A NEW CHARACTERIZATION OF GROMOV HYPERBOLICITY FOR NEGATIVELY CURVED SURFACES

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Abstract

In this paper we show that to check Gromov hyperbolicity of any surface of constant negative curvature, or, Riemann surface, we only need to verify the Rips condition on a very small class of triangles, namely, those obtained by marking three points in a simple closed geodesic. This result is, in fact, a new characterization of Gromov hyperbolicity for Riemann surfaces.

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§1. Introduction

To understand the connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [ARY], [CFPR], [FR2], [HS], [K1], [K2], [K3], [R1], [R2], [So]) Gromov hyperbolic spaces are a useful tool. Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [GH], [G1], [G2] and the references therein).

A geodesic metric space is called hyperbolic (in the Gromov sense) if it satisfies the "Rips condition": there is an upper bound of the distance of every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.3).

But, it is not easy to determine if a given space is Gromov hyperbolic or not. One interesting instance is that of a Riemann surface endowed with the Poincaré metric. With that metric structure a Riemann surface is negatively curved, but not all Riemann surfaces are Gromov hyperbolic, since topological obstacles can impede it: for instance, the two-dimensional jungle-gym (a $\mathbf{Z}^2$-covering of a torus with genus two) is not hyperbolic.

We are interested in studying when Riemann surfaces equipped with their Poincaré metric are Gromov hyperbolic. The following theorem is the main result of this paper, which is a new characterization of Gromov hyperbolicity for Riemann surfaces (see Theorem 5.1):

A Riemann surface S is hyperbolic if and only if the c_0-triangles contained in simple closed geodesics of S satisfy the Rips condition. By a c_0-*triangle* we mean a triangle with continuous injective $(1, c_0)$-quasigeodesic sides, and we require that the vertices and the edges of such triangles are contained in simple closed geodesics of S.

In general, one has to verify the Rips condition for all triangles. Our result is that for Riemann surfaces you only have to verify it for a smaller class of triangles.

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Furthermore this theorem provides a bound for the hyperbolicity constant: if the triangles contained in simple closed geodesics satisfy the Rips condition with constant δ_0, then every geodesic triangle satisfy it with constant $\delta = \max\{11, \delta_0 + 6\}$.

A connected question with our main theorem is when a Euclidean bounded domain with its quasi-hyperbolic metric is Gromov hyperbolic. (Let us recall that in the case of modulated plane domains, quasihyperbolic and Poincaré metrics are equivalent.) Recently, Balogh and Buckley [BB] have made significant progress in this question (see also [BHK] and the references therein).

Theorem 5.1 provides good bounds for the hyperbolicity constants of some classical surfaces such as the punctured disk, the annuli, the Y-pieces and plane domains of finite type (see Lemma 5.4 and corollaries 5.1, 5.2 and 5.3).

It can also be successfully used as a powerful tool to study hyperbolicity of a class of Riemann surfaces by means of its decomposition in Y-pieces and funnels (see Theorem 5.3).

As a consequence of these results, we have obtained interesting examples of hyperbolic Riemann surfaces (see Theorem 5.3 and corollaries 5.1, 5.2 and 5.3), and a result that allows us a better understanding of the role that funnels and half-disks (see Definition 5.4) play in the study of hyperbolicity (see Theorem 5.2). Theorem 5.2 is a useful result which has several applications in [RT2] and [PRT2]. One can think of the following as a natural first result in order to study hyperbolicity: if a Riemann surface has a sequence of funnels $\{F_n\}_n$ with $\lim_{n\to\infty} L(\partial F_n) = \infty$, then it is not hyperbolic. In [RT2] we prove that this reasonable result is false indeed, and an important tool in the proof is Theorem 5.2.

Notations. We denote by X or X_n geodesic metric spaces. By d_X, L_X and B_X we shall denote, respectively, the distance, the length and the balls in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X.

We denote by R, S or S_0 Riemann surfaces. We assume that the metric defined on these surfaces is the Poincaré metric, unless the contrary is specified.

If Ω is a plane domain, we shall denote by λ_Ω the conformal density of the Poincaré metric in Ω, *i.e.* the function such that $ds = \lambda_\Omega(z)|dz|$ is the Poincaré metric in Ω.

We denote by $\Re z$ and $\Im z$ the real and imaginary part of z, respectively.

Finally, we denote by l, c and c_i, positive constants which can assume different values in different theorems.

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§2. Background in Gromov spaces

In our study of hyperbolic Gromov spaces we use the notations of [GH]. We give now the basic facts about these spaces. We refer to [GH] for more background and further results.

Definition 2.1. Let us fix a point w in a metric space (X,d). We define the *Gromov product* of $x,y\in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2}\big(d(x,w)+d(y,w)-d(x,y)\big)\geq 0\,.$$

We say that the metric space (X,d) is δ-*hyperbolic* ($\delta\geq 0$) if

$$(x|z)_w \geq \min\big\{(x|y)_w,(y|z)_w\big\}-\delta\,,$$

for every $x,y,z,w\in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples: (1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic.

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k>0$, is hyperbolic.

(3) Every tree with edges of arbitrary length is 0-hyperbolic.

We refer the reader to [BHK], [GH] and [CDP] for further examples.

Definition 2.2. If $\gamma:[a,b]\longrightarrow X$ is a continuous curve in a metric space (X,d), we can define the *length* of γ as

$$L(\gamma):=\sup\Big\{\sum_{i=1}^{n} d(\gamma(t_{i-1}),\gamma(t_i)) :\, a=t_0<t_1<\cdots<t_n=b\Big\}\,.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]})=d(\gamma(t),\gamma(s))=|t-s|$ for every $s,t\in[a,b]$. We say that γ is a *local geodesic* if for every $t\in[a,b]$ there exists $\varepsilon>0$ such that the restriction of γ to $[t-\varepsilon,t+\varepsilon]\cap[a,b]$ is a geodesic. We say that X is a *geodesic metric space* if for every $x,y\in X$ there exists a geodesic joining x and y; we denote by $[x,y]$ any such geodesic (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient).

Definition 2.3. If X is a geodesic metric space and J is a polygon whose sides are $J_1,J_2,\dots,J_n$, with $J_j\subseteq X$, we say that J is δ-*thin* if for every $x\in J_i$ we have that $d(x,\cup_{j\neq i}J_j)\leq\delta$. If $x_1,x_2,x_3\in X$, a *geodesic triangle* $T=\{x_1,x_2,x_3\}$ is the union of three geodesics $J_1:=[x_1,x_2]$, $J_2:=[x_2,x_3]$ and $J_3:=[x_3,x_1]$. The space X is δ-*thin* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin.

Remark. Every geodesic quadrilateral in a δ-thin geodesic space is 2δ-thin. To see this, it is enough to divide the quadrilateral in two triangles. In general, every geodesic polygon of n sides is $(n-2)\delta$-thin. If we have a triangle with two identical vertices, we call it a "bigon"; obviously, every bigon in a δ-thin space is δ-thin.

A fundamental result is that hyperbolicity is equivalent to the Rips condition:

Theorem A. ([GH, p.41]) *Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin.*

(2) *If X is δ-thin, then it is 4δ-hyperbolic.*

We present now the class of maps which play the main role in the theory.

Definition 2.4. A function between two metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there are constants $a \geq 1$, $b \geq 0$ with

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

Such a function is called an (a, b)-*quasi-isometry.* An (a, b)-*quasigeodesic* in X is an (a, b)-quasi-isometry between an interval of $\mathbf{R}$ and X.

Let us observe that a quasi-isometry does not have to be continuous (for instance, the map $f : \mathbf{R} \longrightarrow \mathbf{Z}$ such that $f([n, n+1)) = n$ for every integer n is a $(1, 1)$-quasi-isometry).

Quasi-isometries are important since they are maps which preserve hyperbolicity:

Theorem B. ([GH, p.88]) *Let us consider an (a, b)-quasi-isometry between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which depends only on δ, a and b.*

Definition 2.5. Let us consider $H > 0$, a metric space X, and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the H-*neighbourhood* of Y in X. The *Hausdorff distance* of Y to Z is defined by $\mathcal{H}(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z),\ Z \subseteq V_H(Y)\}$.

The following is a beautiful and useful result:

Theorem C. ([GH, p.87]) *For each $\delta, b \geq 0$ and $a \geq 1$, there exists a constant $H = H(\delta, a, b)$ with the following property:*

Let us consider a δ-hyperbolic geodesic metric space X and an (a, b)-quasigeodesic g joining x and y. If γ is a geodesic joining x and y, then $\mathcal{H}(g, \gamma) \leq H$.

This property is known as geodesic stability. Mario Bonk has proved that, in fact, geodesic stability is equivalent to hyperbolicity [Bo].

Along this paper we will work with topological subspaces of a geodesic metric space X. There is a natural way to define a distance in these spaces:

Definition 2.6. If X_0 is a subset connected by rectifiable paths of a metric space (X, d), then we associate to it the *inner* or *intrinsic distance*

$$d_{X_0}(x, y) := d_X|_{X_0}(x, y) := \inf\big\{L(\gamma) :\ \gamma \subset X_0 \text{ is a continuous curve joining } x \text{ and } y\big\} \geq d_X(x, y)\,.$$

§3. Results in metric spaces

We are interested in studying when non-exceptional Riemann surfaces equipped with their Poincaré metric are Gromov hyperbolic. However, we have proved several results on hyperbolicity for general metric spaces, which are interesting by themselves and have consequences for Riemann surfaces (see Section 5).

We want to remark that almost every constant appearing in the results of this paper depends just on a small number of parameters (in fact, we give explicit expressions for them). This is a common place in the theory of hyperbolic spaces (see e.g. theorems A, B and C) and is also typical of surfaces with curvature -1 (see the Collar Lemma in [R] and [S], and Theorem 3.1 in [PRT2]).

