GROMOV HYPERBOLICITY OF RIEMANN SURFACES

José M. Rodríguez[(1)] **and Eva Tourís**[(1)(2)]

José M. Rodríguez and Eva Tourís
Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés, Madrid, SPAIN
e-mails: jomaro@math.uc3m.es
etouris@math.uc3m.es

Abstract

In this paper we study the hyperbolicity in the Gromov sense of Riemann surfaces. We deduce the hyperbolicity of a surface from the hyperbolicity of its "building block components". We also prove the equivalence between the hyperbolicity of a Riemann surface and the hyperbolicity of some graph associated to it. These results clarify how the decomposition of a Riemann surface in Y-pieces and funnels affects on the hyperbolicity of the surface. The results simplify the topology of the surface and allow to obtain global results from local information.

Key words: Gromov hyperbolicity, hyperbolic Riemann surface.

2000 AMS Subject Classification: 30F, 30F20, 30F45.

Abbreviated title: GROMOV HYPERBOLICITY OF RIEMANN SURFACES.

(1) Research partially supported by a grant from DGI (BFM 2003-04870), Spain.
(2) Research partially supported by a grant from DGI (BFM 2000-0022), Spain.

§1. Introduction

A good way to understand the important connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [APR], [ARY], [CFPR], [FR2], [HS], [K1], [K2], [K3], [R1], [R2], [So]) is to study Gromov hyperbolic spaces. This approach allows us to establish a general setting to work simultaneously with graphs and manifolds, in the context of metric spaces. Besides, the idea of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [GH], [G1], [G2] and the references therein).

Although there exist some interesting examples of hyperbolic spaces (see the examples after Definition 2.1), the literature gives no good guide about how to determine whether or not a space is hyperbolic. Recently, some interesting results of Balogh and Buckley [BB] about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see also [BHK] and the references therein).

We are interested in studying when non-exceptional Riemann surfaces equipped with their Poincaré metrics are Gromov hyperbolic. We have also proved several theorems on hyperbolicity for general metric spaces, which are interesting by themselves (see Section 2); they are key tools in the study of Riemann surfaces (see Section 3). Although one should expect Gromov hyperbolicity in non-exceptional Riemann surfaces due to its constant curvature -1, this turns out to be untrue in general, since topological obstacles can impede it: for instance, the two-dimensional jungle-gym (a $\mathbf{Z}^2$-covering of a torus with genus two) is not hyperbolic. Let us recall that in the case of modulated plane domains, the quasihyperbolic metric and Poincaré metric are equivalent.

We prove in Section 4 that there is no inclusion relationship between hyperbolic Riemann surfaces and the usual classes of Riemann surfaces, such as O_G, O_{HP}, O_{HB}, O_{HD}, surfaces with hyperbolic isoperimetric inequality, or the complements of these classes (even in the case of plane domains). This fact shows that the study of hyperbolic Riemann surfaces is more complicated and interesting that one might think at first sight. One can find other results on hyperbolicity of Riemann surfaces in [RT1], [RT2] and [PRT2].

Here we present the outline of the main results. We refer to the next sections for the definitions and the precise statements of the theorems.

We can create or delete infinitely many topological obstacles in a metric space, preserving its hyperbolicity (see Theorem 2.2). This fact simplifies the topology of the space (recall that topological obstacles make difficult the hyperbolicity of a space).

One of the important aims in this paper is obtaining global results on hyperbolicity from local information. That was the idea that led us to think of a Riemann surface S as the union of some "pieces" or "building block components" $\{S_n\}$. Theorem 2.1 guarantees the hyperbolicity of some metric spaces which are narrow in some sense. Using this result, we study the role of the decomposition of a Riemann surface in Y-pieces and funnels (or more general bordered surfaces) in its hyperbolicity (see theorems 3.1, 3.2, 3.6, 3.7 and 3.8). In particular, theorems 3.2, 3.7 and 3.8 can be applied even in cases with arbitrarily long simple closed geodesics in the boundary of the Y-pieces. The hyperbolicity constant in Theorem 2.1 is sharp, and this fact allows us to obtain accurate hyperbolicity constants in Theorem 3.1, and propositions 3.1 and 3.2, and good constants in the other results.

We also have results on uniform hyperbolicity of surfaces of finite type (see theorems 3.4 and 3.5, and propositions 3.1 and 3.2). Theorem 3.5 is remarkable, since it guarantees the hyperbolicity of surfaces of finite type, with hyperbolicity constants which only depend on the topology of the surface and some metric restrictions. By this reason it can be viewed as a result on stability of the hyperbolicity of Riemann surfaces.

Theorem 3.7 is also one of the remarkable results of this paper, since it allows us to simplify significantly the study of the hyperbolicity of a Riemann surface S: it shows how to construct explicitly a very simple graph G related to S, such that the hyperbolicity of G guarantees the hyperbolicity of S. In theorems 3.1, 3.2 and 3.6 the uniform hyperbolicity of the pieces gives the hyperbolicity of the surface, since the pieces are joined together following a tree-like design (in which no topological obstacles are created). In Theorem 3.7 we cannot obtain the global hyperbolicity just from local information, since we do not have any restriction on the connections of the pieces; it is necessary to ask for the hyperbolicity of the graph used as a model for the connections. This result simplifies the geometry of the surface, since we only need to study its "skeleton".

Theorem 3.7 can be applied to prove that some deformations of Riemann surfaces preserve the hyperbolicity, such as significant changes in the length of simple closed geodesics (see Theorem 3.8) or "twists" in the Y-pieces (see Corollary 3.3).

We want to remark a last result. It is clear that the funnel F_l with $L(\partial F_l) = l$ has thin constant $\delta_l \geq l/4$; consequently, one can think that a surface with funnels with arbitrarily long simple closed geodesics cannot be hyperbolic. However, Corollary 3.1 shows that this is not true.

We want to remark that almost every constant appearing in the results of this paper depends just on a small number of parameters. This is a common place in the theory of hyperbolic spaces (see e.g. theorems A, B and C) and is also typical of surfaces with curvature -1 (see e.g. theorems D and E, the Collar Lemma in [R] and [S], and Theorem 3.1 in [PRT2]).

Notations. We denote by X or X_n geodesic metric spaces. By d_X, L_X and B_X we shall denote, respectively, the distance, the length and the balls in the metric of X.

We denote by S or S_n non-exceptional Riemann surfaces. We assume that the metric defined on these surfaces is the Poincaré metric, unless the contrary is specified.

Finally, we denote by c_i, k_i, positive constants which can assume different values in different theorems.

Acknowledgements. We would like to thank Professor José Luis Fernández for some helpful suggestions. Also, we would like to thank Venancio Alvarez, María Auxiliadora Márquez and Ana Portilla for some useful discussions. Also, we would like to thank the referee for his/her careful reading of the manuscript and for some helpful suggestions.

§2. Results in metric spaces

In our study of hyperbolic Gromov spaces we use the notations of [GH]. We give now the basic facts about these spaces. We refer to [GH] for more background and further results.

Definition 2.1. Let us fix a point w in a metric space (X,d). We define the *Gromov product* of $x, y \in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big) \geq 0 \,.$$

We say that the metric space (X,d) is δ-*hyperbolic* $(\delta \geq 0)$ if

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta \,,$$

for every $x, y, z, w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

In this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples: (1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic (see e.g. [GH, p. 29]).

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [GH, p. 52]).

(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [GH, p. 29]).

Definition 2.2. If $\gamma : [a,b] \longrightarrow X$ is a continuous curve in a metric space (X,d), we can define the length of γ as

$$L(\gamma) := \sup\Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[x,y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

Definition 2.3. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. A geodesic triangle T is δ-*thin* (or satisfies the *Rips condition* with constant δ) if for every $x \in [x_i, x_j]$ we have that $d(x, [x_j, x_k] \cup [x_k, x_i]) \leq \delta$ for any permutation $\{x_i, x_j, x_k\}$ of $\{x_1, x_2, x_3\}$. The space X is δ-*thin* if every geodesic triangle in X is δ-thin.

Remark. If we have a triangle with two identical vertices, we call it a "bigon". Note that since this is a special case of the definition, every bigon in a δ-thin space is δ-thin.

A basic result is that hyperbolicity is equivalent to Rips condition:

Theorem A. ([GH, p. 41]) *Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin.*

(2) *If X is δ-thin, then it is 4δ-hyperbolic.*

We present now the class of maps which play the main role in the theory.

Definition 2.4. A function between two metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there are constants $a \geq 1$, $b \geq 0$ with

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

Such a function is called an (a, b)-*quasi-isometry.* An (a, b)-*quasigeodesic* in X is an (a, b)-quasi-isometry between an interval of $\mathbf{R}$ and X. An (a, b)-*quasigeodesic segment* in X is an (a, b)-quasi-isometry between a compact interval of $\mathbf{R}$ and X.

Notice that a quasi-isometry can be discontinuous.

Quasi-isometries are important since they are maps which preserve hyperbolicity:

Theorem B. ([GH, p. 88]) *Let us consider two geodesic metric spaces X and Y, and an (a, b)-quasi-isometry f of X onto Y. If Y (respectively X) is δ-hyperbolic, then X (respectively Y) is δ'-hyperbolic, where δ' is a constant which only depends on δ, a and b.*

In this paper we will work with topological subspaces of a geodesic metric space X. There is a natural way to define a distance in these spaces:

Definition 2.5. If X_0 is a path-connected subset of a metric space (X, d), then we associate to it the *restricted distance*

$$d_{X_0}(x, y) := d_X|_{X_0}(x, y) := \inf \big\{ L(\gamma) : \gamma \subset X_0 \text{ is a continuous curve joining } x \text{ and } y \big\} \geq d_X(x, y)\,.$$

We need an additional definition in order to obtain our first result.

Definition 2.6. A geodesic metric space X is c_1-*decomposible* if it verifies:

(1) $X = \cup_{r \in I} X^r$, with I an interval in the real line, $\{X^r\}_{r \in I}$ pairwise disjoint, $A(r)$ a set of indices for each $r \in I$ and $X^r = \cup_{a \in A(r)} X^r_a$, with $\{X^r_a\}_{a \in A(r)}$ pairwise disjoint closed sets, and $\operatorname{diam}_X X^r_a \leq c_1$.

(2) If for each geodesic $\gamma : [0, l] \longrightarrow X$ and $s \in [0, l]$, we denote by $X^{r(s)}_{a(s)}$ the set X^r_a with $\gamma(s) \in X^r_a$, then $\cup_{s \in [0,l]} X^{r(s)}_{a(s)}$ is a closed set.

(3) If $X^{r(0)}_{a(0)} \neq X^{r(l)}_{a(l)}$, then there is $s \in (0, l)$ such that $X \setminus X^{r(s)}_{a(s)}$ is not connected, and x_0, x_l are in different connected components of $X \setminus X^{r(s)}_{a(s)}$, for every $x_0 \in X^{r(0)}_{a(0)}, x_l \in X^{r(l)}_{a(l)}$.

A standard way to obtain a decomposition is to take a continuous function $f : X \longrightarrow \mathbf{R}$, to define $X^r := f^{-1}(\{r\})$ and to consider $\{X^r_a\}_{a \in A(r)}$ as the connected components of X^r. A natural choice of f is $f(x) = d(x, x_0)$, for fixed $x_0 \in X$. This choice gives the first example of decomposible spaces: the trees are 0-decomposible. Non-trivial examples of decomposible spaces appear in propositions 3.1 and 3.2, and in theorems 3.1, 3.2 and 3.4.

Remarks. 1. The item (2) is only a technical topological condition about the "continuity" in r of X^r, which is trivially satisfied in the applications developed in propositions 3.1 and 3.2, and theorems 3.1, 3.2 and 3.4.

2. If $\gamma : [0, l] \longrightarrow X$ is a geodesic, then $\gamma : [\alpha, \beta] \longrightarrow X$ is also a geodesic for any $0 \leq \alpha < \beta \leq l$. Hence conditions (2) and (3) imply, respectively:

$(2')$ For each geodesic $\gamma : [0,l] \longrightarrow X$, the set $\cup_{s\in[\alpha,\beta]} X_{a(s)}^{r(s)}$ is closed for any $0 \le \alpha < \beta \le l$.

