OPERATORS WITH COMMON HYPERCYCLIC SUBSPACES

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Abstract. We provide a reasonable sufficient condition for a countable family of operators to have a common hypercyclic subspace. We also extend a result of the third author and A. Montes [22], thereby obtaining a common hypercyclic subspace for certain countable families of compact perturbations of operators of norm no larger than one.

1. Introduction

It is known that for any separable infinite dimensional Banach space X, there is a continuous linear operator $T : X \to X$ which is hypercyclic; that is, there is a vector x such that the set $\{x, Tx, \ldots, T^n x, \ldots\}$ is norm dense in X ([2], [5]). Moreover, a simple Baire category argument shows that the set $HC(T)$ of such so-called *hypercyclic vectors* x is a dense G_δ in X [21], and its linear structure is well understood: While $HC(T)$ must always contain a dense subspace ([9], [20]), it not always contains a *closed* infinite dimensional one; see [16] for a complete

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characterization of when this occurs. (Throughout, when we say that $HC(T)$ contains a vector space V we mean of course that every $x \in V$ *except* $x = 0$ is hypercyclic for T.) Thus, for example it was shown that for the simplest example of a hypercyclic operator on a Banach space, namely the Rolewicz operator

$$B_2 : \ell_2 \to \ell_2,\ B_2(x_1, x_2, \cdots) = 2(x_2, x_3, \cdots),$$

$HC(B_2)$ contains an infinite dimensional vector space but that this vector space cannot be closed [25, Theorem 3.4].

In recent years, an increasing amount of attention has been paid to the set $\cap_{T\in\mathcal{F}} HC(T)$ of common hypercyclic vectors of a given family $\mathcal{F}$ of hypercyclic operators acting on the same Banach space X. Again, by a Baire category argument $\cap_{T\in\mathcal{F}} HC(T)$ is a dense subset of X whenever $\mathcal{F}$ is countable. Moreover, L. Bernal and C. Moreno [7] showed this set contains a dense vector space if we ask in addition that the members be hereditarily hypercyclic. Finally S. Grivaux proved that this additional hypothesis can be suppressed [17, Proposition 4.3].

Other important recent work is by E. Abakumov and J. Gordon [1], who showed that

$$\cap_{\{\lambda\in\mathbb{C}:\ |\lambda|>1\}} HC(B_\lambda) \neq \emptyset,$$

where B_λ is the Rolewicz operator with 2 replaced by λ. In fact it is simple to derive from this that the above intersection contains a dense subspace of ℓ_2. On the other hand, in [4] F. Bayart showed that under the assumption of a strong form of the hypercyclicity condition, uncountable collections of hypercyclic operators can indeed contain an infinite dimensional *closed* subspace of common hypercyclic vectors. Similar results were obtained by G. Costakis and M. Sambarino [13], who also provided a criterion for the existence of common hypercyclic vectors.

Our interest here will be in the following problem:

Problem 1. *Let $\mathcal{F}$ be a countable family of operators acting on a Banach space X. When does $\cap_{T\in\mathcal{F}} HC(T)$ contain a closed infinite dimensional subspace?*

In Section 2 we show that a family of operators acting on a common Banach space may fail to support a common hypercyclic subspace, even if each operator in the family has a hypercyclic subspace (Example 2.1). Moreover, if the family is uncountable it may even fail to have single common hypercyclic *vector* (Example 2.2). In Section 3 we extend a result of A. Montes [25, Theorem 2.1] by providing a reasonable sufficient condition on a countable family of hypercyclic operators acting on a Banach space to have a common infinite dimensional hypercyclic subspace (Corollary 3.5). We then apply this to extend a result of the third author and A. Montes [22], thereby obtaining a common hypercyclic subspace for certain countable families of operators of the form $T = U + K$ where $\|U\| \leq 1$ and K is compact.

2. Two Examples

Example 2.1 was provided to us by an anonymous referee. An operator T is said to be *hereditarily hypercyclic* with respect to a given increasing sequence of positive integers (n_k) provided $\{T^{n_k}\}_{k\in\mathbb{N}}$ is hereditarily universal (cf. Section 3).

Example 2.1. *Consider the operators $T_1 := (I + B_w) \oplus B_2$ and $T_2 := B_2 \oplus (I + B_w)$ acting on $\ell_2 \oplus \ell_2$, where B_2 and I are the Rolewicz' and the identity operator on ℓ_2, respectively, and B_w is the compact shift on*

ℓ_2 defined by

$$B_w e_n := \begin{cases} \frac{1}{n}\, e_{n-1} & \text{if } n \geq 2 \\ 0 & \text{if } n = 1 \end{cases} \tag{1}$$

We show next that (a) Each of T_1, T_2 has a hypercyclic subspace, and (b) T_1 and T_2 do not support a common hypercyclic subspace.