We need some technical results which we collect in the following lemmas.

Lemma 3.1. *Let us consider a geodesic metric space X and $\varepsilon > 0$. If γ is a continuous curve joining $x, y \in X$ with $L_X(\gamma) \le d_X(x,y) + \varepsilon$, then γ is a $(1,\varepsilon)$-quasigeodesic with its arc-length parametrization.*

Proof. Let us consider γ with its arc-length parametrization $\gamma : [0,l] \longrightarrow X$. Since γ is continuous, it is clear that $d_X(\gamma(t),\gamma(s)) \le L_X(\gamma([t,s])) = |t-s|$. Let us show now $|t-s| \le d_X(\gamma(t),\gamma(s)) + \varepsilon$. We assume that there are $0 \le t,s \le l$ with $|t-s| > d_X(\gamma(t),\gamma(s)) + \varepsilon$. Without loss of generality we can assume $t < s$. We define a curve γ_0 as a concatenation of three curves: $\gamma([0,t])$, a geodesic η connecting $\gamma(t)$ with $\gamma(s)$, and $\gamma([s,l])$. Since γ_0 is a continuous curve connecting x with y, we have that

$$\begin{aligned} d_X(x,y) &\le L_X(\gamma_0) = L_X(\gamma) - L_X(\gamma([t,s])) + d_X(\gamma(t),\gamma(s)) \\ &= L_X(\gamma) - |t-s| + d_X(\gamma(t),\gamma(s)) \\ &< L_X(\gamma) - \varepsilon \le d_X(x,y)\,, \end{aligned}$$

which is a contradiction. □

Corollary 3.1. *Let us consider a geodesic metric space X and $\varepsilon > 0$. If γ is a continuous curve with $L_X(\gamma) \le \varepsilon$, then γ is a $(1,\varepsilon)$-quasigeodesic with its arc-length parametrization.*

Lemma 3.2. *Let us consider a metric space X with a closed geodesic g of length l. If γ is a continuous injective $(1,c)$-quasigeodesic in X with its arc-length parametrization, and it is contained in g, then $L(\gamma) \le (l+c)/2$.*

Remarks. 1. It is clear that every closed geodesic is only a local geodesic, but not a geodesic (see Definition 2.2); however, since there is no possible confusion, we call it closed geodesic instead of closed local geodesic.

2. If γ is a geodesic, it is clear that $L(\gamma) \le l/2$; Lemma 3.2 generalizes this fact.

Proof. Let us consider γ with its arc-length parametrization $\gamma : [0,l_0] \longrightarrow X$. Assume that $l_0 > (l+c)/2$; then $l - l_0 < l_0 - c$. Observe that $d(\gamma(0),\gamma(l_0)) \le l - l_0$, since $g \setminus \gamma$ is a continuous curve of length $l - l_0$ joining $\gamma(0)$ and $\gamma(l_0)$ (γ is continuous and injective). Hence, we have that $l_0 - c \le d(\gamma(0), \gamma(l_0)) \le l - l_0 < l_0 - c$, which is a contradiction. $\square$

Lemma 3.3. *Every (a,b)-quasigeodesic triangle in a δ-hyperbolic geodesic metric space X is $(4\delta + 2H(\delta,a,b))$-thin, where H is the constant in Theorem C.*

Proof. Given an (a,b)-quasigeodesic triangle in X of sides q_1, q_2, q_3, Theorem C gives that there exist geodesics g_1, g_2, g_3, such that g_i has the same end points as q_i and $\mathcal{H}(g_i, q_i) \le H = H(\delta, a, b)$. If $\{i,j,k\}$ is any permutation of $\{1,2,3\}$, and $x \in q_i$, then there is a point $x' \in g_i$ with $d(x,x') \le H$. Since X is 4δ-thin, we can find $y' \in g_j \cup g_k$ with $d(y',x') \le 4\delta$. We also have a point $y \in q_j \cup q_k$ with $d(y',y) \le H$. Consequently $d(x, q_j \cup q_k) \le d(x,y) \le 4\delta + 2H$. $\square$

The following result is a modification of Theorem 2.4 in [RT1] (using a completely different line of argument). Furthermore, this proof gives an explicit expression for the constants involved. It can be applied to the study of hyperbolicity of Riemann surfaces (see Theorem 5.3). In order to state it, we need one definition.

Definition 3.1. We say that the closed geodesic metric spaces $\{X_n\}_{n\in\Lambda}$ are a (c_1,c_2)-*regular decomposition* of the geodesic metric space X if they verify the following conditions:

(a) $X = \cup_{n\in\Lambda} X_n$ and $X_n \cap X_m = \eta_{nm}$, where for each $n \in \Lambda$, $\{\eta_{nm}\}_{m\in\Lambda\setminus\{n\}}$ are pairwise disjoint closed subsets of X_n ($\eta_{nm} = \varnothing$ is allowed); furthermore any geodesic in X with finite length meets at most a finite number of η_{nm}'s.

(b) For any $n, m \in \Lambda$, $\operatorname{diam}_{X_n}(\eta_{nm}) \le c_1$ and if $\eta_{nm} \ne \varnothing$, then $X \setminus \eta_{nm}$ is not connected and a, b are in different connected components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$.

(c) For each $n \in \Lambda$ there exist disjoint sets $A_n, B_n \subseteq \Lambda$, verifying the following properties: if $m \notin A_n \cup B_n$, then $\eta_{nm} = \varnothing$; $\operatorname{diam}_{X_n}(\cup_{m\in A_n} \eta_{nm}) \le c_2$; and every geodesic joining two points in X_n cannot escape from X_n across a η_{nm} with $m \in B_n$,

Remarks. 1. The sets Λ, A_n and B_n do not need to be countable.

2. The hypothesis on $X \setminus \eta_{nm}$ guarantees that the graph $R = (V,E)$ constructed in the following way is a tree: $V = \cup_{n\in\Lambda}\{v_n\}$ and $[v_n, v_m] \in E$ if and only if $\eta_{nm} \ne \varnothing$.

3. We can think that the hypothesis "a geodesic joining two points in X_n cannot escape from X_n across a η_{nm} with $m \in B_n$", is very restrictive; however, Lemma 5.5 below gives a very simple condition which allows one to assure this hypothesis.

4. If X is a Riemann surface, $\{X_n\}_{n\in\Lambda}$ are bordered Riemann surfaces and $\eta_{nm} \subset \partial X_n \cap \partial X_m$, condition "$a, b$ are in different components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$" in ($b$), is a consequence of "$X \setminus \eta_{nm}$ is not connected".

5. We wish to emphasize that condition $\operatorname{diam}_{X_n}(\eta_{nm}) \le c_1$ is not very restrictive: if the space is "wide" at every point (in the sense of long injectivity radius, as in the case of simply connected spaces) or "narrow" at every point (as in the case of trees), it is easier to study its hyperbolicity; if we can found narrow parts (as η_{nm}) and wide parts, the problem is more difficult and interesting.

Theorem 3.1. *Let us consider a (c_1, c_2)-regular decomposition $\{X_n\}_{n\in\Lambda}$ of the geodesic metric space X. If there exists a constant δ_0 such that X_n is δ_0-thin for every $n \in \Lambda$, then X is δ-thin with $\delta = 20\delta_0 + \max\{c_1 + c_2/2, c_2\}$.*

Proof. Let us consider a geodesic triangle $T = \{a, b, c\}$ in X. If $T \subseteq X_n$ for some n, then T is δ_0-thin, by hypothesis. We assume now that T intersects several X_n's. We intend to study T in each of those X_n's separately.

Let us take $z \in T$. If z belongs to two sides of T, there is nothing to prove; if z only belongs to one side of T, we denote by A the union of the sides of T which do not intersect z.

Let us fix $n \in \Lambda$. We assume first that the three sides of T intersect X_n.

We construct a geodesic polygon P_n in X_n modifying $T \cap X_n$ in the following way: Let us consider a side γ_i $(i = 1, 2, 3)$ of T. If $\gamma_i \subseteq X_n$, we define $g_i := \gamma_i$. If γ_i is not contained in X_n, then we consider $\gamma_i : [0, l] \longrightarrow X$. Let us define

$$t_1^i := \min\{0 \le t \le l : \gamma_i(t) \in X_n\}, \qquad t_4^i := \max\{0 \le t \le l : \gamma_i(t) \in X_n\}.$$

If $\gamma_i([t_1^i, t_4^i]) \subseteq X_n$, we consider $g_i := \gamma_i([t_1^i, t_4^i])$. In other case, we define

$$t_2^i := \min\{0 \le t \le l : \gamma_i(t) \in \cup_{m\in A_n}\eta_{nm}\}, \qquad t_3^i := \max\{0 \le t \le l : \gamma_i(t) \in \cup_{m\in A_n}\eta_{nm}\},$$

and $g_i := \gamma_i([t_1^i, t_2^i]) \cup [\gamma_i(t_2^i), \gamma_i(t_3^i)] \cup \gamma_i([t_3^i, t_4^i])$, where we choose a geodesic $[\gamma_i(t_2^i), \gamma_i(t_3^i)]$ in X_n. This minimum and this maximum exist since γ_i is a continuous function in a compact interval and $\gamma_i \cap (\cup_{m\in A_n}\eta_{nm})$ is a compact set: each η_{nm} is a closed set and γ_i meets at most a finite number of η_{nm}'s.

It is possible that $g_1 \cup g_2 \cup g_3$ is not a polygon, since there can exist gaps between two g_i's. Since $\operatorname{diam}_{X_n}(\eta_{nm}) \le c_1$ and $X \setminus \eta_{nm}$ is not connected for any $m \in \Lambda$, we can find three geodesics h_1, h_2, h_3 in X_n of length less or equal than c_1 such that $g_1 \cup h_1 \cup g_2 \cup h_2 \cup g_3 \cup h_3$ is a geodesic polygon P_n in X_n (some h_i can be a point). It is clear that P_n has at most 12 sides, and then it is $10\delta_0$-thin.