$(3')$ For any $0 \le \alpha < \beta \le l$, if $X_{a(\alpha)}^{r(\alpha)} \neq X_{a(\beta)}^{r(\beta)}$, then there is $s \in (\alpha,\beta)$ such that $X \setminus X_{a(s)}^{r(s)}$ is not connected, and x_α, x_β are in different connected components of $X \setminus X_{a(s)}^{r(s)}$, for every $x_\alpha \in X_{a(\alpha)}^{r(\alpha)}, x_\beta \in X_{a(\beta)}^{r(\beta)}$.

Theorem 2.1. *Every c_1-decomposible geodesic metric space is $(3c_1/2)$-thin.*

Proof. The idea of the proof is to show that given a point x in a geodesic triangle T, then there exists a set X_a^r (near x) which intersects two sides of T.

Let us consider a geodesic triangle T with vertices $\{x_1, x_2, x_3\}$ and $x \in T$. Without loss of generality we can assume that $x \in [x_1, x_2]$. If $l := d_X(x_1, x_2)$, we consider the arc-length parametrization of $[x_1, x_2]$, $\gamma : [0,l] \longrightarrow X$. Let us denote by η the union of the two other sides $\eta := [x_2, x_3] \cup [x_3, x_1]$. If $x \in X_a^r$ and $\eta \cap X_a^r \neq \varnothing$, then $d_X(x,\eta) \le c_1$ by (1), and there is nothing else to prove. Assume that $\eta \cap X_a^r = \varnothing$; then we will prove $d_X(x,\eta) \le 3c_1/2$. We consider $s_1 := \gamma^{-1}(x) \in (0,l)$. Let us define

$$s_0 := \inf \left\{ \alpha > 0 : \ X_{a(s)}^{r(s)} \cap \eta = \varnothing \ \ \forall s \in (\alpha, s_1] \right\},$$
$$s_2 := \sup \left\{ \beta < l : \ X_{a(s)}^{r(s)} \cap \eta = \varnothing \ \ \forall s \in [s_1, \beta) \right\}.$$

We show now that $X_{a(s_0)}^{r(s_0)} \cap \eta \neq \varnothing$ and $X_{a(s_2)}^{r(s_2)} \cap \eta \neq \varnothing$. We only deal with the second case; the first one is similar. By definition of s_2 we have only two possibilities: $X_{a(s_2)}^{r(s_2)} \cap \eta \neq \varnothing$ or $X_{a(t_k)}^{r(t_k)} \cap \eta \neq \varnothing$ with $t_k \searrow s_2$. Let us assume that we have the second possibility; then we can choose $x_k \in X_{a(t_k)}^{r(t_k)} \cap \eta$. Since η is a compact set, we can choose a subsequence (which we also denote by x_k) and a point $x_0 \in \eta$ with $x_k \to x_0$. For each $\varepsilon > 0$ there exists N such that $x_k \in \eta \cap \left(\cup_{s\in[s_2,s_2+\varepsilon]} X_{a(s)}^{r(s)} \right)$, for every $k \ge N$. Recall that $\eta \cap \left(\cup_{s\in[s_2,s_2+\varepsilon]} X_{a(s)}^{r(s)} \right)$ is a closed set by $(2')$. Then $x_0 \in \eta \cap \left(\cup_{s\in[s_2,s_2+\varepsilon]} X_{a(s)}^{r(s)} \right)$, for every $\varepsilon > 0$.

First of all we will prove

$$\cap_{\varepsilon>0} \cup_{s\in[s_2,s_2+\varepsilon]} X_{a(s)}^{r(s)} = X_{a(s_2)}^{r(s_2)}.$$

It is clear that $X_{a(s_2)}^{r(s_2)} \subseteq \cap_{\varepsilon>0} \cup_{s\in[s_2,s_2+\varepsilon]} X_{a(s)}^{r(s)}$. In order to check the other inclusion let us consider any $y \in \cap_{\varepsilon>0} \cup_{s\in[s_2,s_2+\varepsilon]} X_{a(s)}^{r(s)}$. Since y belongs to this intersection, there exists a non-increasing sequence $\{u_n\}_n$ converging to s_2, such that $y \in X_{a(u_n)}^{r(u_n)}$ for every n, and then $X_{a(u_n)}^{r(u_n)} = X_{a(u_1)}^{r(u_1)}$ for every n.

Since $X_{a(u_1)}^{r(u_1)}$ is a closed set, $\lim_{n\to\infty} \gamma(u_n) = \gamma(s_2)$ and $\gamma(u_n) \in X_{a(u_1)}^{r(u_1)}$, then $\gamma(s_2) \in X_{a(u_1)}^{r(u_1)}$; therefore $X_{a(u_1)}^{r(u_1)} = X_{a(s_2)}^{r(s_2)}$, since $\gamma(s_2) \in X_{a(s_2)}^{r(s_2)}$ by definition. Consequently, $y \in X_{a(s_2)}^{r(s_2)}$ as we want to check.

Following with the proof of the theorem, we can conclude that $x_0 \in \eta \cap X_{a(s_2)}^{r(s_2)}$; hence, $X_{a(s_2)}^{r(s_2)} \cap \eta \neq \varnothing$.

We prove now that $X_{a(s_0)}^{r(s_0)} = X_{a(s_2)}^{r(s_2)}$. Seeking a contradiction, let us assume that this is not true. By $(3')$ we can take $s \in (s_0, s_2)$ such that $X \setminus X_{a(s)}^{r(s)}$ is not connected, and $\gamma(s_0), \gamma(s_2)$ are in different connected components of $X \setminus X_{a(s)}^{r(s)}$; also by $(3')$, the same is true if we change $\gamma(s_0)$ by any point in $X_{a(s_0)}^{r(s_0)}$ and $\gamma(s_2)$ by any point in $X_{a(s_2)}^{r(s_2)}$.

Consider now a parametrization of the curve $\eta : [0, l_3] \longrightarrow X$. Since $X_{a(s_0)}^{r(s_0)} \cap \eta \neq \varnothing$ and $X_{a(s_2)}^{r(s_2)} \cap \eta \neq \varnothing$, we can choose $0 \le l_1 < l_2 \le l_3$ with $\eta(l_1) \in X_{a(s_0)}^{r(s_0)}$, $\eta(l_2) \in X_{a(s_2)}^{r(s_2)}$ (or viceverse). We denote by

$\eta_0 : [l_1, l_2] \longrightarrow X$ the restriction of η to $[l_1, l_2]$. It is clear that $\eta_0 \cap X_{a(s)}^{r(s)} = \varnothing$ (since $\eta \cap X_{a(s)}^{r(s)} = \varnothing$) and therefore, η_0 joins $\eta(l_1) \in X_{a(s_0)}^{r(s_0)}$ with $\eta(l_2) \in X_{a(s_2)}^{r(s_2)}$ in $X \setminus X_{a(s)}^{r(s)}$, which is a contradiction. Therefore $X_{a(s_0)}^{r(s_0)} = X_{a(s_2)}^{r(s_2)}$.

Since $X_{a(s_0)}^{r(s_0)} = X_{a(s_2)}^{r(s_2)}$, we have that $L_X(\gamma([s_0, s_2])) = d_X(\gamma(s_0), \gamma(s_2)) \le \operatorname{diam}_X \big(X_{a(s_0)}^{r(s_0)}\big) \le c_1$. Recall that $x = \gamma(s_1)$. Consequently $d_X\big(x, X_{a(s_0)}^{r(s_0)}\big) \le \min\{L_X(\gamma([s_0, s_1])), L_X(\gamma([s_1, s_2]))\} \le c_1/2$, and so $d_X(x, \eta) \le d_X\big(x, X_{a(s_0)}^{r(s_0)}\big) + \operatorname{diam}_X \big(X_{a(s_0)}^{r(s_0)}\big) \le c_1/2 + c_1 = 3c_1/2$. □

Definition 2.7. Let us consider a geodesic metric space X and $\{\eta_n^1, \eta_n^2\}_n$ pairwise disjoint compact subsets of X. If c_1, c_2, c_3, c_4 are positive constants, we say that $\{\eta_n^1, \eta_n^2\}_n$ are (c_1, c_2, c_3, c_4)-*identified* if:

(1) there exists a bijective isometry $f_n : (\eta_n^1, d_X|_{\eta_n^1}) \longrightarrow (\eta_n^2, d_X|_{\eta_n^2})$ for each n,

(2) $d_X(p, p') \le c_1$ if $f_n(p) = p'$ for some n,

(3) $d_X(\eta_n^1 \cup \eta_n^2, \eta_m^1 \cup \eta_m^2) \ge c_2$ for every $n \ne m$,

(4) if we denote by X_0 the space obtained by identifying in X the closed sets η_n^1 and η_n^2 by f_n for each n, and by f the canonical projection of X onto X_0, then for each n there exists $i \in \{1, 2\}$ with $d_X(u, v) \le c_3\, d_{X_0}(f(u), f(v)) + c_4$ if u, $v \in \eta_n^i$.

Remarks. 1. Hypothesis (3) guarantees that d_{X_0} (defined by Definition 2.5) is a distance.

2. Conditions (2) and (4) are satisfied if $\operatorname{diam}_X(\eta_n^1 \cup \eta_n^2) \le c$ for every n.

The following theorem allows us to create infinitely many topological obstacles in a metric space ("genus", if the space is a surface), preserving its hyperbolicity.

There is a more useful point of view to appreciate the next theorem: we can delete infinitely many topological obstacles in a metric space, preserving its hyperbolicity. This fact allows a great simplification in the topology of the space (recall that topological obstacles make difficult the hyperbolicity).

Theorem 2.2. *Let us consider a geodesic metric space X and $\{\eta_n^1, \eta_n^2\}_n$ (c_1, c_2, c_3, c_4)-identified. Then the canonical projection f of X onto X_0 is a quasi-isometry with constants which only depend on c_1, c_2, c_3, c_4. Consequently, if X_0 is a geodesic metric space, then X is hyperbolic if and only if X_0 is hyperbolic. In particular, if X (respectively X_0) is δ-hyperbolic, then X_0 (respectively X) is δ'-hyperbolic, with δ' a universal constant which only depends on δ, c_1, c_2, c_3 and c_4.*

Remarks. 1. It is possible to prove (we can apply a similar argument to the one in the proof of [RT1, Theorem 2.1]) that X_0 is a geodesic metric space if each ball in X intersects only a finite number of η_n^i's (this is the case if X is proper).

2. If η_n^i are simple closed curves, the condition that $(\eta_n^1, d_X|_{\eta_n^1})$ and $(\eta_n^2, d_X|_{\eta_n^2})$ are isometric is equivalent to $L_X(\eta_n^1) = L_X(\eta_n^2)$.

3. Theorem 2.2 is an improvement of [RT1, Theorem 2.2]; furthermore, it uses simpler and shorter arguments.

Proof. We have that $d_X|_{\eta_n^1}, d_X|_{\eta_n^2}$ and d_{X_0} (defined by Definition 2.5) are distances.

It is clear that for every curve σ in X we have $L_X(\sigma) = L_{X_0}(f(\sigma))$. Then for every $x, y \in X$ we have $d_{X_0}(f(x), f(y)) \le d_X(x, y)$, since there are more curves joining $f(x)$ and $f(y)$ in X_0 than curves joining x and y in X.

In order to prove the other inequality, let us fix $x, y \in X$ and let us consider a geodesic $\gamma_0 : [0,l] \longrightarrow X_0$ joining $f(x)$ and $f(y)$, if there exists such geodesic (if this was not so, we can take γ_k with $L_{X_0}(\gamma_k) \le d_{X_0}(f(x), f(y)) + 1/k$). Let us define $\eta_n := f(\eta_n^1) = f(\eta_n^2)$. Then $d_{X_0}(\eta_n, \eta_m) \ge c_2$ if $n \ne m$, by (3).

