ and $[m + N, n - N] \subset [\sigma(m), \sigma(n)]$; then, in $\sigma([m,n]) \setminus [\sigma(m), \sigma(n)]$ there are at most $4N$ integers.

If $A'_n(h) \ge m$, then $l'_k > h$ for every $k \in (A'_n(h), n)$, and

$$\big(\Gamma^0_{nA'_n(h)}\big)'(h) = e^h \sum_{k=A'_n(h)+1}^{n} e^{-l'_k} \le e^h \sum_{\substack{k\in[m,n]\\ l_{\sigma(k)}\ge h}} e^{-l_{\sigma(k)}} = e^h \sum_{\substack{j\in\sigma([m,n])\\ l_j\ge h}} e^{-l_j} \le \sum_{\substack{j\in\sigma([m,n])\setminus[\sigma(m),\sigma(n)]\\ l_j\ge h}} e^{h-l_j} + e^h \sum_{\substack{j=\sigma(m)\\ l_j\ge h}}^{\sigma(n)} e^{-l_j}$$

$$\le 4N+1+e^h \sum_{j=\sigma(m)+1}^{\sigma(n)} e^{-l_j} \le 4N+1+\Gamma^0_{\sigma(n)\sigma(m)}(h) \le 4N+1+K^0.$$

If $A'_n(h) < m$, then $l'_k > h$ for every $k \in [m,n)$, and

$$\big(\Gamma^0_{nm}\big)'(h) = l'_m - h + e^h \sum_{k=m}^{n} e^{-l'_k} = l_{\sigma(m)} - h + e^h \sum_{k\in[m,n]} e^{-l_{\sigma(k)}} = l_{\sigma(m)} - h + e^h \sum_{j\in\sigma([m,n])} e^{-l_j}$$

$$\le \sum_{j\in\sigma([m,n])\setminus[\sigma(m),\sigma(n)]} e^{h-l_j} + l_{\sigma(m)} - h + e^h \sum_{j=\sigma(m)}^{\sigma(n)} e^{-l_j} \le 4N + \Gamma^0_{\sigma(n)\sigma(m)}(h) \le 4N + K^0.$$

Hence, $\big(K^0\big)' \le 4N+1+K^0$, and Theorem 4.12 gives (7). □

References

[1] Aikawa, H., Positive harmonic functions of finite order in a Denjoy type domain, *Proc. Amer. Math. Soc.* **131** (2003), 3873-3881.

[2] Alvarez, V., Pestana, D., Rodríguez, J. M., Isoperimetric inequalities in Riemann surfaces of infinite type, *Rev. Mat. Iberoamericana* **15** (1999), 353-427.

[3] Alvarez, V., Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity of Denjoy domains, *Geom. Dedicata.* In press.

[4] Alvarez, V., Rodríguez, J.M., Yakubovich, D.V., Subadditivity of p-harmonic "measure" on graphs, *Michigan Math. J.* **49** (2001), 47-64.

[5] Balogh, Z. M., Buckley, S. M., Geometric characterizations of Gromov hyperbolicity, *Invent. Math.* **153** (2003), 261-301.

[6] Basmajian, A., Constructing pair of pants, *Ann. Acad. Sci. Fenn. Series AI* **15** (1990), 65-74.

[7] Basmajian, A., Hyperbolic structures for surfaces of infinite type, *Trans. Amer. Math. Soc.* **336** (1993), 421-444.

[8] Benoist, Y., Convexes hyperboliques et fonctions quasisymtriques, *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), 181-237.

[9] Bonk, M., Heinonen, J., Koskela, P., Uniformizing Gromov hyperbolic spaces. Astérisque No. 270 (2001).

[10] Cantón, A., Fernández, J. L., Pestana, D., Rodríguez, J. M., On harmonic functions on trees, *Potential Analysis* **15** (2001), 199-244.

[11] Chavel, I., *Eigenvalues in Riemannian Geometry.* Academic Press, New York, 1984.

[12] Fenchel, W., *Elementary Geometry in Hyperbolic Space.* Walter de Gruyter, Berlin-New York, 1989.

[13] Fernández, J. L., Rodríguez, J. M., Area growth and Green's function of Riemann surfaces, *Arkiv för matematik* **30** (1992), 83-92.

[14] Garnett, J., Jones, P., The Corona theorem for Denjoy domains, *Acta Math.* **155** (1985), 27-40.

[15] Ghys, E., de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics, Volume 83. Birkhäuser. 1990.

[16] González, M. J., An estimate on the distortion of the logarithmic capacity, *Proc. Amer. Math. Soc.* **126** (1998), 1429-1431.

[17] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[18] Gromov, M. (with appendices by M. Katz, P. Pansu and S. Semmes), Metric Structures for Riemannian and Non-Riemannnian Spaces. Progress in Mathematics, vol. 152. Birkhäuser, 1999.

[19] Haas, A., Dirichlet points, Garnett points and infinite ends of hyperbolic surfaces I, *Ann. Acad. Sci. Fenn. Series AI* **21** (1996), 3-29.

[20] Hästö, P. A., Gromov hyperbolicity of the j_G and $\widetilde{j}_G$ metrics, *Proc. Amer. Math. Soc.* In press.

[21] Holopainen, I., Soardi, P. M., p-harmonic functions on graphs and manifolds, *Manuscripta Math.* **94** (1997), 95-110.

[22] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391-413.

[23] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227-238.

[24] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122-137.

[25] Karlsson, A., Noskov, G. A., The Hilbert metric and Gromov hyperbolicity, *Enseign. Math.* **48** (2002), 73-89.

[26] Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123-149.
[27] Portilla, A., Rodríguez, J. M., Tourís, E., The topology of balls and Gromov hyperbolicity of Riemann surfaces, *Diff. Geom. Appl.* **21** (2004), 317-335.
[28] Portilla, A., Rodríguez, J. M., Tourís, E., The role of funnels and punctures in the Gromov hyperbolicity of Riemann surfaces. *Proc. Edinburgh Math. Soc.* **49** (2006), 399-425.
[29] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publicacions Matemàtiques* **38** (1994), 243-253.
[30] Rodríguez, J. M., Two remarks on Riemann surfaces, *Publicacions Matemàtiques* **38** (1994), 463-477.
[31] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004), 53-84.
[32] Rodríguez, J. M., Tourís, E., A new characterization of Gromov hyperbolicity for Riemann surfaces, *Publicacions Matemàtiques* **50** (2006), 249-278.
[33] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica.* In press.
[34] Soardi, P. M., Rough isometries and Dirichlet finite harmonic functions on graphs, *Proc. Amer. Math. Soc.* **119** (1993), 1239-1248.
[35] Väisälä, J., Hyperbolic and uniform domains in Banach spaces. Preprint.
[Available at http://www.helsinki.fi/j̃vaisala/preprints.html]