Without loss of generality we can assume that $z \in g_1$. In order to simplify the notation, we define $x_j := \gamma_1(t_j^1)$ for $1 \le j \le 4$.

If $g_1 := \gamma_1([t_1^1, t_4^1]) = [x_1, x_4]$, then there exists $w' \in P_n \setminus \operatorname{int} g_1$ with $d_{X_n}(z, w') \le 10\delta_0$, where $\operatorname{int} g_1$ denotes g_1 without its end points. If $w' \in A$, then $d_X(z, A) \le 10\delta_0$; if $w' \notin A$, then there exists $w \in P_n \cap A$ with $d_{X_n}(w, w') \le \max\{c_1, c_2/2\}$, and therefore $d_X(z, A) \le 10\delta_0 + \max\{c_1, c_2/2\}$.

Let us assume now that $g_1 := [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_4]$. Recall that $[x_1, x_2] \cup [x_3, x_4] \subseteq \gamma_i \subseteq T$, and $L_X([x_2, x_3]) \le c_2$. We denote by $a_1 \in [x_1, x_2]$ the point farther of x_2 such that $d_{X_n}(a_1, [x_2, x_3]) \le 10\delta_0$; in a similar way, we define $a_2 \in [x_3, x_4]$ as the point farther of x_3 such that $d_{X_n}(a_2, [x_2, x_3]) \le 10\delta_0$; then $d_{X_n}(a_1, a_2) \le 20\delta_0 + c_2$.

Let us consider $b_1 \in [a_1, x_1]$ the point farther of a_1 such that $d_{X_n}(b_1, [x_3, x_4]) \le 10\delta_0$ (if this b_1 does not exist, we take $b_1 := a_1$) and $b_2 \in [a_2, x_4]$ the point farther of a_2 such that $d_{X_n}(b_2, [x_1, x_2]) \le 10\delta_0$ (if this b_2 does not exist, we take $b_2 := a_2$). If $b_1 \ne a_1$, then $d_X(b_1, x_3) = L_X([b_1, x_3]) = d_X(b_1, [x_3, x_4]) \le 10\delta_0$; in a similar way, if $b_2 \ne a_2$, then $d_X(b_2, x_2) \le 10\delta_0$. We consider now the next four possibilities:

If $b_1 = a_1$ and $b_2 = a_2$, we have seen that $d_X(b_1, b_2) \le 20\delta_0 + c_2$.

If $b_1 \ne a_1$ and $b_2 \ne a_2$, then $d_X(b_1, b_2) \le L_X([b_1, x_3]) + L_X([b_2, x_2]) \le 20\delta_0$.

If $b_1 \ne a_1$ and $b_2 = a_2$, then there is a point $z_0 \in [x_2, x_3]$ with $d_X(b_2, z_0) \le 10\delta_0$; since there is some x_j $(j = 2, 3)$ with $d_X(x_j, z_0) \le c_2/2$, we obtain that $d_X(b_1, b_2) \le d_X(b_1, x_j) + d_X(x_j, z_0) + d_X(z_0, b_2) \le 20\delta_0 + c_2/2$.

If $b_1 = a_1$ and $b_2 \ne a_2$, we obtain in a similar way that $d_X(b_1, b_2) \le 20\delta_0 + c_2/2$.

Therefore, in the four situations we have $d_X(b_1, b_2) \le 20\delta_0 + c_2$. If $z \in [b_1, x_1] \cup [b_2, x_4]$, then $d_X(z, A) \le 10\delta_0 + \max\{c_1, c_2/2\}$. If $z \in [b_1, b_2]$, we can take b_i with $d_X(z, b_i) \le 10\delta_0 + c_2/2$; since $d_X(b_i, A) \le 10\delta_0 + \max\{c_1, c_2/2\}$, we obtain $d_X(z, A) \le 20\delta_0 + \max\{c_1 + c_2/2, c_2\}$. Let us remark that if we consider $z' \in [b_1, b_2]$, with $z' \notin X_n$, the same argument gives $d_X(z', A) \le 20\delta_0 + \max\{c_1 + c_2/2, c_2\}$.

Let us assume now that only two sides of T intersect X_n. As in the previous case, we can replace each γ_i which intersect X_n by g_i. Then we can construct in a similar way a geodesic polygon P_n in X_n with at most 8 sides, which is $6\delta_0$-thin. Hence the previous argument gives the same result with even sharper constant.

Finally, let us assume that only one side of T intersects X_n. Then z belongs to some $[b_1, b_2]$, and the same inequality holds.

Consequently, X is δ-thin with $\delta := 20\delta_0 + \max\{c_1 + c_2/2, c_2\}$. □

The same proof of Theorem 3.1 gives sharper constants in some particular cases.

Corollary 3.2. *Under the hypothesis of Theorem* 3.1*, we have that:*

(1) *We can take* $\delta := \max\{2\delta_0 + c_2, 6\delta_0 + c_2/2, 3c_2/2\}$*, if* $B_n = \varnothing$ *for every* $n \in \Lambda$.

(2) *We can take* $\delta := 4\delta_0 + c_1$*, if* $A_n = \varnothing$ *for every* $n \in \Lambda$.

§4. Background in Riemann surfaces

We collect here some definitions concerning Riemann surfaces.

An *open non-exceptional* Riemann surface (or a non-exceptional Riemann surface without boundary) S is a Riemann surface whose universal covering space is the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk

$$ds = \lambda_{\mathbf{D}}(z)|dz| = \frac{2\,|dz|}{1 - |z|^2}\ ,$$

or, equivalently, the upper half plane $\mathbf{U} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, with the metric $ds = \lambda_{\mathbf{U}}(z)|dz| = |dz|/\operatorname{Im} z$. Observe that, with this definition, every compact non-exceptional Riemann surface without boundary is open. With this metric, S is a complete Riemannian manifold with constant curvature -1; therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. It is easy to study hyperbolicity of these particular cases.

It is well-known (see e.g. [An, p.100], [B, p.131], [JS, p.227], [N, p.18]) that

$$(4.1) \qquad d_{\mathbf{D}}(0,z) = \log \frac{1+|z|}{1-|z|} = 2\,\mathrm{Arctanh}\,|z|\,, \qquad \sinh^2 \frac{d_{\mathbf{U}}(z,w)}{2} = \frac{|z-w|^2}{4\,\mathrm{Im}\,z\,\mathrm{Im}\,w}\,.$$

A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S "bounded" by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ; such a collar is equal to $\{p \in S:\ d_S(p,\gamma) < d\}$, for some positive constant d. The constant d is called the *width* of the collar. The Collar Lemma [R] says that there exists a collar of γ of width d, for every $0 < d \le d_0$, where $\cosh d_0 = \coth(L_S(\gamma)/2)$, or similarly $\sinh d_0 = \mathrm{cosech}(L_S(\gamma)/2)$.

As we remarked after Lemma 3.2, every closed geodesic is a local geodesic, but not a geodesic; however, since there is no possible confusion, we call it closed geodesic instead of closed local geodesic.

A puncture in a non-exceptional Riemann surface is a doubly connected end in which we can find homotopically non-trivial curves with arbitrarily small length. A puncture is an isolated point in ∂S in the case that $S \subset \mathbf{C}$. We can think of a puncture as a boundary geodesic of zero length.

We say that S is a *bordered non-exceptional Riemann surface* (or a non-exceptional Riemann surface with boundary) if it can be obtained by deleting an open set V of an open non-exceptional Riemann surface R, with $d_S := d_R|_S$ (recall Definition 2.6). Any such surface S is a bordered orientable Riemannian manifold of dimension 2 and its Riemannian metric has constant negative curvature -1. It is not difficult to see that if any ball in R intersects at most a finite number of connected components of V, and the boundary of S is locally Lipschitz, then S is a geodesic metric space.

A *funnel* is a bordered non-exceptional Riemann surface which is topologically a cylinder and whose boundary is a simple closed geodesic. Given a positive number a, there is a unique (up to conformal mapping) funnel such that its boundary curve has length a. Every funnel is conformally equivalent, for some $\beta > 1$, to the subset $\{z \in \mathbf{C}:\ 1 \le |z| < \beta\}$ of the annulus $\{z \in \mathbf{C}:\ 1/\beta < |z| < \beta\}$.

Every doubly connected end of an open non-exceptional Riemann surface is a puncture (if there are homotopically non-trivial curves with arbitrary small length) or a funnel (in other case).

A *Y-piece* is a bordered non-exceptional Riemann surface which is conformally equivalent to a sphere minus three open disks and whose boundary curves are simple closed geodesics (and then it is triply connected). Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see e.g. [Ra, p.410]). They are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use are given in [Bu, chapter 1] and [C, chapter X.3].

A *generalized Y-piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere without n open disks and m points, with integers $n, m \ge 0$ such that $n+m = 3$, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Observe that a generalized Y-piece is topologically the union of a Y-piece and m cylinders.

§5. Results in Riemann surfaces

Although one should expect Gromov hyperbolicity in non-exceptional Riemann surfaces due to its constant curvature -1, this turns out to be untrue in general, since topological obstacles can impede it: for instance, the two-dimensional jungle-gym (a $\mathbf{Z}^2$-covering of a torus with genus two) is not hyperbolic.

In [RT2] we prove that there is no inclusion relationship between hyperbolic Riemann surfaces and the usual classes of Riemann surfaces, such as O_G, O_{HP}, O_{HB}, O_{HD}, surfaces with linear isoperimetric inequality, or the complements of these classes (even in the case of plane domains). This fact shows that the study of hyperbolic Riemann surfaces is more complicated and interesting than one might think at first sight. One can find other results on hyperbolicity of Riemann surfaces in [RT1], [RT2], [PRT1] and [PRT2].