If $L_{X_0}(\gamma_0) = d_X(x,y)$, then $d_{X_0}(f(x), f(y)) = d_X(x,y)$ and we are done. So suppose this is not so. We shall construct a continuous curve g in X joining x and y, related to γ_0. If $L_{X_0}(\gamma_0) < d_X(x,y)$, then γ_0 meets some η_n. In this case let us choose a finite union of curves γ in X as follows: Since $d_{X_0}(\eta_n, \eta_m) \ge c_2$, γ_0 intersects only a finite number of η_n's. Let us define

$$t_1^1 := \min\{0 \le t \le l : \gamma_0(t) \in \cup_n \eta_n\} .$$

There exists this minimum since γ_0 is a continuous function in a compact interval and $\gamma_0 \cap (\cup_n \eta_n)$ is a compact set (γ_0 intersects only a finite number of η_n's).

Then there is n_1 such that $\gamma_0(t_1^1) \in \eta_{n_1}$, and we define

$$t_1^2 := \max\{0 \le t \le l : \gamma_0(t) \in \eta_{n_1}\} .$$

In a similar way, we define recursively

$$t_j^1 := \min\{t_{j-1}^2 < t \le l : \gamma_0(t) \in \cup_n \eta_n\} ;$$

if $\gamma_0(t_j^1) \in \eta_{n_j}$, for some n_j, we take

$$t_j^2 := \max\{t_{j-1}^2 < t \le l : \gamma_0(t) \in \eta_{n_j}\} .$$

We can continue this choice for $1 \le j \le r$. We define a finite union of curves γ in X as the restriction of $f^{-1}(\gamma_0)$ to the closed set $[0, t_1^1] \cup [t_1^2, t_2^1] \cup \cdots \cup [t_{r-1}^2, t_r^1] \cup [t_r^2, l]$; since f is not injective, we take $\gamma(t_j^1) := \lim_{t \to (t_j^1)^-} \gamma(t)$ and $\gamma(t_j^2) := \lim_{t \to (t_j^2)^+} \gamma(t)$; if $t_1^1 = 0$ (and/or $t_r^2 = l$), we take $\gamma(0) = x$ (and/or $\gamma(l) = y$).

Notice that $\gamma(t_j^1) \in \eta_{n_j}^i, \gamma(t_j^2) \in \eta_{n_j}^k$, with $i, k \in \{1,2\}$, and i can be equal or not to k.

Now, let us choose continuous curves g_j connecting $\gamma(t_j^1)$ and $\gamma(t_j^2)$ in X in the following way: by (4), we can take $i \in \{1,2\}$ and a geodesic h_j in X joining $\gamma(t_j^1)^*$ and $\gamma(t_j^2)^*$, with $L_X(h_j) = d_X(\gamma(t_j^1)^*, \gamma(t_j^2)^*) \le c_3\, d_{X_0}(f(\gamma(t_j^1)), f(\gamma(t_j^2))) + c_4$, where $\gamma(t_j^k)^* \in \eta_{n_j}^i \cap \{f^{-1}(\gamma_0(t_j^k))\}$ for $k = 1, 2$. Then g_j is the union of h_j and at most two curves of length less or equal than c_1, by (2); therefore $L_X(g_j) \le c_3\, d_{X_0}(\gamma_0(t_j^1), \gamma_0(t_j^2)) + 2c_1 + c_4$.

We define $g := \gamma \cup g_1 \cup g_2 \cup \cdots \cup g_r$, which is a continuous curve in X joining x and y. Consequently we have

$$\begin{aligned} d_X(x,y) \le L_X(g) &= L_X(\gamma) + \sum_{j=1}^{r} L_X(g_j) \\ &\le d_{X_0}(\gamma_0(0), \gamma_0(t_1^1)) + \sum_{j=1}^{r-1} d_{X_0}(\gamma_0(t_j^2), \gamma_0(t_{j+1}^1)) + d_{X_0}(\gamma_0(t_r^2), \gamma_0(l)) \\ &\quad + \sum_{j=1}^{r} \big(c_3\, d_{X_0}(\gamma_0(t_j^1), \gamma_0(t_j^2)) + 2c_1 + c_4 \big). \end{aligned}$$

Notice that

$$(r-1)(2c_1+c_4) \le \frac{2c_1+c_4}{c_2} \sum_{j=1}^{r-1} d_{X_0}(\gamma_0(t_j^2), \gamma_0(t_{j+1}^1)),$$

since $d_{X_0}(\eta_n, \eta_m) \ge c_2$. Then

$$\begin{aligned} d_X(x,y) &\le d_{X_0}(\gamma_0(0), \gamma_0(t_1^1)) + \Big(1 + \frac{2c_1+c_4}{c_2}\Big) \sum_{j=1}^{r-1} d_{X_0}(\gamma_0(t_j^2), \gamma_0(t_{j+1}^1)) \\ &\quad + d_{X_0}(\gamma_0(t_r^2), \gamma_0(l)) + c_3 \sum_{j=1}^{r} d_{X_0}(\gamma_0(t_j^1), \gamma_0(t_j^2)) + 2c_1 + c_4 \\ &\le c\, d_{X_0}(f(x), f(y)) + 2c_1 + c_4 \,, \end{aligned}$$

where $c := \max\{1 + (2c_1+c_4)/c_2, c_3\}$. We conclude that f is a quasi-isometry with constants which only depend on c_1, c_2, c_3, c_4. The conclusions about hyperbolicity are a consequence of this fact and Theorem B. □

We finish this section with a theorem which will be very useful in the proof of the main results of this paper. In order to state it, we need a definition.

Definition 2.8. We say that a geodesic metric space X has a *decomposition*, if there exists a family of geodesic metric spaces $\{X_n\}_{n\in\Lambda}$ with $X = \cup_{n\in\Lambda} X_n$ and $X_n \cap X_m = \eta_{nm}$, where for each $n \in \Lambda$, $\{\eta_{nm}\}_m$ are pairwise disjoint closed subsets of X_n ($\eta_{nm} = \varnothing$ is allowed); furthermore any geodesic segment in X meets at most a finite number of η_{nm}'s.

We say that X_n, with $n \in \Lambda$, is a (k_1, k_2, k_3)-*tree-piece* if it satisfies the following properties:

(a) If $\eta_{nm} \ne \varnothing$, then $X \setminus \eta_{nm}$ is not connected and a, b are in different connected components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$.

(b) $\operatorname{diam}_{X_n}(\eta_{nm}) \le k_1$ for every $m \ne n$, and there exists $A_n \subseteq \Lambda$, such that $\operatorname{diam}_{X_n}(\eta_{nm}) \le k_2\, d_{X_n}(\eta_{nm}, \eta_{nk})$ if $m \ne k$ and $m, k \in A_n$, and $\sum_{m\notin A_n} \operatorname{diam}_{X_n}(\eta_{nm}) \le k_3$.

We say that a geodesic metric space X has a (k_1, k_2, k_3)-*tree-decomposition* if it has a decomposition such that every X_n, with $n \in \Lambda$, is a (k_1, k_2, k_3)-tree-piece.

We wish to emphasize that condition $\operatorname{diam}_{X_n}(\eta_{nm}) \le k_1$ is not very restrictive: if the space is "wide" at every point (in the sense of long injectivity radius, as in the case of simply connected spaces) or "narrow" at every point (as in the case of trees), it is easier to study its hyperbolicity; if we can find narrow parts (as η_{nm}) and wide parts, the problem is more difficult and interesting.

Remarks. 1. Obviously, condition (b) is required only for $\eta_{nm}, \eta_{nk} \ne \varnothing$.

2. The sets Λ and A_n do not need to be countable.

3. The hypothesis $\operatorname{diam}_{X_n}(\eta_{nm}) \le k_2\, d_{X_n}(\eta_{nm}, \eta_{nk})$ holds if we have $d_{X_n}(\eta_{nm}, \eta_{nk}) \ge k_2'$, since $\operatorname{diam}_{X_n}(\eta_{nm}) \le k_1$.

4. Condition (a) for every $n \in \Lambda$ guarantees that the graph $R = (V, E)$ constructed in the following way is a tree: $V = \cup_{n\in\Lambda}\{v_n\}$ and $[v_n, v_m] \in E$ if and only if $\eta_{nm} \ne \varnothing$.

5. If X is a Riemann surface, $\{X_n\}_{n\in\Lambda}$ are bordered Riemann surfaces and $\eta_{nm} \subset \partial X_n \cap \partial X_m$, then the condition "$a, b$ are in different components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$" in ($a$), is a consequence of "$X \setminus \eta_{nm}$ is not connected".

The following result can be applied to the study of the hyperbolicity of Riemann surfaces (see theorems 3.4, 3.6 and 3.8). In [PRT1] explicit expressions for the constants involved are supplied.

Theorem C. ([PRT1, Theorem 1]) *Let us consider a (k_1, k_2, k_3)-tree-decomposition $\{X_n\}_{n\in\Lambda}$ of a geodesic metric space X. Then X is δ-hyperbolic if and only if there exists a constant k_4 such that X_n is k_4-hyperbolic for every $n \in \Lambda$. Furthermore, if X is δ-hyperbolic, then k_4 only depends on δ, k_1, k_2 and k_3; if there exists k_4, then δ only depends on k_1, k_2, k_3 and k_4.*

§3. Results in Riemann surfaces

In this section we always work with the Poincaré metric; consequently, curvature is always -1. In fact, many concepts appearing here (as punctures or funnels) only make sense with the Poincaré metric.

The intuition would say that negative curvature must imply hyperbolicity; in fact this is what happens when there are no topological "obstacles" (as in the case of the Poincaré disk $\mathbf{D}$) or if there is a finite number of them (see Theorem 3.5). However, if there are infinitely many topological "obstacles", the hyperbolicity can fail, as in the case of the two-dimensional jungle gym (a $\mathbf{Z}^2$-covering of a torus with genus two).

The results in this section are useful since they not only provide many examples of hyperbolic Riemann surfaces, but also allow to establish criteria in order to decide whether a Riemann surface is hyperbolic or not.

Below we collect some definitions concerning to Riemann surfaces which will be referred to afterwards.

An *open non-exceptional* Riemann surface (or a non-exceptional Riemann surface without boundary) S is a Riemann surface whose universal covering space is the unit disk $\mathbf{D} = \{z \in \mathbf{C} : \ |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1 - |z|^2)$, or, equivalently, the upper half plane $\mathbf{U} = \{z \in \mathbf{C} : \ \operatorname{Im} z > 0\}$, with the metric $ds = |dz|/\operatorname{Im} z$. Notice that, with this definition, every compact non-exceptional Riemann surface without boundary is open. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1, and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases.

Let S be an open non-exceptional Riemann surface with a puncture or cusp q (if $S \subset \mathbf{C}$, every isolated point in ∂S is a puncture). A *collar* in S about q is a doubly connected domain in S "bounded" both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q.

A collar in S about q of area α will be called an *α-collar* and it will be denoted by $C_S(q, \alpha)$. A theorem of Shimizu [S] gives that for every puncture in any open non-exceptional Riemann surface, there exists an α-collar for every $0 < \alpha \le 2$ (see also [Bu, Chapter 4.4]).

We say that a curve is *homotopic to a puncture* q if it is freely homotopic to $\partial C_S(q,\alpha)$ for some (and then for every) $0<\alpha<2$.

We have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously cannot be minimizing geodesics. We will continue using the word geodesic with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S "bounded" by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ; such collar is equal to $\{p\in S:\, d_S(p,\gamma)<d\}$, for some positive constant d. The constant d is called the *width* of the collar. The Collar Lemma [R] says that there exists a collar of γ of width d, for every $0<d\le d_0$, where $\cosh d_0=\coth(L_S(\gamma)/2)$.

We say that S is a *bordered non-exceptional Riemann surface* (or a non-exceptional Riemann surface with boundary) if it can be obtained by deleting an open set V of an open non-exceptional Riemann surface R, such that:

(1) S is connected and $d_S:=d_R|_S$ (recall Definition 2.5),

(2) any ball in R intersects at most a finite number of connected components of V,

(3) the boundary of S is locally Lipschitz.

Any such surface S is a bordered orientable Riemannian manifold of dimension 2 and its Riemannian metric has constant negative curvature -1. It is not difficult to see that S is a geodesic metric space.

A *funnel* is a bordered non-exceptional Riemann surface which is topologically a cylinder and whose boundary is a simple closed geodesic. Given a positive number a, there is a unique (up to conformal mapping) funnel such that its boundary curve has length a. Every funnel is conformally equivalent, for some $\beta>1$, to the subset $\{z\in \mathbf{C}:\, 1\le |z|<\beta\}$ of the annulus $\{z\in \mathbf{C}:\, 1/\beta<|z|<\beta\}$.

Every doubly connected end of an open non-exceptional Riemann surface is a puncture (if there are homotopically non-trivial curves with arbitrary small length) or a funnel (if this was not so).

A *Y-piece* is a bordered non-exceptional Riemann surface which is conformally equivalent to a sphere without three open disks and whose boundary curves are simple closed geodesics. Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see e.g. [Bu, p. 109]). They are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [C, Chapter X.3] and [Bu, Chapter 3].

A *generalized Y-piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere without n open disks and m points, with integers $n,m\ge 0$ such that $n+m=3$, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Notice that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0\le m\le 3$.

We deduce now several applications of Theorem 2.1 which guarantee the hyperbolicity of many Riemann surfaces, with good control of the hyperbolicity constants.

Proposition 3.1. *Any generalized Y-piece Y_0 with $L(\gamma) \le l$, for every simple closed geodesic $\gamma \subseteq \partial Y_0$, is $(4r_0 + l)$-decomposible and $3(4r_0 + l)/2$-thin, where $r_0 := 2\,\mathrm{Arcsinh}(1/2)$.*

Proof. Let us denote by $\gamma_1, \gamma_2, \gamma_3$, the simple closed geodesics in ∂Y_0 (as usual, we identify a puncture with a geodesic of zero length). If $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$, let us consider the geodesic B_i in Y_0 which is orthogonal to γ_j and γ_k. If we split Y_0 along the curves B_i, we obtain two isometric convex right-angle hexagons H_1, H_2, with consecutive sides of length $L(\gamma_1)/2, L(B_3), L(\gamma_2)/2, L(B_1), L(\gamma_3)/2, L(B_2)$.

Let us consider the middle point x_i of the side with length $L(\gamma_i)/2$ in H_1, and the geodesic triangle $T = \{x_1, x_2, x_3\} \subset H_1$. We can draw a ball $B(z_0, r)$ contained in H_1, which is tangent to some $y_1 \in [x_2, x_3]$, $y_2 \in [x_1, x_3]$ and $y_3 \in [x_1, x_2]$. We have that $\pi \ge A(T) > A(B(z_0, r)) = 4\pi \sinh^2(r/2)$, and then $r < r_0$.

Let us consider the geodesics $\alpha_1, \alpha_2, \alpha_3$ in H_1 starting in z_0 and finishing respectively in x_1, x_2, x_3.

For each point $p \in [x_i, x_j] \subset T$, we consider the geodesic a_p in H_1 which starts in p orthogonally to $[x_i, x_j]$ and finishes in $\alpha_i \cup \alpha_j$, and the geodesic b_p in H_1 which starts orthogonally to B_k and finishes in p. It is clear that $L(a_p) + L(b_p) < r_0 + l/4$.

Therefore we can draw in H_1 curves joining two of the sets B_1, B_2, B_3, with diameter less than $2r_0 + l/2$; in fact, the "curve" containing z_0 is the union of three curves and joins the three sets.

We can do the same design in H_2, since it is isometric to H_1. If we paste these hexagons, we have that Y_0 is $(4r_0 + l)$-decomposible. Theorem 2.1 gives that Y_0 is $3(4r_0 + l)/2$-thin. □

Many Riemann surfaces can be decomposed in a union of funnels and generalized Y-pieces (see [FM, Theorem 4.1] and [AR]). The following results use this decomposition in order to obtain hyperbolicity.

Theorem 3.1. *Let us consider a non-exceptional Riemann surface S (with or without boundary) without genus (S can be viewed as a plane domain). If there is a decomposition of S in a union of generalized Y-pieces $\{Y_n\}_{n\in N}$ with $L_S(\gamma) \le l$ for every simple closed geodesic $\gamma \subset \cup_n \partial Y_n$, then S is $(4r_0 + l)$-decomposible and $3(4r_0 + l)/2$-thin, where $r_0 := 2\,\mathrm{Arcsinh}(1/2)$.*

Proof. By Proposition 3.1 we know that each Y_n is $(4r_0 + l)$-decomposible. Since S is a plane domain, the union in n of the curves constructed in Proposition 3.1 in each Y_n gives that S is also $(4r_0 + l)$-decomposible, since any of such curves disconnects S. Consequently, Theorem 2.1 gives that S is $3(4r_0 + l)/2$-thin. □

With an additional idea we can improve Proposition 3.1 and Theorem 3.1.

Proposition 3.2. *Any generalized Y-piece Y_0 with $L(\gamma) \le l$, for at least two simple closed geodesics $\gamma \subseteq \partial Y_0$, is $(2r_1 + l)$-decomposible and $3(2r_1 + l)/2$-thin, where*

$$r_1 := \max\left\{ \mathrm{Arcsinh}(\coth(l/4)), 4\,\mathrm{Arcsinh}(1/2)\right\}.$$

Remark. This is the best result that we can obtain about Y-pieces: If $L(\gamma) \le l$ for one simple closed geodesic $\gamma \subseteq \partial Y_0$, δ can be arbitrarily long, as shows the example after the proof of Proposition 3.2.

Proof. Let us denote by $\gamma_1, \gamma_2, \gamma_3$, the simple closed geodesics in ∂Y_0 (as usual, we identify a puncture with a geodesic of zero length). Without loss of generality, we can assume that $L(\gamma_3) > l$, since if this was not so, we can apply Proposition 3.1 (we have $r_1 \geq 2r_0$).

If $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$, let us consider the geodesic B_i in Y_0 which is orthogonal to γ_j and γ_k. If we split Y_0 along the curves B_i, we obtain two isometric convex right-angle hexagons H_1, H_2.

For each point $p \in g_3 := \gamma_3 \cap H_1$, we consider the geodesic a_p in H_1 which starts in p and finishes orthogonally to B_3. We want to obtain a bound for $L(a_p)$; in order to do this, let us consider first the simple case in which γ_1 and γ_2 are punctures. Then H_1 is a quadrilateral with two right angles and three sides of infinite length; if p_0 is the middle point of g_3, we can split H_1 by deleting a_{p_0} in two isometric quadrilaterals Q_1, Q_2, with three right angles and two sides of infinite length. The other sides have length $L(\gamma_3)/4$ and $L(a_{p_0})$, with $\sinh(L(\gamma_3)/4) \sinh L(a_{p_0}) = 1$ (see e.g. [B, p. 157], [F, p. 89], [Ra, p. 96]). If p_0, p_1 are the end points of g_3 in Q_1, we can split Q_1 by deleting a_{p_1} in a triangle and a quadrilateral Q_{11} with three right angles and four finite sides. We have (see e.g. [F, p. 88]) $\sinh L(a_{p_1}) = \sinh L(a_{p_0}) \cosh(L(\gamma_3)/4) = \coth(L(\gamma_3)/4)$, and consequently $L(a_p) \leq L(a_{p_1}) < \operatorname{Arcsinh}(\coth(l/4)) \leq r_1$, since $L(\gamma_3) > l$.

It is clear now that in the general case (with $0 \leq L(\gamma_1), L(\gamma_2) \leq l$) we have $L(a_p) < r_1 + l/2$.

Let us denote by b_p the curve in Y_0 obtained by the union of a_p and its symmetric curve $a_{p'}$; each b_p joins γ_3 with itself and have length less than $2r_1 + l$. It is clear that the set of the points of Y_0 which are not in the union of the b_p's has two connected components, which are tubular neighborhoods N_1 of γ_1 and N_2 of γ_2 in Y_0. It is plain that we can draw in N_i curves freely homotopic to γ_i with length less than $2r_1 + l$.

Then we have that Y_0 is $(2r_1 + l)$-decomposible and $3(2r_1 + l)/2$-thin, by Theorem 2.1. □

Example. *The sharp hyperbolicity constant of the generalized Y-piece Y_t with one puncture and two simple closed geodesics of length $2t$ goes to infinity as $t \to \infty$.*

Let us denote by γ_1, γ_2, the simple closed geodesics of Y_t. The idea that lies behind the proof is that given two points in γ_i, the distance between them is approximately the length of a subcurve of γ_i joining them. Let us denote by $p_1 \in \gamma_1, p_2 \in \gamma_2$, the points with $d(p_1, p_2) = d(\gamma_1, \gamma_2) =: s$. We choose the points $q_1 \in \gamma_1, q_2 \in \gamma_2$, with $d(p_1, q_1) = d(p_2, q_2) = t$. If we split Y_t along the geodesics which start orthogonally to γ_1 in p_1 and q_1, and to γ_2 in q_2, we obtain two isometric right-angled hexagons H_1, H_2. Each H_i has sides with length $t, s, t, \infty, 0, \infty$.

Standard hyperbolic trigonometry (see e.g. [B, p. 161]) gives

$$\cosh s = 1 + \frac{2}{\sinh^2 t}\,, \qquad \sinh s = \frac{2\cosh t}{\sinh^2 t}\,.$$

Let us consider the geodesic γ_0 in H_1 which gives the distance between $[p_1, q_1]$ and the side A of infinite length which does not intersect with it; we define $x := \gamma_0 \cap [p_1, q_1]$ and $y := \gamma_0 \cap A$. The geodesic γ_0 splits H_1 into a right-angled pentagon and a quadrilateral with three right-angles and a degenerated angle. Hyperbolic trigonometry for pentagons (see e.g. [B, p. 159]) gives $\cosh L(\gamma_0) = \sinh s \sinh t = 2\coth t$; then $\lim_{t\to\infty} \cosh L(\gamma_0) = 2$ and $\lim_{t\to\infty} \sinh L(\gamma_0) = \sqrt{3}$. We also have that

$\sinh L(\gamma_0) \sinh d(x, q_1) = 1$ and $\lim_{t\to\infty} d(x, q_1) = \text{Arcsinh}(1/\sqrt{3}\,)$. Consequently, $d(x, q_1)$ is bounded when t goes to ∞.

With this computations in mind, we consider the geodesic bigon γ_1 in Y_t with vertices $\{p_1, q_1\}$ (it is geodesic by the symmetry of Y_t). Let us choose the point $z \in \gamma_1 \cap H_1$ such that $d(z, p_1) = d(z, q_1) = t/2$. Then there exists some constant c with $d(z, \gamma_1 \cap H_2) \geq t/2 - c$ if t is large enough, as a consequence of the above computations; hence $\lim_{t\to\infty} \delta(Y_t) = \infty$.

In order to prove our next theorem, we need the following result appearing in [RT2]; in fact, Theorem 5.2 in [RT2] provides a better result than Theorem D, but this version is enough for our purpose.

Theorem D. ([RT2, Theorem 5.2]) *Given an open non-exceptional Riemann surface S, let us denote by F the union of some funnels of S. Let S_0 be the bordered non-exceptional Riemann surface obtained by deleting from S the interior of F. Then S is hyperbolic if and only if S_0 is hyperbolic. Furthermore, if S_0 is δ_0-hyperbolic, then S is δ-hyperbolic, with δ a constant which only depends on δ_0.*

In order to state the next theorem we need a definition.

Definition 3.1. Let us consider a non-exceptional Riemann surface S of finite type (with or without boundary); if S is bordered, we also require that ∂S is the union of local geodesics (closed or non-closed). An *outer loop* in S is a simple closed geodesic which is the boundary curve of a funnel or is contained in ∂S. An *inner loop* in S is a simple closed geodesic which is not an outer loop. The *characteristic* of S is $a = 2g - 2 + n$, where g is the genus of S and n is the sum of the number of punctures of S and the number of outer loops of S.

We also need the following beautiful theorem of Bers.