The main result in this paper is Theorem 5.1, which allows us to reduce drastically the triangles in which we have to check the Rips condition in Riemann surfaces. In [FR1, Lemma 1.2] it is proved that in order to check the linear isoperimetric inequality in a Riemann surface it is enough to deal with domains whose boundary is a finite union of simple closed geodesics; this fact is interesting by itself and has important consequences, as the stability of linear isoperimetric inequality under quasiconformal maps (see [FR1, Theorem 1]), and the equivalence of linear isoperimetric inequality and Poincaré's inequality (see [FR1, Theorem 2]). Here we prove that if the triangles contained in simple closed geodesics of a Riemann surface S satisfy the Rips condition, then S is hyperbolic (see Theorem 5.1).

The results in this section give many examples of hyperbolic Riemann surfaces, and provide criteria in order to decide whether a Riemann surface is hyperbolic or not.

Definition 5.1. By a *simply connected polygon* in a non-exceptional Riemann surface we mean a polygon isometric to a polygon in the Poincaré disk. We say that two sides in a polygon *are disjoint* if their interiors are disjoint.

We collect in the following lemmas some technical results which we need in order to clarify the proof of Theorem 5.1.

Lemma 5.1. *Let us consider a simply connected locally geodesic quadrilateral in a non-exceptional Riemann surface S with pairwise disjoint sides A, C, B and η, of lengths a, c, b and l_0, respectively. Let us assume also that C meets orthogonally the sides A and B. We have that:*

(1) $\cosh l_0 = \cosh a \cosh b \cosh c - \sinh a \sinh b$.

(2) *Let us fix $c_0 > 0$. If $c \geq c_0$, then $a + b + c - c_1 \leq l_0$, with $c_1 := 3\log 2 - 2\log(1 - e^{-c_0})$.*

(3) *If $c_0 := \log(5 + 2\sqrt{6}\,)$, then $c_1 = c_0$.*

Remark. It is clear by the triangle inequality that $l_0 \leq a + b + c$.

Proof. Since the quadrilateral is simply connected, we can assume that it is contained in the unit disk **D**. Part (1) can be found in [F, p.88].

We show part (2). Let us observe that the function $f(t) := 2(\cosh t - 1)e^{-t} = (1-e^{-t})^2$ is increasing in $[0,\infty)$. Then $f(c) \geq f(c_0) = (1-e^{-c_0})^2$, for $c \geq c_0$, i.e. $\cosh c - 1 \geq \frac{1}{2} f(c_0)e^c$. Consequently, if $c \geq c_0$,

$$e^{l_0} \geq \cosh l_0 = \cosh a \cosh b \cosh c - \sinh a \sinh b \geq (\cosh c - 1)\cosh a \cosh b \geq \frac{1}{8}\, f(c_0)e^{a+b+c},$$

and we deduce $l_0 \geq a+b+c+\log \frac{1}{8}(1-e^{-c_0})^2 = a+b+c-c_1$.

A direct computation gives (3). □

Lemma 5.2. *Let us consider a simply connected self-intersecting locally geodesic quadrilateral in a non-exceptional Riemann surface S with sides A, C, B and η, of lengths a, c, b and l_0, respectively. Let us assume also that C meets orthogonally the sides A and B. If η and C are not disjoint, then we have that:*

(1) $\cosh l_0 = \cosh a \cosh b \cosh c + \sinh a \sinh b$.

(2) $a+b+c-3\log 2 \leq l_0$.

Proof. Since the quadrilateral is simply connected, we can assume that it is contained in the unit disk **D**. Part (1) can be found in [F, p.89].

We show part (2). The inequality is a consequence of

$$e^{l_0} \geq \cosh l_0 = \cosh a \cosh b \cosh c + \sinh a \sinh b \geq \cosh a \cosh b \cosh c \geq \frac{1}{8}\, e^{a+b+c}. \quad \square$$

Lemma 5.3. *Let us consider $c_0 > 0$ and a simply connected locally geodesic quadrilateral Q in a non-exceptional Riemann surface S with pairwise disjoint sides A, C, B and η, of lengths a, c, b and l_0, respectively. Let us assume also that C meets orthogonally the sides A and B. If $c \geq c_0$, then we have that*

$$d(z,\eta) < c_2 := \operatorname{Arcsinh} \frac{e^{c_0}+1}{e^{c_0}-1} = \operatorname{Arcsinh}\Big(\operatorname{cotanh} \frac{c_0}{2}\Big),$$

for every $z \in A \cup B \cup C$.

Proof. Since Q is simply connected, we can assume that it is contained in the upper half plane **U**. Without loss of generality we can assume that Q is the quadrilateral with vertices $i, it, ie^{-i\theta}, ie^{-i\phi}t$, with $0 < \theta, \phi < \pi/2$ and $t = e^c \geq e^{c_0}$.

It is clear that $d(z,\eta) \leq \max\{d(i,\eta), d(it,\eta)\}$. Without loss of generality we can assume that $d(i,\eta) = \max\{d(i,\eta), d(it,\eta)\}$ (if it is not the case, we can change the roles of θ and ϕ).

It is obvious that $d(i,\eta)$ is less than the distance of i to the geodesic η_0 joining 1 and t.

The Möbius transformation $Tz := (z-t)/(z-1)$ maps η_0 onto the imaginary half-axis I, and $Ti = (t+1+i(t-1))/2$. A computation gives (see e.g. [B, p.162])

$$d(z,\eta) < d(i,\eta_0) = d(Ti, I) = \operatorname{Arcsinh} \frac{t+1}{t-1} \leq \operatorname{Arcsinh} \frac{e^{c_0}+1}{e^{c_0}-1}\,,$$

since $t \geq e^{c_0}$. □

Lemma 5.4. *Let us consider the annulus A_l such that its simple closed geodesic has length l; we denote by A_0 the limit case $A_0 := \mathbf{D}^* := \mathbf{D} \setminus \{0\}$. Then A_l is $\delta(l)$-thin for any $l \geq 0$, where $\delta(l) := \max\big\{l + 2\log(1+\sqrt{2}\,), d(l) + 3\log(1+\sqrt{2}\,), d(l)/2 + 6\log(1+\sqrt{2}\,)\big\}$, with $d(l) :=$ $\operatorname{Arcsinh}\big(\sinh(l/2)\operatorname{cotanh}(l/6)\big)$ if $l > 0$ and $d(0) := \operatorname{Arcsinh} 3$. In particular, $\delta(0) := \frac{1}{2}\operatorname{Arcsinh} 3 +$ $6\log(1+\sqrt{2}\,) < 6.1975$.*

Proof. Let us consider a geodesic triangle $T = \{a, b, c\}$ in A_l. If T is homotopic to a point, then it is the boundary of a simply connected closed set E, and consequently E, with its intrinsic distance, is isometric to some subset of $\mathbf{D}$; this implies that T is δ_0-thin, with $\delta_0 := \log(1+\sqrt{2}\,)$, since $\mathbf{D}$ is δ_0-thin (see [An, p.130]). Then we can assume that T is freely homotopic to the simple closed geodesic g, if $l > 0$, or to the puncture, if $l = 0$.

Let us assume first that $l > 0$ and $T \cap g \neq \varnothing$. We denote by F^1 and F^2 the two funnels whose union is A_l (the closures of the two connected components of $A_l \setminus g$).

Let us observe that the funnels are geodesically convex (every geodesic connecting two points of the funnel is contained in the funnel). Hence, without loss of generality we can assume that a is in the interior of F^1 and b, c are in the interior of F^2 (the case in which there is some vertex in g is easier). We define $B := [a, b] \cap g$ and $C := [a, c] \cap g$. There are two local geodesics $g_1, g_2 \subset g$ joining B and C; let us observe that $L_{A_l}(g_i) \leq l$.

Let us consider the triangle $T_1 = \{a, B, C\}$, where we choose as $[B, C]$ the local geodesic $g_i \subset g$ such that $[a, B] \cup [B, C] \cup [C, a]$ is homotopic to a point; since T_1 is homotopic to a point, the above argument implies that T_1 is δ_0-thin. Given $x \in [a, B]$ there is some $y \in [a, C] \cup [B, C]$ with $d_{A_l}(x, y) \leq \delta_0$; if $y \in [a, C]$, then $d_{A_l}(x, [a, C]) \leq \delta_0$; if $y \in [B, C]$, we have $d_{A_l}(x, [a, C]) \leq d_{A_l}(x, y) + d_{A_l}(y, C) \leq \delta_0 + l$. If $x \in [a, C]$, we obtain a similar result.

Let us consider the quadrilateral $Q_1 = \{b, c, C, B\}$, where we choose as $[B, C]$ the local geodesic $g_i \subset g$ such that $[b, c] \cup [c, C] \cup [C, B] \cup [B, b]$ is homotopic to a point; since Q_1 is homotopic to a point, the above argument implies that Q_1 is $2\delta_0$-thin. In a similar way to the case of T_1, given any point in $T \cap Q_1$ there is a point $y \in T \cap Q_1$ (in other side of T) with $d_{A_l}(x, y) \leq 2\delta_0 + l$. Then T is $(2\delta_0 + l)$-thin.

Let us assume now that $l > 0$ and $T \cap g = \varnothing$. Next, we find an upper bound for $d_{A_l}(T, g)$. Given a point w of T, we denote by w_0 the point in g with $d_{A_l}(w, w_0) = d_{A_l}(w, g)$. If $T = \{a, b, c\}$, we have that $d_{A_l}(a_0, b_0) + d_{A_l}(b_0, c_0) + d_{A_l}(c_0, a_0) = l$. Hence, without loss of generality we can assume that $d_{A_l}(a_0, b_0) \geq l/3$. Let us consider the point $x \in [a, b]$ with $d_{A_l}(x, g) = d_{A_l}([a, b], g)$. We consider first the case $x \in (a, b)$. We can assume that $t := d_{A_l}(a_0, x_0) \geq l/6$.