Theorem E. ([Be, Theorem 1]) *Let us consider a non-exceptional Riemann surface S of finite type* (*with or without boundary*); *if S is bordered, we also require that ∂S is the union of local geodesics* (*closed or non-closed*). *If S has characteristic a, the length of its shortest inner loop* (*if any*) *is bounded from above by a constant $J = J(a, L)$ depending only on a and on the length L of the longest outer loop* (*if any*).

Remark. There exists such inner loop if $3g - 3 + n > 0$.

In fact, Theorem E is proved in [Be] only for surfaces without boundary, but the other case is direct from this one.

If we use Proposition 3.2 instead of Proposition 3.1 in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let us consider a non-exceptional Riemann surface S* (*with or without boundary*) *without genus. If there is a decomposition of S into a union of funnels $\{F_m\}_{m\in M}$ and generalized Y-pieces $\{Y_n\}_{n\in N}$ ($N \neq \emptyset$) with $L_S(\gamma) \leq l$ for at least two simple closed geodesics $\gamma \subseteq \partial Y_n$ for every $n \in N$, then S is δ-hyperbolic, where δ is a constant which only depends on l.*

Proof. By Theorem D, we can assume that $M=\varnothing$. Let us consider Y_n with $L_S(\gamma)>l$ for some simple closed geodesic $\gamma\subseteq\partial Y_n$. Let us assume that there exists some $n_0\neq n$ with $\gamma\subseteq\partial Y_{n_0}$; we can apply Theorem E to the bordered Riemann surface $Y_n\cup Y_{n_0}$ with characteristic $a=2$, and then we have an inner loop γ' in $Y_n\cup Y_{n_0}$ with $L_S(\gamma')\le J(2,l)$. If we split $Y_n\cup Y_{n_0}$ by γ', we obtain two generalized Y-pieces Y_n',Y_{n_0}', such that $Y_n\cup Y_{n_0}=Y_n'\cup Y_{n_0}'$, and $L_S(\sigma)\le l_0:=\max\{l,J(2,l)\}$ for every simple closed geodesic $\sigma\subseteq\partial Y_n'\cup\partial Y_{n_0}'$.

Consequently, without loss of generality, we can assume that the decomposition of S in the union of generalized Y-pieces $\{Y_n\}_{n\in N}$ verifies the following property: if $L_S(\gamma)>l_0$ for some simple closed geodesic $\gamma\subseteq\cup_n\partial Y_n$, then γ is in the boundary of just one generalized Y-piece.

By Proposition 3.2 we have that each Y_n is $(2r_1+l_0)$-decomposible, with

$$r_1:=\max\big\{\operatorname{Arcsinh}(\coth(l_0/4)),4\operatorname{Arcsinh}(1/2)\big\}.$$

Since S is a plane domain, the union in n of the curves constructed in Proposition 3.2 in each Y_n gives that S is also $(2r_1+l_0)$-decomposible, since any of such curves disconnects S. Consequently, Theorem 2.1 gives that S is $3(2r_1+l_0)/2$-thin. □

Since the funnel F_l with $L(\partial F_l)=l$ has thin constant $\delta_l\ge l/4$, one can think that a surface with funnels with arbitrarily long simple closed geodesics cannot be hyperbolic. However, Theorem 3.2 allows us to prove the following surprising result.

Corollary 3.1. *There exist hyperbolic plane domains with funnels with arbitrarily long simple closed geodesics.*

Proof. For each positive integer n we consider a Y-piece Y_n with two boundary geodesics of length 1 and a boundary geodesic of length n. We denote by Z_1 the union of Y_1 and two funnels with boundary geodesics of length 1, and by Z_n $(n>1)$ the union of Y_n and a funnel with boundary geodesic of length n.

Let us denote by Ω the union of $\{Z_n\}_{n=1}^{\infty}$ identifying the boundary geodesics (Z_1 is connected with Z_2, and Z_n is connected with Z_{n-1} and Z_{n+1}, if $n>1$). It is clear that Ω has funnels with arbitrarily long simple closed geodesics, and Theorem 3.2 gives that it is hyperbolic. □

Definition 3.2. We say that a non-exceptional Riemann surface S (with or without boundary) is *of finite type* if its fundamental group is finitely generated.

Definition 3.3. Let us consider a non-exceptional Riemann surface S with boundary and $\{\eta_n^1,\eta_n^2\}_n\subseteq\partial S$ pairwise disjoint simple closed geodesics in S. If c_1 is a positive constant, we say that $\{\eta_n^1,\eta_n^2\}_n$ are c_1-*identified* if $L_S(\eta_n^1)=L_S(\eta_n^2)\le c_1$ and $d_S(\eta_n^1,\eta_n^2)\le c_1$ for every n.

If we apply Theorem 2.2 to the context of Riemann surfaces, we obtain the following result. It will be an important tool in the proof of Theorem 3.5.

Theorem 3.3. *Let us consider a non-exceptional Riemann surface S with boundary and $\{\eta_n^1,\eta_n^2\}_n$ c_1-identified. Then S is hyperbolic if and only if S_0 is hyperbolic. In particular, if S (respectively S_0) is δ-hyperbolic, then S_0 (respectively S) is δ'-hyperbolic, with δ' a universal constant which only depends on δ and c_1.*

Proof. This result is a direct consequence of Theorem 2.2 and the following facts:

Every non-exceptional Riemann surface (with or without boundary) is a geodesic metric space.

Since η_n^i are simple closed curves, the condition that $(\eta_n^1, d_S|_{\eta_n^1})$ and $(\eta_n^2, d_S|_{\eta_n^2})$ are isometric is equivalent to $L_S(\eta_n^1) = L_S(\eta_n^2)$.

We have $\operatorname{diam}_S(\eta_n^1 \cup \eta_n^2) \le L_S(\eta_n^1)/2 + d_S(\eta_n^1, \eta_n^2) + L_S(\eta_n^2)/2 \le 2c_1$.

If γ_1, γ_2, are disjoint simple closed geodesics contained in an open non-exceptional Riemann surface, with length less or equal than a, the Collar Lemma [R] says that there exist disjoint collars of γ_i of width d_0, where $\cosh d_0 = \coth(a/2)$. Therefore, $d(\gamma_1, \gamma_2) \ge 2 \operatorname{Arccosh}(\coth(a/2))$; it is clear that this inequality is also true if S is bordered, since then S is contained in an open non-exceptional Riemann surface. □

In order to prove the next theorems we need some definitions.

Definition 3.4. Given a Riemann surface S with finite genus g, we say that the simple closed geodesics $a_1, \dots, a_g, b_1, \dots, b_g$ are *generators of the genus of* S if $S \setminus a_j$ and $S \setminus b_j$ are connected, $a_j \cap b_j$ is a single point, and $(a_j \cup b_j) \cap (\cup_{k \ne j}(a_k \cup b_k)) = \varnothing$.

Given $c > 0$, we say that a Riemann surface S with finite genus g has c-*small genus* if there exist $a_1, \dots, a_g, b_1, \dots, b_g$ generators of the genus of S such that $L_S(a_j) \le c$, $L_S(b_j) \le c$, for $j = 1, \dots, g$. We say that any plane domain (a surface without genus) has 0-*small genus*.

Definition 3.5. For each $l, c \ge 0$ and each non-negative integer a, we denote by $\mathcal{S}_G(a, l, c)$ the set of non-exceptional Riemann surfaces of finite type S verifying the following properties: if S is bordered, then ∂S is the union of local geodesics (closed or non-closed), S has characteristic less or equal than a and c-small genus, and every outer loop has length less or equal than l.

We denote by $\mathcal{S}_G(a, l)$ the set of plane domains in $\mathcal{S}_G(a, l, c)$.

The two following theorems guarantee the hyperbolicity of the surfaces of finite type, with hyperbolicity constants which only depend on just two or three topological and metric parameters.

Theorem 3.4. *For each $l \ge 0$ and each non-negative integer a, there exists a constant $\delta = \delta(a, l)$, which only depends on a and l, such that every surface in $\mathcal{S}_G(a, l)$ is δ-hyperbolic.*

Proof. We prove the result by induction on a.

Let us consider first the case $a = 0$. If $S \in \mathcal{S}_G(0, l)$, it is the punctured disk, an annulus or a funnel. Lemma 5.4 and Corollary 5.1 in [RT2] give the result for the punctured disk and the annuli; the case of the funnel is a consequence both of this fact and of the funnel being a geodesically convex subset of an annulus.

We consider now the case $a = 1$. If $S \in \mathcal{S}_G(1, l)$, it is the union of a generalized Y-piece and at most three funnels. Since every simple closed geodesic of S is an outer loop, Theorem 3.2 gives the result.

Let us assume now that the result is true for $a - 1$, with $a \ge 2$, and let us prove it for a. Let us consider a surface $S \in \mathcal{S}_G(a, l)$. By Theorem E we can find an inner loop γ with length less or equal than $J(a, l)$ (there exists such inner loop since $g = 0$ and $a - 1 > 0$). Then S is the union of two surfaces S_1, S_2, with $S_1 \cap S_2 = \gamma$, since S has genus 0; notice that $\gamma \subseteq \partial S_1, \partial S_2$. If we define

$l_a := J(a,l)/2$ (which only depends on a and l), $A_1 := A_2 := \varnothing$, we see that $\{S_1, S_2\}$ is a $(l_a, 0, l_a)$-tree-decomposition of S (see Definition 2.8). It is clear that S_1 and S_2 have characteristic less than a, and every outer loop has length less or equal than $\max\{l, J(a,l)\}$; then we have that they are δ_0-hyperbolic, with δ_0 a constant which only depends on a and l, by the induction hypothesis. Then Theorem C gives that there exists a constant $\delta = \delta(a,l)$, which only depends on a and l, such that S is δ-hyperbolic. □

We can improve this last result in the following theorem, in which we deal with the case of surfaces with genus.

Theorem 3.5. *For each $l, c \geq 0$ and each non-negative integer a, there exists a constant $\delta = \delta(a,l,c)$, which only depends on a, l and c, such that every surface in $\mathcal{S}_G(a,l,c)$ is δ-hyperbolic.*

Proof. Let us fix a, l, c, and let us consider $S \in \mathcal{S}_G(a,l,c)$. If $S \in \mathcal{S}_G(a,l)$, we only need to apply Theorem 3.4. If this was not so, we choose $a_1, \dots, a_g, b_1, \dots, b_g$ generators of the genus of S. Then we consider the bordered surface S_1 obtained by cutting S along $a_1, \dots, a_g$, and we define $t := \max\{l, c\}$. To cut along a_j decreases the genus by 1 and increases the number of outer loops by 2; therefore, the characteristic remains unchanged. It is clear that $S_1 \in \mathcal{S}_G(a,t)$, and then we have by Theorem 3.4 that S_1 is δ_0-hyperbolic, with δ_0 a constant which only depends on a and t. Notice that $L_S(b_j) \leq c$; hence the two copies of a_j in ∂S_1 are c-identified; then Theorem 3.3 gives that there exists a constant $\delta = \delta(\delta_0, c)$, which only depends on δ_0 and c, such that S is δ-hyperbolic. □

The conclusion of Theorem 3.5 is not true without the hypothesis of c-small genus, as shows by the following example:

Example. There exist open non-exceptional Riemann surfaces of finite type S_t with genus 1 and characteristic 1, a puncture, and $\lim_{t\to 0^+} \delta(S_t) = \infty$: For each $t > 0$, let us consider the generalized Y-piece Y_t with a puncture and two simple closed geodesics γ_1, γ_2, of length $2t$. Splitting Y_t into two isometric hexagons (with a side of zero length), standard hyperbolic trigonometry (see e.g. [B, p. 161]) gives

$$d_{Y_t}(\gamma_1, \gamma_2) = \operatorname{Arccosh}\Big(1 + \frac{2}{\sinh^2 t}\Big) =: g(t)\,.$$

Let us denote by S_t the Riemann surface of finite type with genus 1 and a puncture obtained from Y_t by identifying γ_1 with γ_2. It is clear that there exists a simple closed geodesic with length $g(t)$ in S_t "surrounding" the genus; then we have that if S_t is δ-thin, then $\delta \geq g(t)/4$.