We consider now the geodesic quadrilateral $Q := \{a, a_0, x_0, x\}$ with three right angles (known as Lambert quadrilateral). If $s := d_{A_l}(x_0, x)$ and ϕ is the angle of $[a, a_0]$ and $[a, x]$ in a, the trigonometric formulas give $\sinh s \sinh t = \cos\phi$ (see e.g. [B, p.157], [C, p.263]). Then

$$\sinh s = \frac{\cos\phi}{\sinh t} < \frac{1}{\sinh t} \leq \frac{1}{\sinh(l/6)}\,.$$

Therefore, we have that

$$d_{A_l}(T,g) < \operatorname{Arcsinh} \frac{1}{\sinh(l/6)}\,. \tag{5.1}$$

If $x = a$ or $x = b$, a similar argument with $t := d_{A_l}(a_0, b_0)$ gives $\sinh s \sinh t < 1$, and we obtain $\sinh s < 1/\sinh(l/3)$, which also implies (5.1).

Without loss of generality we can assume that $d_{A_l}(T,g) = d_{A_l}(x,g) = d_{A_l}(x,x_0) = s$. Let us consider the local geodesic g_x starting and finishing in x, which is freely homotopic to g. We consider first the case $x \in (a,b)$. We denote by $2d_x$ the length of g_x and by y the point in g_x at distance d_x of x.

We consider the geodesic quadrilateral $R := \{x, x_0, y_0, y\}$ with three right angles. Since $d_{A_l}(x_0,y_0) = l/2$, the trigonometric formulas give (see e.g. [F, p.88])

$$\begin{aligned}\sinh d_x &= \sinh(l/2)\cosh s = \sinh(l/2)\sqrt{1+\sinh^2 s}\\ &< \sinh(l/2)\sqrt{1+\operatorname{cosech}^2(l/6)} = \sinh(l/2)\operatorname{cotanh}(l/6)\,.\end{aligned}$$

Let us assume now that $l = 0$, i.e. that we deal with the case $A_0 = \mathbf{D}^*$; then T is freely homotopic to the puncture. We consider the universal covering map $\pi : \mathbf{U} \longrightarrow \mathbf{D}^*$, given by $\pi(z) = \exp(2\pi i z)$. It is clear that π maps bijectively $U_0 := \{z \in \mathbf{U} :\, 0 \le \Re z < 1\})$ in $\mathbf{D}^*$. Without loss of generality we can assume that $\pi(z_1) = a$, $\pi(z_2) = b$ and $\pi(z_3) = c$, with $\Re z_1 = 0$ and $1/3 \le \Re z_2 \le \Re z_3 < 1$. Since $\Re(z_2 - z_1) \ge 1/3$, there exists a point $z \in [z_1, z_2]$ with $\Im z > 1/6$; then $\max\big\{\Im z :\, \pi(z) \in T\big\} > 1/6$. We denote by z_0 a point of U_0 in which this maximum is attained.

Let us consider the local geodesic g_0 in $\mathbf{D}^*$ starting and finishing in $\pi(z_0)$, which is freely homotopic to the puncture; if we denote by $2d_{\pi(z_0)}$ the length of g_0, (4.1) gives that

$$\sinh^2 d_{\pi(z_0)} = \sinh^2 \frac{d_{\mathbf{U}}(z_0, 1+z_0)}{2} < \sinh^2 \frac{d_{\mathbf{U}}(i/6, 1+i/6)}{2} = 9\,,$$

and consequently $d_{\pi(z_0)} < \operatorname{Arcsinh} 3$.

Recall that $d(l) := \operatorname{Arcsinh}\big(\sinh(l/2)\operatorname{cotanh}(l/6)\big)$ if $l > 0$ and $d(0) := \operatorname{Arcsinh} 3$. Then there exists a point $p \in T$ such that the local geodesic g_p in A_l starting and finishing in p, which is freely homotopic to g or to the puncture, has length $2d_p < 2d(l)$.

Let us assume first that p is not a vertex of T; without loss of generality we can assume also that $p \in [a,c]$. Since g_p is freely homotopic to T, we have a geodesic pentagon $P' := \{a', b', c', p_1', p_2'\}$ in $\mathbf{D}$, which is isometric to the pentagon P made of $[a,b]$, $[b,c]$, $[c,p]$, g_p and $[p,a]$, if we identify p_1' with p_2' (we have chosen P' such that $d_{\mathbf{D}}(a',b') = d_{A_l}(a,b)$, $d_{\mathbf{D}}(b',c') = d_{A_l}(b,c)$, $d_{\mathbf{D}}(c',p_1') = d_{A_l}(c,p)$, $d_{\mathbf{D}}(p_1',p_2') = L_{A_l}(g_p)$ and $d_{\mathbf{D}}(p_2',a') = d_{A_l}(p,a)$).

It is clear that if x', y', are the corresponding points in P' to the points $x, y \in P$, we have $d_{A_l}(x,y) \le d_{\mathbf{D}}(x',y')$.

Now we use a similar argument to the one in the proof of Theorem 3.1.

Since P' is a geodesic pentagon in $\mathbf{D}$, we have that it is $3\delta_0$-thin. Let us consider the point α_1' in the oriented geodesic $[p_1', c']$, defined by $\alpha_1' := \max\{z \in [p_1', c'] :\, d_{\mathbf{D}}(z, [p_1', p_2']) \le 3\delta_0\}$, and the point α_2' in the oriented geodesic $[p_2', a']$, defined by $\alpha_2' := \max\{z \in [p_2', a'] :\, d_{\mathbf{D}}(z, [p_1', p_2']) \le 3\delta_0\}$.

If α_j is the corresponding point in P to α'_j, we have that $L_{A_l}([\alpha_1,\alpha_2]) = d_{A_l}(\alpha_1,\alpha_2) \le 6\delta_0 + d(l)$, since $d_{A_l}(\alpha_j, g_p) \le 3\delta_0$ and $\operatorname{diam}_{A_l}(g_p) \le d_p < d(l)$.

We define now $\beta'_1 := \max\big(\{\alpha'_1\} \cup \{z \in [p'_1, c'] : d_{\mathbf{D}}(z, [p'_2, a']) \le 3\delta_0\}\big)$, $\beta'_2 := \max\big(\{\alpha'_2\} \cup \{z \in [p'_2, a'] : d_{\mathbf{D}}(z, [p'_1, c']) \le 3\delta_0\}\big)$. Let us denote by β_j the corresponding point in P to β'_j.

If $\beta_1 \ne \alpha_1$, then $d_{A_l}(\beta_1, p) = L_{A_l}([\beta_1, p]) = d_{A_l}(\beta_1, [p, a]) \le 3\delta_0$; in a similar way, if $\beta_2 \ne \alpha_2$, then $d_{A_l}(\beta_2, p) \le 3\delta_0$. We consider now the next four possibilities:

If $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$, we have seen that $d_{A_l}(\beta_1, \beta_2) \le 6\delta_0 + d(l)$.

If $\beta_1 \ne \alpha_1$ and $\beta_2 \ne \alpha_2$, then $d_{A_l}(\beta_1, \beta_2) \le d_{A_l}(\beta_1, p) + d_{A_l}(p, \beta_2) \le 6\delta_0$.

If $\beta_1 \ne \alpha_1$ and $\beta_2 = \alpha_2$, then there is a point $z_0 \in [p'_1, p'_2]$ with $d_{\mathbf{D}}(\beta'_2, z_0) \le 3\delta_0$; since there is some p'_i with $d_{\mathbf{D}}(p'_i, z_0) \le d(l)$, we obtain that $d_{A_l}(\beta_1, \beta_2) \le d_{\mathbf{D}}(\beta'_1, \beta'_2) \le d_{\mathbf{D}}(\beta'_1, p'_i) + d_{\mathbf{D}}(p'_i, z_0) + d_{\mathbf{D}}(z_0, \beta'_2) \le 6\delta_0 + d(l)$.

If $\beta_1 = \alpha_1$ and $\beta_2 \ne \alpha_2$, we obtain in a similar way that $d_{A_l}(\beta_1, \beta_2) \le 6\delta_0 + d(l)$.

Therefore, in the four situations we have $d_{A_l}(\beta_1, \beta_2) \le 6\delta_0 + d(l)$. If $x \in [\beta_1, c] \cup [\beta_2, a]$, then $d_{A_l}(x, [a, b] \cup [b, c]) \le 3\delta_0$. If $x \in [\beta_1, \beta_2]$, we can take β_i with $d_{A_l}(x, \beta_i) \le 3\delta_0 + d(l)/2$; since $d_{A_l}(\beta_i, [a, b] \cup [b, c]) \le 3\delta_0$, we obtain $d_{A_l}(x, [a, b] \cup [b, c]) \le 6\delta_0 + d(l)/2$.

If $x \in [a, b]$, there exists a point $y' \in P' \setminus (a', b')$ with $d_{\mathbf{D}}(x', y') \le 3\delta_0$. If $y' \notin [p'_1, p'_2]$, then $d_{A_l}(x, [b, c] \cup [c, a]) \le 3\delta_0$. If $y' \in [p'_1, p'_2]$, there is p'_i with $d_{\mathbf{D}}(y', p'_i) \le d(l)$, and hence $d_{\mathbf{D}}(x', p'_i) \le 3\delta_0 + d(l)$. Since $p \in [a, c]$, we have that $d_{A_l}(x, [b, c] \cup [c, a]) \le 3\delta_0 + d(l)$. A similar result is true if $x \in [b, c]$. These facts give that T is $\max\{3\delta_0 + d(l), 6\delta_0 + d(l)/2\}$-thin.