Theorems 3.5 and C give the following result.

Theorem 3.6. *Let us consider a non-exceptional Riemann surface S (with or without boundary). If there exists a tree-decomposition of S into a union of bordered surfaces $\{S_m\}_{m\in M} \subset \mathcal{S}_G(a,l,c)$, then S is δ-hyperbolic, where δ is a constant which only depends on a, l and c.*

Remark. The condition "$\{S_m\}_{m\in M}$ is a tree-decomposition of S" is verified if "$\{S_m\}_{m\in M}$ is a decomposition of S such that for every $m, n \in M$, $\partial S_m \cap \partial S_n$ is the empty set or an outer loop γ of S_m and S_n, and $S \setminus \gamma$ is not connected if $\gamma = \partial S_m \cap \partial S_n$"; it is sufficient to take $k_1 = l/2$, $A_n = \varnothing$, $k_2 = 0$ and $k_3 = (a+2)l/2$, which are constants only depending on a and l.

This remark and Theorem 3.6 give the following result.

Corollary 3.2. *Let us consider a non-exceptional Riemann surface S (with or without boundary) without genus. If there exists a decomposition of S into a union of bordered surfaces $\{S_m\}_{m\in M} \subset \mathcal{S}_G(a,l)$, then S is δ-hyperbolic, where δ is a constant which only depends on a and l.*

Now, we want to obtain the equivalence of the hyperbolicity of an extensive class of Riemann surfaces and some graphs. We start with a definition.

Definition 3.6. Let us consider a generalized Y-piece Y_0, with $L(\gamma_i) = l_i \le l$, for every simple closed geodesic $\gamma_i \subseteq \partial Y_0$. We say that a tree $G := (V,E)$ is the *l-skeleton* of Y_0 if G has vertices $V = \{v, v_1, v_2, v_3\}$ and edges $E := \cup_{i=1}^3 [v, v_i]$, such that $L([v, v_i]) = \operatorname{Arccosh}(\coth(l_i/2))$ for $i = 1, 2, 3$.

Let us consider a generalized Y-piece Y_0, with $L(\gamma_i) = l_i$, for every simple closed geodesic $\gamma_i \subseteq \partial Y_0$, $l_1, l_2 \le l$ and $l_3 > l$. We say that a tree G is the *l-skeleton* of Y_0 if G has just one edge $[v_1, v_2]$, such that

$$L([v_1, v_2]) = \operatorname{Arccosh}\Big(\frac{\cosh(l_3/2) + \cosh(l_1/2)\cosh(l_2/2)}{\sinh(l_1/2)\sinh(l_2/2)}\Big).$$

Remark. If $L(\gamma_i) = l_i = 0$ (i.e., if γ_i is a puncture), we choose as $[v, v_i]$ a halfline starting in v. $L([v_1, v_2])$ is the distance between γ_1 and γ_2 (see e.g. [B, p. 161]).

Proposition 3.3. *Given any generalized Y-piece Y_0 with $L(\gamma_i) = l_i \le l$, for at least two simple closed geodesic $\gamma_i \subseteq \partial Y_0$, there exists a $(1, M)$-quaiisometry of Y_0 onto its l-skeleton G, with*

$$M := \max\Big\{\frac{3}{2}\Big(\log\Big(2\cosh\frac{l}{2}\Big(1+\cosh\frac{l}{2}\Big)\Big) + \frac{l}{2}\coth\frac{l}{2}\Big),\ 2\operatorname{Arcsinh}\Big(\coth\frac{l}{4}\Big) + l\Big\}.$$

Proof. Let us denote by $\gamma_1, \gamma_2, \gamma_3$, the simple closed geodesics in ∂Y_0 (as usual, we identify a puncture with a geodesic of zero length).

We deal first with the case $L(\gamma_i) = l_i \le l$, for $i = 1, 2, 3$. The Collar Lemma gives that, for each geodesic γ_i, there exists a collar C_{γ_i} of width $d_i = \operatorname{Arccosh}(\coth(l_i/2))$, with boundary curves γ_i and η_i; the closed curve η_i verifies $L(\eta_i) = L(\gamma_i)\cosh d_i = l_i \coth(l_i/2)$ for $i = 1, 2, 3$. When γ_i is a puncture, we have $l_i = 0$, $d_i = \infty$ and $L(\eta_i) = 2$ (see [Bu, Chapter 4.4]).

If $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$, let us consider the geodesic segment B_i in Y_0 which is orthogonal to γ_j and γ_k. If we split Y_0 along the curves B_i, we obtain two isometric convex right-angle hexagons H_1, H_2, with consecutive sides of length $L(\gamma_1)/2, L(B_3), L(\gamma_2)/2, L(B_1), L(\gamma_3)/2, L(B_2)$, such that

$$L(B_i) = \operatorname{Arccosh}\Big(\frac{\cosh(l_i/2) + \cosh(l_j/2)\cosh(l_k/2)}{\sinh(l_j/2)\sinh(l_k/2)}\Big)$$

(see [B, p. 161]). Now, we define the hexagon $H_1^* := H_1 \setminus \cup_{i=1}^3 C_{\gamma_i}$ in H_1 (similarly H_2^* in H_2), with consecutive sides of length $L(\eta_1)/2, \alpha_3, L(\eta_2)/2, \alpha_1, L(\eta_3)/2, \alpha_2$, verifying

$$\begin{aligned}\alpha_i :=& L(B_i) - (d_j + d_k)\\ =& \operatorname{Arccosh}\Big(\frac{\cosh(l_i/2) + \cosh(l_j/2)\cosh(l_k/2)}{\sinh(l_j/2)\sinh(l_k/2)}\Big) - \Big(\operatorname{Arccosh}(\coth(l_j/2)) + \operatorname{Arccosh}(\coth(l_k/2))\Big)\\ \le& \log\Big(2\,\frac{\cosh(l_i/2) + \cosh(l_j/2)\cosh(l_k/2)}{\sinh(l_j/2)\sinh(l_k/2)}\Big) - \Big(\log(\coth(l_j/2)) + \log(\coth(l_k/2))\Big)\\ =& \log\Big(2\,\frac{\cosh(l_i/2) + \cosh(l_j/2)\cosh(l_k/2)}{\cosh(l_j/2)\cosh(l_k/2)}\Big) \le \log\big(2\cosh(l/2)(1+\cosh(l/2))\big).\end{aligned}$$

When γ_i is a puncture, we obtain the same inequality by a limit process (see [Bu, Chapter 4.4]).

On the other hand, the function $g(t) = t\coth(t/2)$ is increasing in $t \in (0,\infty)$; therefore $L(\eta_i) \le l\coth(l/2)$. Consequently, $L(\partial H_i^*) \le 2M$ and $\operatorname{diam}(H_i^*) \le M$ for $i = 1, 2$.

Let us define $B_2' := B_2 \cap C_{\gamma_1}$, $B_3' := B_3 \cap C_{\gamma_2}$ and $B_1' := B_1 \cap C_{\gamma_3}$.

We consider the continuous function $f : Y_0 \longrightarrow G$, with $f(H_1^* \cup H_2^*) = v$, which is an isometry between B_2' and the edge $[v, v_1]$, between B_3' and $[v, v_2]$, and between B_1' and $[v, v_3]$. In the other points of Y_0, if $a \in C_{\gamma_1}$, we define $f(a) = f(a') \in [v, v_1]$, where a' is the point in B_2' such that $d(a,\gamma_1) = d(a',\gamma_1)$; we define f in similar way in C_{γ_2} and C_{γ_3}.

First of all, we have $d_G(f(x), f(y)) \le d_{Y_0}(x, y)$ for every $x, y \in Y_0$.

We also have $d_{Y_0}(x, y) \le d_G(f(x), f(y)) + M$ for every $x, y \in Y_0$.

Therefore, we have that f is a $(1,M)$-quasi-isometry of Y_0 onto G.

We deal now with the case $l_1, l_2 \le l$ and $l_3 > l$. We have that $G = [v_1, v_2]$ and that $L([v_1, v_2])$ is equal to the length of the geodesic segment B_3 in Y_0 joining γ_1 and γ_2. We have seen in the proof of Proposition 3.2 that any point in Y_0 has a point of B_3 at distance less or equal than $\operatorname{Arcsinh}(\coth(l/4)) + l/2 \le M/2$; consequently, the map $f_1 : Y_0 \longrightarrow B_3$ such that $f_1(x)$ is the nearest point to x in B_3 verifies $d_{B_3}(f_1(x), f_1(y)) \le d_{Y_0}(x, y) \le d_{B_3}(f_1(x), f_1(y)) + M$ for every $x, y \in Y_0$. Therefore, we have that $f := f_2 \circ f_1$ is a $(1,M)$-quasi-isometry of Y_0 onto G, if f_2 is an isometry between B_3 and G. $\square$

Definition 3.7. Let us consider $l > 0$ and a Riemann surface S (with or without boundary), such that there is a decomposition of S into a union of generalized Y-pieces $\{Y_n\}_{n\in N}$ and funnels $\{F_m\}_{m\in M}$, with $L_S(\gamma) \le l$ for at least two simple closed geodesics $\gamma \subseteq \partial Y_n$ in each n. We say that a graph G is a *l-skeleton* of S if it is the union of $\{G_n\}_{n\in N}$ with the following properties:

(a) G_n is the l-skeleton of Y_n for $n \in N$.

(b) If $Y_n \cap Y_m = \cup_{i\in I_{nm}} \gamma_{nm}^i$ (with $\gamma_{nm}^i = \gamma_{mn}^i$), then $G_n \cap G_m = \cup_{i\in I_{nm}} v_{nm}^i$, where v_{nm}^i is the vertex associated to γ_{nm}^i, and we identify v_{nm}^i with v_{mn}^i in order to obtain G.

Remarks. 1. As a consequence of (b), we have:

(P) if $L_S(\gamma) > l$ for some simple closed geodesic $\gamma \subseteq \cup_n \partial Y_n$, then γ is in the boundary of just one generalized Y-piece.

We want to remark that (P) is not a restriction at all, since if $\{Y_n\}_n$ does not satisfy this property, we can change $\{Y_n\}_n$ by $\{Y_n'\}_n$, as in the proof of Theorem 3.2, such that $\{Y_n'\}_n$ verifies (P) with $\max\{l, J(2,l)\}$ instead of l. Consequently, if S has a decomposition into a union of generalized Y-pieces

$\{Y_n\}_{n\in N}$ and funnels $\{F_m\}_{m\in M}$, with $L_S(\gamma)\le l$ for at least two simple closed geodesics $\gamma\subseteq\partial Y_n$ in each n, then it has a $\max\{l,J(2,l)\}$-skeleton.

2. Notice that $\operatorname{card} I_{nm}\le 3$, and $G_n\cap G_m=\varnothing$ if and only if $Y_n\cap Y_m=\varnothing$.

Theorem 3.7 below let us move the study of the hyperbolicity of a Riemann surface S to a graph G with much simpler structure, so long as between them there exists the precise relationship described in Definition 3.7.

Theorem 3.7. *Let us consider a non-exceptional Riemann surface S (with or without boundary), with a l-skeleton G. If S (respectively G) is δ-hyperbolic, then G (respectively S) is δ'-hyperbolic, with δ' a constant which only depends on δ and l.*

Proof. Without loss of generality, we can assume that S does not have funnels by Theorem D. We see now that there exists a $(1+c,2M)$-quaiisometry of S onto G, with M the constant in Proposition 3.3 and $c:=M/(2\operatorname{Arccosh}(\coth(l/2)))$.

For each $n\in N$, we have $\partial Y_n=\cup_{mi}\gamma_{nm}^i$ (as usual, we identify a puncture with a geodesic of zero length). Let us consider the $(1,M)$-quasi-isometry $f_n:Y_n\longrightarrow G_n$ defined in the proof of Proposition 3.3. Let us define $f:S\longrightarrow G$ such that $f|_{Y_n}:=f_n$; we will show now that f is a $(1+c,2M)$-quasi-isometry.

First of all, we have that $d_G(f(x),f(y))\le d_S(x,y)$ for every $x,y\in S$.

We prove now the reverse inequality. If x and y are in the same Y_n, we apply Proposition 3.3. If x and y are not in the same Y_n, let us consider a geodesic g joining $f(x)$ and $f(y)$ in G. Since g meets at most a finite number of G_n's, we denote them by $G_{n_1},G_{n_2},\dots,G_{n_r}$, where $f(x)\in G_{n_1}$, $f(y)\in G_{n_r}$, and g meets $G_{n_{k+1}}$ after G_{n_k}. Now, we take a continuous curve γ in S joining x and y, such that $f(\gamma)=g$ and $\gamma\cap Y_n$ is a geodesic in Y_n; then γ meets each simple closed curve $\sigma\subseteq\cup_n\partial Y_n$ at most in a point, γ only meets the pieces $Y_{n_1},Y_{n_2},\dots,Y_{n_r}$, and γ meets $Y_{n_{k+1}}$ after Y_{n_k}.

First of all, recall that $d_{G_n}(v_{nm_1}^{i_1},v_{nm_2}^{i_2})\ge 2\operatorname{Arccosh}(\coth(l/2))$, by the Collar Lemma. Consequently, if $a\in\gamma_{nm_1}^{i_1}, b\in\gamma_{nm_2}^{i_2}$, we obtain (using Proposition 3.3)

$$d_{Y_n}(a,b)\le d_{G_n}(v_{nm_1}^{i_1},v_{nm_2}^{i_2})+M=d_{G_n}(v_{nm_1}^{i_1},v_{nm_2}^{i_2})+2c\operatorname{Arccosh}(\coth(l/2))\le(1+c)d_{G_n}(v_{nm_1}^{i_1},v_{nm_2}^{i_2}).$$

If we define $x_k:=\gamma\cap\partial Y_{n_k}\cap\partial Y_{n_{k+1}}$, for $k=1,\dots,r-1$, we have

$$\begin{aligned} d_S(x,y)\le L_S(\gamma)&=d_{Y_{n_1}}(x,x_1)+\sum_{k=2}^{r-1}d_{Y_{n_k}}(x_{k-1},x_k)+d_{Y_{n_r}}(x_{r-1},y)\\ &\le d_{G_{n_1}}(f(x),f(x_1))+M+(1+c)\sum_{k=2}^{r-1}d_{G_{n_k}}(f(x_{k-1}),f(x_k))+d_{G_{n_r}}(f(x_{r-1}),f(y))+M\\ &\le(1+c)d_G(f(x),f(y))+2M. \end{aligned}$$

Therefore, $f:S\longrightarrow G$ is a $(1+c,2M)$-quasi-isometry, and Theorem B finishes the proof. □

Next we prove that the hyperbolicity is stable under significant metric changes (even with non-quasi-isometric deformations), as long as the topology is preserved. The following definition describes the outstanding parameters involved in the kind of deformations studied in Theorem 3.8.

Definition 3.8. Given a positive constant l, we say that two Riemann surfaces S and S' (with or without boundary) have *similar l-skeletons* if there are decompositions $\{Y_n\}_{n\in N} \cup \{F_m\}_{m\in M}$ of S and $\{Y'_n\}_{n\in N} \cup \{F'_m\}_{m\in M'}$ of S', with associated l-skeletons G and G' respectively, such that:

(a) $Y_n \cap Y_m = \cup_{i\in I_{nm}} \gamma^i_{nm}$ (with $\gamma^i_{nm} = \gamma^i_{mn}$) and $Y'_n \cap Y'_m = \cup_{i\in I_{nm}} \eta^i_{nm}$ (with $\eta^i_{nm} = \eta^i_{mn}$).

(b) If we define $c_1 := \inf\{L_S(\gamma) : \gamma \subseteq (\cup_n \partial Y_n) \setminus \partial S$ and $S \setminus \gamma$ is connected$\}$ and $c'_1 := \inf\{L_{S'}(\eta) : \eta \subseteq (\cup_n \partial Y'_n) \setminus \partial S'$ and $S' \setminus \eta$ is connected$\}$, then $c_1 = 0$ if and only if $c'_1 = 0$.

(c) If we define $c_2 := \sup\{L_S(\gamma) : \gamma \subseteq (\cup_m \partial F_m) \cup \partial S, \gamma \subseteq \partial Y_n$ and $S \setminus Y_n$ is connected$\}$ and $c'_2 := \sup\{L_{S'}(\eta) : \eta \subseteq (\cup_m \partial F'_m) \cup \partial S', \eta \subseteq \partial Y'_n$ and $S' \setminus Y'_n$ is connected$\}$, then $c_2 = \infty$ if and only if $c'_2 = \infty$.

Theorem 3.8. *Let us consider two non-exceptional Riemann surfaces S and S' (with or without boundary) with similar l-skeletons. Then S is hyperbolic if and only if S' is hyperbolic. Furthermore, if S is δ-hyperbolic, then S' is δ'-hyperbolic, with δ' a constant which only depends on δ, l, c_j and c'_j $(j = 1, 2)$.*

Proof. Without loss of generality, we can assume that S and S' do not have funnels, by Theorem D. If $c_1 = 0$, then there exist geodesics γ^i_{nm} (which do not disconnect S), with lengths $l^i_{nm} \to 0$; then Theorem 3.3 in [PRT2] gives that S is not hyperbolic (since $c'_1 = 0$, we also have that S' is not hyperbolic).

If $c_2 = \infty$, then there exist generalized Y-pieces Y_n (which do not disconnect S), with $l^{i_1}_{nm_1}, l^{i_2}_{nm_2} \le l$ and $d_{Y_n}(\gamma^{i_1}_{nm_1}, \gamma^{i_2}_{nm_2}) \to \infty$; then Theorem 2.2 in [PRT2] gives that S is not hyperbolic (since $c'_2 = \infty$, we also have that S' is not hyperbolic).

Let us assume now that $c_1, c'_1 > 0$ and $c_2, c'_2 < \infty$. First, we prove the result if $S \setminus \gamma$ and $S' \setminus \eta$ are connected for every $\gamma \subseteq \cup_n \partial Y_n$ and $\eta \subseteq \cup_n \partial Y'_n$. If G and G' are the l-skeletons of S and S' respectively, Theorem 3.7 gives that there exist two surjective $(1+c, 2M)$-quasi-isometries $f : S \longrightarrow G$ and $f' : S' \longrightarrow G'$, where M and c only depend on l. By Theorem B, we only need to prove that if G is δ_0-hyperbolic, then G' is δ'_0-hyperbolic, with δ'_0 a constant which only depends on δ_0, l, c_j and c'_j (since S and S' play symmetric roles).

We say that an edge e in a graph is a *leaf* if a vertex of e has degree one. Now, let us consider the graph G_0 (respectively G'_0) obtained by removing from G (respectively G') its set of leaves. Let us remark that $\delta(G_0) = \delta(G)$ and $\delta(G'_0) = \delta(G')$.

We define a function $F : G_0 \longrightarrow G'_0$, in the following way:

Let us consider a generalized Y-piece Y_n such that its three curves in ∂Y_n have length less or equal than l. If $f(\gamma^i_{nm}) = v^i_{nm}$ and $f'(\eta^i_{nm}) = w^i_{nm}$, F is a dilatation of $[v_n, v^i_{nm}] \in G_0$ onto $[w_n, w^i_{nm}] \in G'_0$.

If Y_n has a curve in ∂Y_n with length greater than l, and the other boundary curves are $\gamma^{i_1}_{nm_1}, \gamma^{i_2}_{nm_2}$, F is a dilatation of $[v^{i_1}_{nm_1}, v^{i_2}_{nm_2}] \in G_0$ onto $[w^{i_1}_{nm_1}, w^{i_2}_{nm_2}] \in G'_0$.

Let us prove now that F is a bijective $(\alpha, 0)$-quasi-isometry, beeing α a constant which only depends on l, c_j and c'_j: Since $c_1, c'_1 > 0$ and $c_2, c'_2 < \infty$, and there are no leaves either in G_0 or in G'_0, then $c_1 \le l^i_{nm} := L_S(\gamma^i_{nm}) \le \max\{l, c_2\}$ and $c'_1 \le L^i_{nm} := L_{S'}(\eta^i_{nm}) \le \max\{l, c'_2\}$, if $v^i_{nm} \in G_0$ (recall that $v^i_{nm} \in G_0$ if and only if $w^i_{nm} \in G'_0$). Hence $L_G([v_n, v^i_{nm}]) = \operatorname{Arccosh}\big(\coth(l^i_{nm}/2)\big)$ and $L_{G'}([w_n, w^i_{nm}]) = \operatorname{Arccosh}\big(\coth(L^i_{nm}/2)\big)$ are comparable with constants which only depend on l, c_j

and c_j' (if every curve in ∂Y_n has length less or equal than l). The same is true for $L_G([v^{i_1}_{nm_1}, v^{i_2}_{nm_2}])$ and $L_{G'}([w^{i_1}_{nm_1}, w^{i_2}_{nm_2}])$ (if Y_n has a curve in ∂Y_n with length greater than l).

Let us assume now that there are geodesics γ^i_{nm} such that $S \setminus \gamma^i_{nm}$ is not connected (then we also have $S' \setminus \eta^i_{nm}$ is not connected). In this case, we can decompose $S = \cup_r S_r$ (respectively $S' = \cup_r S'_r$), where $\{S_r\}_r$ are the connected components which we obtain by splitting S for every simple closed geodesic $\gamma \subseteq \cup_n \partial Y_n$ with $S \setminus \gamma$ not connected; then any simple closed geodesic $\gamma \subseteq (\cup_n \partial Y_n) \cap S_r$ (respectively $\eta \subseteq (\cup_n \partial Y'_n) \cap S'_r$) does not disconnect S_r (respectively S'_r). Consequently $\{S_r\}_r$ is a $(k_1, k_2, 0)$-tree-decomposition of S with $A_n = \Lambda$, $k_1 = \frac{l}{2}$ and $k_2 = \frac{l}{4\,\text{Arccosh}(\coth(l/2))}$ (see Definition 2.8; in order to estimate $d_{S_r}(\gamma^i_{nm}, \gamma^u_{st})$ we can use the Collar Lemma, since $\gamma^i_{nm}, \gamma^u_{st}$ are disjoint simple closed geodesics). Similarly $\{S'_r\}_r$ is also a $(k_1, k_2, 0)$-tree-decomposition of S'.

Then Theorem C gives that if S is δ-hyperbolic then S_r is δ_1-hyperbolic for every r, with δ_1 a constant which only depends on δ and l. Now, we can apply the last argument to S_r and S'_r, and therefore S'_r is δ_2-hyperbolic with δ_2 a constant which only depends on δ, l, c_j and c'_j. Finally, we use again Theorem C to assure that S' is δ'-hyperbolic, with δ' only depending on δ, l, c_j and c'_j.

This finishes the proof because of the symmetry between S and S'. □

Remark. After the proofs of theorems 3.7 and 3.8, it is clear that the conclusions of these theorems also hold if we define the l-skeleton of a Y-piece in the following similar way:

If $L(\gamma_i) = l_i \le l$, for $i = 1, 2, 3$, we define $L([v, v_i]) := \log(1 + l_i^{-1})$ for $i = 1, 2, 3$. If $l_1, l_2 \le l$ and $l_3 > l$, we take $L([v_1, v_2]) := \log(1 + l_1^{-1}) + \log(1 + l_2^{-1}) + l_3$.

As a consequence of Theorem 3.8, we obtain that hyperbolicity is a property stable under "twisting", for Riemann surfaces with l-skeletons (the result is not true without this hypothesis).

Notice that if two non-exceptional Riemann surfaces have the same l-skeleton G, they have the same decomposition $\{Y_n\}_{n \in N} \cup \{F_m\}_{m \in M}$, and they are obtained by gluing the pieces following the same design of G, after applying a twist to the curves in $\cup_n \partial Y_n$.