If p is a vertex of T, the proof is easier since we construct a quadrilateral instead of a pentagon, and we do not need to split a side of T. This finishes the proof of Lemma 5.4. □

The following is the main result of this paper; it allows one to check the Rips condition only for triangles contained in simple closed geodesics. We would like to remark the simplification that Theorem 5.1 means in the applications: Let us consider an annulus A with simple closed geodesic γ. A generic triangle T in A is determined by the coordinates of three points, i.e., by six real coordinates; however, a generic triangle T_0 in the simple closed geodesic γ is determined by three real coordinates. Therefore Theorem 5.1 is a remarkable improvement of Rips condition in the context of Riemann surfaces.

Definition 5.2. By a c_0-*triangle* we mean a triangle with continuous injective $(1, c_0)$-quasigeodesic sides, with its arc-length parametrization.

We define the constants

$$c_0 := \log(5 + 2\sqrt{6}\,) < 2.2925\,, \qquad K := 2\log(1 + \sqrt{2}\,) + \log(5 + 2\sqrt{6}\,) + \log\frac{\sqrt{6} + \sqrt{10}}{2} < 5.0869\,.$$

Theorem 5.1. *Let us consider a non-exceptional Riemann surface S (with or without boundary); if S has boundary, we also require that ∂S is the union of local geodesics (closed or non-closed). Then S is hyperbolic if and only if every c_0-triangle contained in a simple closed geodesic in S is δ_0-thin.*

Furthermore, if every c_0-triangle contained in a simple closed geodesic in S is δ_0-thin, then S is δ-thin, with $\delta = \max\{\delta(4c_0), \delta_0 + K\}$, where $\delta(t)$ is the constant in Lemma 5.4 (*it verifies* $\delta(4c_0) < 10.9325$).

Remarks. 1. Although one can think of quasigeodesic triangles as an artificial technical device, the example after the proof of Theorem 5.1 shows that they are essential.

2. Even though this theorem reduces drastically the triangles in which we have to check the Rips condition, we must "pay" for it by working with quasigeodesic triangles; however the situation is advantageous since the class of quasigeodesics that we need is very restrictive: recall that we only consider continuous injective $(1, c_0)$-quasigeodesics, and Lemma 3.2 gives a bound of its length which will be good enough in the applications (see Theorem 5.3 and corollaries 5.2 and 5.3).

Proof. The heart of the proof of Theorem 5.1 is to relate any geodesic triangle T in S with a c_0-triangle contained in a simple closed geodesic γ in S. In some way, we can consider T and γ as "subsets" of the annulus A_l (with $l := L_S(\gamma)$). The geodesic triangles in the simple closed geodesic of A_l are $(l/4)$-thin, and this value is sharp (it is enough to consider a triangle with sides of lengths $l/4$, $l/4$ and $l/2$). However, the problem in a general Riemann surface is more difficult (and recall that we can find simple closed geodesics arbitrarily long). Therefore, if l is big we need a narrow metric relationship between T and γ.

If S is hyperbolic, Lemma 3.3 guarantees that every c_0-triangle in S is δ_0-thin.

Let us assume that every c_0-triangle contained in a simple closed geodesic in S is δ_0-thin. First, we want to remark that if S has boundary, the hypothesis on ∂S gives that it is the union of pairwise disjoint simple local geodesics (closed or non-closed).

In this case, we can construct an open non-exceptional Riemann surface R by pasting to S a funnel in each simple closed geodesic, and a half-disk in each non-closed simple geodesic.

Since S is geodesically convex in R (every geodesic connecting two points of S is contained in S), then $d_R(z, w) = d_S(z, w)$ for every $z, w \in S$, and any simple closed geodesic in R is contained in S.

Let us consider a geodesic triangle T in S. By Lemma 2.1 in [RT1], we can assume that T is a simple closed curve.

We have three possibilities: T is homotopic to a point, T is homotopic to a puncture, or T is freely homotopic to a simple closed geodesic in S. This is well known if S has no boundary; if S has boundary, it is enough to apply the result to R, since R has not additional topological obstacles (the fundamental groups of S and R are isomorphic).

If T is homotopic to a point, then it is the boundary of a simply connected closed set E, and consequently E, with its intrinsic distance, is isometric to some subset of $\mathbf{D}$; this implies that T is $\log(1 + \sqrt{2})$-thin, since $\mathbf{D}$ is $\log(1 + \sqrt{2})$-thin (see [An, p.130]).

If T is homotopic to a puncture, then it is the boundary of a closed doubly connected set, which is, with its intrinsic distance, isometric to some subset of $\mathbf{D}^* := \mathbf{D} \setminus \{0\}$; this implies that T is $\delta(0)$-thin, with $\delta(0)$ the constant in Lemma 5.4. Since every geodesic triangle in $\mathbf{D}$ is isometric to some geodesic triangle in $\mathbf{D}^*$, we have that $\log(1+\sqrt{2}\,) \le \delta(0)$.

In other case, T is freely homotopic to a simple closed geodesic γ in S.

If $L(\gamma) < 4c_0$, let us consider the annulus $A_{L(\gamma)}$ with a simple closed geodesic g of length $L(\gamma)$. We have that $A_{L(\gamma)}$ is $\delta(L(\gamma))$-thin, with $\delta(L(\gamma))$ the constant in Lemma 5.4. Since

$$d = d(l) = \operatorname{Arcsinh}\Big(\frac{\sinh(l/2)}{\sinh(l/6)}\cosh(l/6)\Big),$$

if $l > 0$ and $d(0) = \lim_{l\to 0} d(l)$, we have that $d = d(l)$ is an increasing function for $l \ge 0$; then we also have that $\delta(0) \le \delta(L(\gamma)) < \delta(4c_0)$, with

$$\begin{aligned}\delta(4c_0) = \max\Big\{ & 4c_0 + 2\log(1+\sqrt{2}\,), \operatorname{Arcsinh}\big(\sinh(2c_0)\operatorname{cotanh}(2c_0/3)\big) + 3\log(1+\sqrt{2}\,),\\ & \frac{1}{2}\operatorname{Arcsinh}\big(\sinh(2c_0)\operatorname{cotanh}(2c_0/3)\big) + 6\log(1+\sqrt{2}\,)\Big\}\\ = {} & 4c_0 + 2\log(1+\sqrt{2}\,) < 10.9325\,.\end{aligned}$$

In this case, the closed set in S bounded by T and γ is, with its intrinsic distance, isometric to a set in $A_{L(\gamma)}$, bounded by g and a triangle T_0. These facts give that T is $\delta(4c_0)$-thin.

We consider now the case $L(\gamma) \ge 4c_0$.

First, we assume that $\gamma \cap T = \varnothing$. If η is a side of T, we associate to it two curves η', η'', in the following way. We consider a simply connected locally geodesic quadrilateral Q in S with pairwise disjoint sides A, C, B and η, of lengths a, c, b and l_0, respectively, with the following conditions: (i) $C \subset \gamma$, (ii) C meets orthogonally the sides A and B. Q is uniquely determined by these conditions. If $c \ge c_0$, the arc $\eta' := A \cup C \cup B$ is a continuous injective $(1, c_0)$-quasigeodesic with its arc-length parametrization by lemmas 3.1 and 5.1. If $c < c_0$, we take $\eta' := \eta$, which is a geodesic. (Observe that we have $c < c_0$ for at most one side of T, since $L(\gamma) \ge 4c_0$; in other case, T would not be a geodesic triangle.) In both cases, we define $\eta'' := C \subset \gamma$. We have that η'' is always a continuous injective $(1, c_0)$-quasigeodesic with its arc-length parametrization: this is clear if $c \ge c_0$ (since $\eta'' \subset \eta'$), and it is a consequence of Corollary 3.1 if $c < c_0$.

If T is the union of the geodesics η_1, η_2, η_3, we consider the $(1, c_0)$-quasigeodesic triangle T' defined as the union of the $(1, c_0)$-quasigeodesics $\eta_1', \eta_2', \eta_3'$. We consider also the $(1, c_0)$-quasigeodesic triangle $T'' \subset \gamma$ defined as the union of the $(1, c_0)$-quasigeodesics $\eta_1'', \eta_2'', \eta_3''$.

By hypothesis, T'' is δ_0-thin. We prove now that T' is δ_1-thin, with $\delta_1 := \max\{\delta_0, 2\log(1+\sqrt{2}\,)\} + c_0$.

If $\eta_i' \ne \eta_i$, for $i = 1, 2, 3$, then T' is δ_0-thin, since every point in $T' \setminus T''$ belongs to two sides of T'.

If it is not the case, there is only one i with $\eta_i' = \eta_i$; we can assume $\eta_1' = \eta_1$. Let us consider the quadrilateral Q_1 with sides A_1, C_1, B_1 and η_1; we have that $L(C_1) < c_0$. Since Q_1 is simply connected, it is isometric to a quadrilateral in $\mathbf{D}$ which is $2\log(1+\sqrt{2}\,)$-thin.

Then for each $z \in \eta_1' = \eta_1$, there exists $w \in A_1 \cup C_1 \cup B_1$ with $d(z, w) \le 2\log(1+\sqrt{2}\,)$. If $w \in A_1 \cup B_1$, then $d(z, \eta_2' \cup \eta_3') \le 2\log(1+\sqrt{2}\,)$. If $w \in C_1$, then there exists $w' \in A_1 \cup B_1$ with $d(w, w') \le c_0$ (since $L(C_1) < c_0$), and we have $d(z, \eta_2' \cup \eta_3') \le 2\log(1+\sqrt{2}\,) + c_0$.