Corollary 3.3. *Let us consider two non-exceptional Riemann surfaces S, S' (with or without boundary), with the same l-skeleton. If S is δ-hyperbolic, then S' is δ'-hyperbolic, with δ' a constant which only depends on δ and l.*

§4. The hyperbolicity in the Classification Theory of Riemann surfaces

We prove in this section that there is no inclusion relationship between hyperbolic Riemann surfaces and the usual classes of Riemann surfaces, such as O_G (surfaces without Green's function), surfaces with hyperbolic isoperimetric inequality, or the complements of these classes (even in the case of plane domains). This fact is important since it points out that hyperbolic Riemann surfaces are a class completely different from the more usual classes of Riemann surfaces. This makes the study of hyperbolic Riemann surfaces more complicated and interesting.

We have the same result for the classes O_{HP} (surfaces without non-constant harmonic positive functions), O_{HB} (surfaces without non-constant harmonic bounded functions), and O_{HD} (surfaces without non-constant harmonic functions with finite Dirichlet integral), since in the case of plane domains (and even for surfaces with finite genus) we have $O_G = O_{HP} = O_{HB} = O_{HD}$ (see e.g. [AS, p. 208]).

Let us denote by H the class of hyperbolic Riemann surfaces and by HII the class of Riemann surfaces with hyperbolic isoperimetric inequality. We are going to provide some plane domains which can be used so as to prove the following facts:

(a) HII is not contained in H.

(b) $(HII)^c$ is not contained in H.

(c) H is not contained in HII.

(d) H is not contained in $(HII)^c$.

(e) O_G is not contained in H.

(f) $(O_G)^c$ is not contained in H.

(g) H is not contained in O_G.

(h) H is not contained in $(O_G)^c$.

Notice that we only need to prove (a), (d), (e) and (h), since $HII \subset (O_G)^c$.

Definition 4.1. We can define the *modulus* of an annulus $\{r < |z - z_0| < R\}$ as R/r. We say that the annulus $A = \{r < |z - z_0| < R\}$ *separates the boundary* of the plane domain Ω if $A \subseteq \Omega$ and each connected component of the complement of A meets $\partial\Omega$. We say that a plane domain Ω is *modulated* if there is an upper bound for the modulus of every annulus which separates the boundary of Ω.

Any modulated plane domain belongs to HII (see e.g. [FR1, Theorem 3]). Let us recall that a plane domain Ω belongs to O_G if and only if $\partial\Omega$ has zero logarithmic capacity (see e.g. [AS, p. 249]). Hence, the plane domains in O_G (and in O_{HP}, O_{HB}, O_{HD}) can be characterized by the size of their boundaries. However, when we deal with hyperbolic plane domains the situation is much more sophisticated.

(a) There exists a plane domain $\Omega_1 \in HII \cap H^c$.

For each natural number n we consider a real number $a_0 := 0$ (if $n = 0$) and $a_n \in (0, 1/2)$ (if $n > 0$), the sets $E_n := \{|z - 2n| = 1/2\}$, $F_n := \{|z - 2n| = 1/2 + 1/(n + 2), |\operatorname{Im} z| \ge a_n, \operatorname{Re} z \le 2n\}$ and $G_n := \{|z - 2n| = 1/2 + 1/(n + 2), |\operatorname{Im} z| \ge a_{n+1}, \operatorname{Re} z \ge 2n\}$. Let us denote by H_n^+ the segment contained in $\{\operatorname{Im} z = a_{n+1}\}$ joining the point $G_n \cap \{\operatorname{Im} z = a_{n+1}\}$ with $F_{n+1} \cap \{\operatorname{Im} z = a_{n+1}\}$; we denote by H_n^- the conjugated of H_n^+.

We define Ω_1 as the unique plane domain whose boundary is equal to $\cup_n(E_n \cup F_n \cup G_n \cup H_n^+ \cup H_n^-)$.

We have that $\Omega_1 \in HII$ since it is a modulated domain.

If we denote by γ_n the simple closed geodesic in Ω_1 freely homotopic to the ideal boundary E_n, we choose the sequence $\{a_n\}_n$ small enough to guarantee that $\gamma_n \cap \{\operatorname{Im} z \ge 0\}$ is a geodesic in Ω_1. Then, for each n, we can choose a geodesic triangle contained in γ_n with thin constant greater or equal than $L(\gamma_n)/4$. Since $L(\gamma_n) \longrightarrow \infty$, we deduce that Ω_1 is not hyperbolic.

(d) The unit disk is a plane domain contained in $H \cap HII$.

(e) There exists a plane domain $\Omega_2 \in O_G \cap H^c$.

For each positive integer n we consider isometric generalized Y-pieces $\{Y_n^r\}_{r=1}^{2N_n}$ with two boundary geodesics of length n and a puncture. Let us denote by Z_n the union of $\{Y_n^r\}_{r=1}^{2N_n}$ identifying the boundary geodesics (we join Y_n^r with Y_n^{r-1} and Y_n^{r+1}, if $1 < r < 2N_n$); we paste these pieces without "twist", i.e. if α_n^r is the geodesic joining the two boundary geodesics of Y_n^r, we identify a point of α_n^r with a point of α_n^{r+1}.

We denote by Y_1 the generalized Y-piece with two punctures and a boundary geodesic of length 1; for each positive integer $n > 1$ we consider a generalized Y-piece Y_n with a puncture and two boundary geodesics of length n and $n-1$.

Let us define Ω_2 as the union of $\{Z_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ identifying boundary geodesics of equal length (we join Z_n with Y_n and Y_{n+1} without "twist").

We denote by β_n the geodesic $\beta_n := \cup_{n=1}^{2N_n} \alpha_n$ joining the two boundary geodesics of Z_n.

Let us consider the "central" geodesic of Z_n, $\gamma_n := Y_n^{N_n} \cap Y_n^{N_n+1}$. The symmetry of Z_n guarantee that there is a geodesic bigon (see the remark after Definition 2.3) in Ω_2 contained in γ_n: we choose as vertices $u_n := \beta_n \cap \gamma_n = \alpha_n^{N_n} \cap \alpha_n^{N_n+1}$ and $v_n \in \gamma_n$ with $d_{\gamma_n}(u_n, v_n) = n/2$. Let us choose N_n large enough in order to have $d_{Z_n}(\gamma_n, \partial Z_n) \geq n$; this inequality gives that this bigon has sharp thin-constant equal to $n/4$. Hence, Ω_2 is not hyperbolic.

It is not very difficult to see that Ω_2 is in O_G: Let us consider the simple closed geodesic σ_n that joins Y_n with Z_n, Φ_n^r the family of curves joining the two simple closed geodesics in Y_n^r, and Γ_n the family of curves joining σ_1 with σ_{n+1} in $\cup_{j=1}^{n-1}(Z_j \cup Y_{j+1})$. In order to see that $\Omega_2 \in O_G$, it is sufficient to see that $\lim_{n\to\infty} \Lambda(\Gamma_n) = \infty$, by [AS, p. 229], where $\Lambda(\Gamma_n)$ denotes the extremal length of Γ_n (see [AS, pp. 220-223] for the definition and properties of extremal length). Since $\Lambda(\Phi_j^r)$ does not depend on r, the second theorem in [AS, p. 222] gives that $\Lambda(\Gamma_n) \geq \sum_{j=1}^{n-1} 2N_j \Lambda(\Phi_j^r)$. If we choose $N_j \geq 1/\Lambda(\Phi_j^r)$, we obtain $\lim_{n\to\infty} \Lambda(\Gamma_n) \geq \lim_{n\to\infty}(2n-2) = \infty$, and consequently $\Omega_2 \in O_G$.

(h) The twice puncture plane $\mathbf{C} \setminus \{0, 1\}$ (the generalized Y-piece with three punctures) is a plane domain contained in $H \cap O_G$. It is hyperbolic by Proposition 3.1, since it is a generalized Y-piece, and it is in O_G since a finite number of points has zero logarithmic capacity.

References.

[AS] Ahlfors, L., Sario, L., *Riemann surfaces.* Princeton Univ. Press, Princeton, New Jersey, 1960.

[APR] Alvarez, V., Pestana, D., Rodríguez, J. M., Isoperimetric inequalities in Riemann surfaces of infinite type, *Rev. Mat. Iberoamericana* **15** (1999), 353-427.

[AR] Alvarez, V., Rodríguez, J. M., Structure theorems for Riemann and topological surfaces, *J. London Math. Soc.* **69** (2004), 153-168.

[ARY] Alvarez, V., Rodríguez, J.M., Yakubovich, V.A., Subadditivity of p-harmonic "measure" on graphs, *Michigan Math. J.* **49** (2001), 47-64.

[BB] Balogh, Z. M., Buckley, S. M., Geometric characterizations of Gromov hyperbolicity, *Invent. Math.* **153** (2003), 261-301.

[B] Beardon, A. F., *The geometry of discrete groups.* Springer-Verlag, New York, 1983.

[Be] Bers, L., An Inequality for Riemann Surfaces. Differential Geometry and Complex Analysis. H. E. Rauch Memorial Volume. Springer-Verlag. 1985.

[BHK] Bonk, M., Heinonen, J., Koskela, P., Uniformizing Gromov hyperbolic spaces. Astérisque No. 270 (2001).

[Bu] Buser, P., *Geometry and Spectra of Compact Riemann Surfaces.* Birkhäuser, Boston, 1992.

[CFPR] Cantón, A., Fernández, J. L., Pestana, D., Rodríguez, J. M., On harmonic functions on trees, *Potential Anal.* **15** (2001), 199-244.

[C] Chavel, I., *Eigenvalues in Riemannian Geometry.* Academic Press, New York, 1984.

[F] Fenchel, W., *Elementary Geometry in Hyperbolic Space.* Walter de Gruyter, Berlin-New York, 1989.

[FM] Fernández, J. L., Melián, M. V., Escaping geodesics of Riemannian surfaces, *Acta Math.* **187** (2001), 213-236.

[FR1] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces. Bass Riemann surfaces. *Ann. Acad. Sci. Fenn. Series AI* **15** (1990), 165-183.

[FR2] Fernández, J. L., Rodríguez, J. M., Area growth and Green's function of Riemann surfaces, *Arkiv Mat.* **30** (1992), 83-92.

[GH] Ghys, E., de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics, Volume 83. Birkhäuser. 1990.

[G1] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[G2] Gromov, M. (with appendices by M. Katz, P. Pansu and S. Semmes), Metric Structures for Riemannian and Non-Riemannian Spaces. Progress in Mathematics, vol. 152. Birkhäuser, 1999.

[HS] Holopainen, I., Soardi, P. M., p-harmonic functions on graphs and manifolds, *Manuscripta Math.* **94** (1997), 95-110.

[K1] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391-413.

[K2] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227-238.

[K3] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122-137.

[P] Paulin, F., On the critical exponent of a discrete group of hyperbolic isometries, *Diff. Geom. Appl.* **7** (1997), 231-236.

[PRT1] Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123-149.

[PRT2] Portilla, A., Rodríguez, J. M., Tourís, E., The topology of balls and Gromov hyperbolicity of Riemann surfaces. *Diff. Geom. Appl.* In press.

[R] Randol, B., Cylinders in Riemann surfaces. *Comment. Math. Helv.* **54** (1979), 1-5.

[Ra] Ratcliffe, J.G., *Foundations of Hyperbolic Manifolds.* Springer-Verlag, New York, 1994.

[R1] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publicacions Matemàtiques* **38** (1994), 243-253.

[R2] Rodríguez, J. M., Two remarks on Riemann surfaces, *Publicacions Matemàtiques* **38** (1994), 463-477.

[RT1] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004), 53-84.

[RT2] Rodríguez, J. M., Tourís, E., A new characterization of Gromov hyperbolicity for Riemann surfaces. Preprint.

[S] Shimizu, H., On discontinuous groups operating on the product of upper half-planes. *Ann. Math.* **77** (1963), 33-71.

[So] Soardi, P. M., Rough isometries and Dirichlet finite harmonic functions on graphs, *Proc. Am. Math. Soc.* **119** (1993), 1239-1248.