If $z \in \eta_2'$, we consider three cases. If $z \in \eta_2' \cap \gamma = \eta_2''$, then $d(z, \eta_1' \cup \eta_3') \le d(z, \eta_3'') \le d(z, \eta_1'' \cup \eta_3'') + c_0 \le \delta_0 + c_0$. If $z \in \eta_2' \cap \eta_3'$, then $d(z, \eta_1' \cup \eta_3') = 0$. In other case, $z \in A_1 \cup B_1$ (we can assume that $A_1 \subset \eta_2'$

and $B_1 \subset \eta_3'$); then there exists $w \in B_1 \cup C_1 \cup \eta_1$ with $d(z,w) \le 2\log(1+\sqrt{2}\,)$; since $L(C_1) < c_0$, there exists $w' \in B_1 \cup \eta_1 \subset \eta_3' \cup \eta_1'$ with $d(w,w') \le c_0$, and we have $d(z, \eta_1' \cup \eta_3') \le d(z,w') \le 2\log(1+\sqrt{2}\,)+c_0$.

Consequently, T' is δ_1-thin, with $\delta_1 := \max\{\delta_0, 2\log(1+\sqrt{2}\,)\} + c_0$.

The case $z \in \eta_3'$ is similar to $z \in \eta_2'$.

We show now that T is δ_2-thin, with

$$\delta_2 := \delta_1 + 2\log(1+\sqrt{2}\,) + c_2, \qquad \text{and} \qquad c_2 := \operatorname{Arcsinh}\Big(\operatorname{cotanh}\frac{c_0}{2}\Big) = \log\frac{\sqrt{6}+\sqrt{10}}{2}\,.$$

Let us consider $x \in T$; we can assume that $x \in \eta_1$. If $\eta_1 \ne \eta_1'$, then $\eta_1 \cup \eta_1'$ is a simply connected geodesic quadrilateral, and therefore there exists $x' \in \eta_1'$ with $d(x,x') \le 2\log(1+\sqrt{2}\,)$. If $\eta_1 = \eta_1'$, we take $x' = x$. Then there exists $y' \in \eta_2' \cup \eta_3'$ with $d(x',y') \le \delta_1$; without loss of generality we can assume that $y' \in \eta_2'$. If $\eta_2 \ne \eta_2'$, Lemma 5.3 gives that there exists $y \in \eta_2$ with $d(y,y') < c_2$. If $\eta_2 = \eta_2'$, we take $y = y'$. Consequently we have that $d(x,y) < \delta_2 := \delta_1 + 2\log(1+\sqrt{2}\,) + c_2 = \max\{\delta_0, 2\log(1+\sqrt{2}\,)\} + K$.

Therefore T is δ-thin, with $\delta := \max\{\delta(4c_0), \delta_0 + K, 2\log(1+\sqrt{2}\,) + K\} = \max\{\delta(4c_0), \delta_0 + K\}$, since $\delta(4c_0) > 10 > 2\log(1+\sqrt{2}\,) + K$.

We assume now that $\gamma \cap T \ne \varnothing$.

If $\gamma \cap T$ has only one connected component, the same argument works.

If $\gamma \cap T$ has two connected components, the argument is similar, using now Lemma 5.2 instead of Lemma 5.1. The constant in this case is smaller, since $3\log 2 < c_0$. $\square$

The following example shows that the quasigeodesic triangle T'' in the proof of Theorem 5.1 does not need to be geodesic.

Example. *There is a geodesic triangle T in a triply connected Riemann surface S_0 such that T'' is not geodesic.*

Given $x_0 < \operatorname{Arcsinh} 1$, there exists $y > 0$ with $\sinh(x_0 + y) > \cosh y$. Then $\sinh(x+y) > \cosh y$ for any $x_0 \le x < \operatorname{Arcsinh} 1$, and consequently we can choose some $x < \operatorname{Arcsinh} 1$ such that $\sinh x \sinh(x+y) > \cosh y$.

If we define $\varepsilon := \operatorname{Arcsinh}(1/\sinh x) - x > 0$, we have that $\sinh x \sinh(x+\varepsilon) = 1$. Let us consider a geodesic quadrilateral V with three right angles and an angle equal to zero, such that the two finite sides have length x and $x + \varepsilon$ (see e.g. [B, p.157], [F, p.89]). If we paste four quadrilaterals isometric to V, we obtain a generalized Y-piece Y_0 with two punctures and a simple closed geodesic γ with $L(\gamma) = 4(x+\varepsilon)$. We obtain S_0 by gluing Y_0 with a funnel F whose simple closed geodesic has length $4(x+\varepsilon)$.

Let us denote by μ_0 the geodesic in Y_0 with $L(\mu_0) = 2x$, joining γ with itself which is not homotopic to any curve contained in γ. We denote by p'', q'' the end points of μ_0. Let us consider the non bounded geodesic μ in S_0 which contains μ_0, and the two points $p, q \in \mu \cap F$ at distance y of γ.

Let us define the triangle T as the union of the two geodesics α, β in F joining p and q (in fact, T is a geodesic "bigon"). The length of the segment of μ between p and q is $2x + 2y$; by [F, p.88] we have $\sinh(L(\alpha)/2) = \sinh(x+\varepsilon)\cosh y = \cosh y/\sinh x < \sinh(x+y)$; then we obtain $L(\alpha) < 2x + 2y$, and consequently α, β are in fact geodesics in S_0. However, $T'' = \{p'', q''\}$ is contained in γ and then $L(\alpha'') = L(\beta'') = 2x + 2\varepsilon > 2x = L(\mu_0)$; hence α'', β'' are not geodesics in S_0.

From now on we will obtain several consequences of Theorem 5.1.

Corollary 5.1. *The annulus A_l such that its simple closed geodesic has length $l \geq 4c_0$ is $(l/4+K)$-thin, with $K < 5.0869$ the constant in Theorem 5.1. The same is true for each funnel of A_l.*

Remark. This bound of the hyperbolicity constant for the annulus is asymptotically sharp: we have that the best thin constant of A_l is greater than or equal to $l/4$, since we have a geodesic triangle contained in the simple closed geodesic with sides of lengths $l/2, l/4, l/4$.

Proof. Let us observe that the last part of the proof of Theorem 5.1 gives that A_l is δ_2-thin, if $l \geq 4c_0$.

In this case the hypothesis "any continuous injective $(1, c_0)$-quasigeodesic triangle contained in a simple closed geodesic in S is δ_0-thin", can be changed by "any geodesic triangle contained in the simple closed geodesic γ of A_l is δ_0-thin", since T'' is a geodesic triangle in A_l if T is a geodesic triangle in A_l. Since the sides of any geodesic triangle contained in γ have length less than or equal to $l/2$, any geodesic triangle contained in γ is δ_0-thin, with $\delta_0 = \delta_0(A_l) = l/4$. Consequently, we obtain that A_l is δ_2-thin with

$$\delta_2 = \max\left\{\frac{l}{4}, 2\log(1+\sqrt{2})\right\} + K = \frac{l}{4} + K,$$

since $l/4 \geq c_0 > 2 > 2\log(1+\sqrt{2})$. The same is true for each funnel of A_l. □

Definition 5.3. We say that a non-exceptional Riemann surface S (with or without boundary) is *of finite type* if its fundamental group is finitely generated.

Corollary 5.2. *Let us consider a non-exceptional Riemann surface S (with or without boundary) of genus 0; if S has boundary, we also require that ∂S is the union of local geodesics (closed or non-closed). If S is of finite type, then it is hyperbolic. In fact, if every simple closed geodesic γ in S verifies $L(\gamma) \leq l$, then S is δ-thin, with $\delta = \max\{\delta(4c_0), K+(l+c_0)/4\}$ and $c_0, \delta(4c_0), K$ the constants in Theorem 5.1.*

Proof. The set of simple closed geodesics in S is finite: $\{\gamma_1, \dots, \gamma_k\}$, and we have $L(\gamma_j) \leq l$. Every continuous injective $(1, c_0)$-quasigeodesic with its arc-length parametrization $g \subset \gamma_j$ verifies $L(g) \leq (l + c_0)/2$ by Lemma 3.2; hence $d(z, \partial g) \leq (l + c_0)/4$ for every $z \in g$. Then the hypothesis of Theorem 5.1 is verified with $\delta_0 := (l+c_0)/4$. Hence S is δ-thin with $\delta = \max\{\delta(4c_0), K+(l+c_0)/4\}$. □

A consequence of this corollary is the following result.

Corollary 5.3. *Every generalized Y-piece Y with $L(\gamma_i) \leq l$, where γ_i $(i = 1, 2, 3)$ are the simple closed geodesics in ∂Y, is δ-thin, with $\delta = \max\{\delta(4c_0), K + (l + c_0)/4\}$.*

Remark. As usual we see a puncture as a simple closed geodesic with zero length.

In order to prove the following result we need one definition.

Definition 5.4. A *half-disk* is a bordered non-exceptional Riemann surface which is topologically a closed half-plane and whose boundary is a simple geodesic. Every half-disk is conformally equivalent to the subset $\{z \in \mathbf{D} : \Re z \ge 0\}$ of the hyperbolic disk $\mathbf{D}$.

It is clear that a funnel contains infinitely many half-disks.

Two additional consequences which are important in the study of hyperbolicity of Riemann surfaces can be deduced from Theorem 5.1. The first one (see Theorem 5.2 below) allows us to simplify the topology: it assures that to delete funnels and half-disks does not change the hyperbolicity of a Riemann surface. Theorem 5.2 is a useful result which has several applications in [RT2] and [PRT2].

One can think of the following as a natural first result in order to study hyperbolicity: if a Riemann surface has a sequence of funnels $\{F_n\}_n$ with $\lim_{n\to\infty} L(\partial F_n) = \infty$, then it is not hyperbolic. In [RT2] we prove that this reasonable result is false indeed, and an important tool in the proof is Theorem 5.2.

Our recent research let us expect that Theorem 5.2 will be a key tool in the characterization of hyperbolic Denjoy domains.

Theorem 5.2. *Let us consider a non-exceptional Riemann surface S (with or without boundary); if S has boundary, we also require that ∂S is the union of local geodesics (closed or non-closed). Let us denote by F the union of some pairwise disjoint funnels and half-disks of S. Let S_0 be the bordered non-exceptional Riemann surface obtained by deleting from S the interior of F. Then S is hyperbolic if and only if S_0 is hyperbolic.*

Furthermore, if S is δ-thin (hyperbolic), then S_0 is δ-thin (hyperbolic); if S_0 is δ'-hyperbolic, then S is δ-thin, with $\delta = \max\{\delta(4c_0), 4\delta' + 2H(\delta', 1, c_0) + K\}$, $c_0, \delta(4c_0), K$ the constants in Theorem 5.1, and H the constant in Theorem C.

Remark. We want to emphasize that there is no hypothesis about the length of the boundary curves of the funnels. This is an important fact since there are hyperbolic Riemann surfaces containing funnels F_n with $L(\partial F_n) \longrightarrow \infty$ as $n \to \infty$ (see the examples in Section 4 of [RT2]).

Proof. Let us assume that S is δ-thin (hyperbolic). As S_0 is geodesically convex in S (every geodesic connecting two points of S_0 is contained in S_0), then $d_S(z,w) = d_{S_0}(z,w)$ for every $z, w \in S_0$. Therefore S_0 is also δ-thin (hyperbolic).

Let us assume now that S_0 is δ'-hyperbolic. By Lemma 3.3, every $(1, c_0)$-quasigeodesic triangle T in S_0 is $(4\delta' + 2H(\delta', 1, c_0))$-thin, where H is the constant in Theorem C. Let us observe that any simple closed geodesic in S is contained in S_0. Since $d_S(z,w) = d_{S_0}(z,w)$ for every $z, w \in S_0$, every $(1, c_0)$-quasigeodesic triangle in S (contained in a simple closed geodesic in S) is also a $(1, c_0)$-quasigeodesic triangle in S_0. Let us observe also that $H \ge 1 > \log(1 + \sqrt{2}\,)$. Then Theorem 5.1 gives that S is δ-thin, with $\delta = \max\{\delta(4c_0), 4\delta' + 2H(\delta', 1, c_0) + K\}$. $\square$

The following result on geodesically convex subsets of Riemann surfaces is a consequence of the Collar Lemma. It will be useful in the proof of Theorem 5.3.

Lemma 5.5. *Let us consider a non-exceptional Riemann surface S (with or without boundary), a simple closed geodesic η of S such that $S \setminus \eta$ is not connected, and the closure S_0 of a connected component of $S \setminus \eta$. We define $L_0 := 4 \operatorname{Arccosh} t_0$, where t_0 is the unique solution greater than 1 of the equation $2t^3 - 2t - 1 = 0$:*

$$t_0 := \sqrt[3]{\frac{9+\sqrt{33}}{36}} + \frac{1}{3}\sqrt[3]{\frac{36}{9+\sqrt{33}}} < 1.1915\,.$$

If $L(\eta) < L_0$, then every geodesic connecting two points of S_0 is contained in S_0, and consequently $d_S(z,w) = d_{S_0}(z,w)$ for every $z,w \in S_0$.

Proof. We assume first that S is open. If $L := L(\eta)$, then there exists a collar of η of width d_0 with $\sinh d_0 \sinh(L/2) = 1$, by the Collar Lemma (see [R]). Hence $\sinh d_0 \sinh(L_0/2) > 1$, since $L < L_0$.

We take $z,w \in S_0$. In order to prove the lemma, without loss of generality we can assume that $z,w \in \eta$; therefore $d_{S_0}(z,w) \le L/2$.

In order to obtain a contradiction, let us assume that there exists a geodesic γ in S joining z,w, and not contained in S_0; then $2d_0 \le L(\gamma) \le L/2$ and we conclude that $4d_0 \le L$. Let us observe that $2t^3 - 2t - 1 < 0$ for every $1 < t < t_0$; this implies that $2\cosh^3(L/4) - 2\cosh(L/4) < 1$, since $L < L_0$. Then we have

$$2\cosh\frac{L_0}{4}\,\sinh^2\frac{L_0}{4} < 1\,, \qquad \sinh\frac{L_0}{4}\,\sinh\frac{L_0}{2} < 1\,, \qquad \sinh\frac{L_0}{4} < \frac{1}{\sinh\frac{L_0}{2}} < \sinh d_0\,,$$

and hence we obtain $L < 4d_0$, which is a contradiction.

If S has boundary, then it is contained in a Riemann surface R and $d_S = d_R|_S$. If γ is a geodesic in S joining z,w, and not contained in S_0, then there is a geodesic in R joining z,w, which is not contained in S_0, and we have seen that it is a contradiction. □

Remark. If we follow the proof of Lemma 5.5, we can deduce that if $L(\eta) = L_0$, it is possible for γ to escape from S_0, but then $L(\eta) = 2d_0 = L/2$, and we also have $d_S(z,w) = d_{S_0}(z,w)$ for every $z,w \in S_0$.

Many Riemann surfaces can be decomposed in a union of funnels and generalized Y-pieces (see [FM, Theorem 4.1] and [AR]). The following result uses this decomposition in order to obtain hyperbolicity. A part of this result appears in [RT1], but here we give a new proof which allows one to obtain an explicit bound for the hyperbolicity constant.

Theorem 5.3. *Let us consider a non-exceptional Riemann surface S (with or without boundary) without genus (S can be viewed as a plane domain). If there is a decomposition of S in a union of funnels $\{F_m\}_{m\in M}$ and generalized Y-pieces $\{Y_n\}_{n\in N}$ with $L_S(\gamma) \le l$ for every simple closed geodesic*

$\gamma \subset (\cup_n \partial Y_n) \cup (\cup_m \partial F_m)$, then S is δ-hyperbolic, where $\delta := 20\delta_0 + l + K_0$, $\delta_0 := \max\{\delta(4c_0), K + (l+c_0)/4\}$ and

$$K_0 := \operatorname{Arccosh}\Big(\frac{\cosh(l/2)\,(1+\cosh(l/2))}{\sinh^2(L_0/2)}\Big),$$

with $c_0, \delta(4c_0), K$ the constants in Theorem 5.1 and L_0 the constant in Lemma 5.5. In fact, if $l < L_0$, we can take $\delta := 4\delta_0 + l/2$.

Proof. First of all, let us observe that Y_n is δ_0-thin, with $\delta_0 := \max\{\delta(4c_0), K + (l+c_0)/4\}$, by Corollary 5.3. Lemma 5.4 and Corollary 5.1 give that F_m is also δ_0-thin.

We denote by L_i for $i = 1, 2, 3$, the three lengths of the simple closed geodesics in ∂Y_n ($L_i = 0$ if its corresponding "geodesic" is a puncture).

If $L_0 \le L_i \le l$ for at least two geodesics, without loss of generality we can assume that $L_2 = L_{Y_n}(\eta_{nm}) \ge L_0$ and $L_3 = L_{Y_n}(\eta_{nk}) \ge L_0$. We consider the geodesic $g_{mk} \subset Y_n$, which joins η_{nm} and η_{nk}, and we put $t = L_{Y_n}(g_{mk})$. We denote by η_{nr} the geodesic in ∂Y_n with length L_1; if we consider the geodesics g_{mr}, g_{kr}, joining respectively η_{nm} and η_{nr}, and η_{nk} and η_{nr}, we can split Y_n in two isometric right-angle hexagons. By standard hyperbolic trigonometry (see e.g. [B, p.161], [Ra, p.100]), we have that

$$\cosh t = \frac{\cosh(L_1/2) + \cosh(L_2/2)\cosh(L_3/2)}{\sinh(L_2/2)\sinh(L_3/2)} \le \frac{\cosh(l/2)\,(1+\cosh(l/2))}{\sinh^2(L_0/2)}\,,$$

and therefore, $t \le K_0$.

We are going to consider different cases according to the values of L_i.

(1) If $L_0 \le L_i \le l$ for $i = 1, 2, 3$, then the distance between any two simple closed geodesics of ∂Y_n is less than or equal to K_0; therefore $\operatorname{diam}_{Y_n}(\cup_m \eta_{nm}) \le l/2 + K_0 + l/2 = l + K_0$. Then we are in the hypothesis of Theorem 3.1, with $c_2 = l + K_0$ and $B_n = \varnothing$.

(2) If $L_1 < L_0 \le L_2, L_3 \le l$, then the distance between the simple closed geodesics of ∂Y_n of length L_2, L_3, (say η_{nm}, η_{nk}) is less than or equal to K_0; then $\operatorname{diam}_{Y_n}(\eta_{nm} \cup \eta_{nk}) \le l + K_0$. Then we are in the hypothesis of Theorem 3.1, with $c_1 = l/2$, $c_2 = l + K_0$ and $A_n = \{m, k\}$.

(3) If $L_1, L_2 < L_0$, then we are in the hypothesis of Theorem 3.1, with $c_1 = l/2$ and $A_n = \varnothing$.

The case of F_m is similar to (3), with $c_1 = l/2$ and $A_n = \varnothing$.

Then, Theorem 3.1 (with $c_1 = l/2$ and $c_2 = l + K_0$) gives that S is δ-thin, with $\delta := 20\delta_0 + \max\{l/2 + (l+K_0)/2, l + K_0\} = 20\delta_0 + l + K_0$.

In fact, if $l < L_0$, we only need to consider (3), and then Corollary 3.2 gives that we can take $\delta := 4\delta_0 + l/2$. □

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