To see (a), notice that B_2 is hereditarily hypercyclic with respect to the entire sequence (n), and $I + B_w$ is hereditarily hypercyclic with respect to some sequence (n_k) [22, Lemma 4.5]. *Hence T_1 and T_2 are hereditarily hypercyclic with respect to some sequence (n_k) and by* [23, Theorem 2.1] *it suffices to verify that the essential spectrum of T_i intersects the closed unit disk $(i = 1, 2)$. Now, the sequence $(e_n \oplus 0)$ is orthonormal in $\ell_2 \oplus \ell_2$. Also, $(T_1 - I)(e_n \oplus 0) = \frac{1}{n} e_{n-1} \oplus 0$ converges to zero in norm as n tends to infinity. This means (cf.* [12] *XI 2.3) that* 1 *belongs to the essential spectrum of T_1. Similarly, 1 belongs to the essential spectrum of T_2. So each of T_1, T_2 has a hypercyclic subspace.*

To show (b) assume, to the contrary, that there exists a closed, infinite dimensional subspace Z of $\ell_2 \oplus \ell_2$ such that every non-zero vector $(x, y) \in Z$ is hypercyclic for $(I + B_w) \oplus B_2$ and $B_2 \oplus (I + B_w)$. In particular, both x and y must be hypercyclic for B_2.

Now, a simple Hilbert space argument shows that (at least) one of the coordinate projections $P_1(Z)$ and $P_2(Z)$ must contain a closed infinite dimensional subspace. Indeed, given an orthonormal sequence in Z one can find a subsequence such that its sequence (x_n) of i-th coordinate projections $(i = 1$ or $2)$ is linearly independent, bounded, and bounded away from zero. Next one can find a subsequence (x_{n_k}) of (x_n) that is equivalent as a basic sequence to an orthonormal sequence, what gives that $P_i(Z)$ contains the closed linear span of the sequence (x_{n_k}).

But this implies that B_2 has a hypercyclic subspace, which is not the case [25, Theorem 3.4]. *So T_1 and T_2 have no common hypercyclic subspace.*

Example 2.2. *Let $X = H$ be a separable, infinite-dimensional Hilbert space, and let S_H be the unit sphere of H. Let (w_n) be a sequence of positive scalars satisfying*

$$\lim_{n\to\infty} \inf_k \left(\prod_{j=1}^{n} w_{k+j}\right)^{\frac{1}{n}} \leq 1 \quad \textit{and} \quad \limsup \prod_{j=1}^{n} w_j = \infty.$$

For each h in S_H, let $\{e(h)_n : n \geq 1\}$ be a basis of H with $e(h)_1 = h$, and let $T_h : H \to H$ be the corresponding unilateral weighted backward shift defined by

$$T_h e(h)_n = \begin{cases} 0 & \textit{if } n = 1 \\ w_n\, e(h)_{n-1} & \textit{if } n \geq 2, \end{cases} \tag{2}$$

So T_h has a hypercyclic subspace [23, Corollary 2.3]. *Also, notice that $\mathcal{F} = \{T_h : \ h \in S_H\}$ satisfies that for all $0 \neq y$ in H,*

$$T_{\frac{y}{\|y\|}} y = 0.$$

That is, $\mathcal{F}$ is a family of operators, each one having a hypercyclic subspace, but such that there is no hypercyclic vector common to all members of $\mathcal{F}$.

Let us also observe that in [1] the authors mention that there is no common hypercyclic vector for the family of hypercyclic operators $\{\lambda B \oplus \delta B \ : \ |\lambda|, |\delta| > 1\}$. It is easy to see that no operator in this family admits a hypercyclic subspace.

3. A sufficient condition for a common hypercyclic subspace

We prove the main result in the more general setting of universality. Given a sequence $\mathcal{F} = \{T_j\}_{j\in\mathbb{N}}$ of bounded operators acting on a Banach space X, we say that a vector $x \in X$ is *universal* for $\mathcal{F}$ if $\{ Tx : T \in \mathcal{F} \}$ is dense in X; the set of such universal vectors is denoted $HC(\mathcal{F})$. The sequence $\mathcal{F}$ is said to be *universal* (respectively, *densely universal*) provided $HC(\mathcal{F})$ is non-empty (respectively, dense in X). $\mathcal{F}$ is called *hereditarily universal* (respectively, *hereditarily densely universal*) provided $\{T_{n_k}\}_{k\in\mathbb{N}}$ is universal (respectively, densely universal) for each increasing sequence (n_k) of positive integers. For more on the notion of universality, see [15] and [19]. A result similar to the following theorem is proved in [10] for a (single) sequence of universal operators in the context of Fréchet spaces.

Theorem 3.1. *Let $T_{n,j}$ $(n, j \in \mathbb{N})$ be bounded operators on a Banach space X, and let Y be a closed subspace of X of infinite dimension. Suppose that for each $n \in \mathbb{N}$*

i) *$\{T_{n,j}\}_{j\in\mathbb{N}}$ is hereditarily densely universal, and*
ii) *$\lim_{j\to\infty} \|T_{n,j}x\| = 0$ for each x in Y.*

Then there exists a closed, infinite dimensional subspace X_1 of X so that $\{T_{n,j}x\}_{j\in\mathbb{N}}$ is dense in X for each non-zero $x \in X_1$ and $n \in \mathbb{N}$. That is, X_1 is a universal subspace of $\{T_{n,j}\}_{j\in\mathbb{N}}$ for each $n \in \mathbb{N}$.

Lemma 3.2. *Let $T_{n,j}$ $(n, j \in \mathbb{N})$ be bounded operators on a Banach space X so that for each fixed integer n the family $\{T_{n,j}\}_{j\geq 1}$ is densely universal. Then the set $\cap_{n=1}^{\infty} HC(\{T_{n,j}\}_{j\geq 1})$ of common universal vectors to every sequence $\{T_{n,j}\}_{j\in\mathbb{N}}$ is dense in X.*

Proof. $\cap_{n=1}^{\infty} HC(\{T_{n,j}\}_{j\geq 1})$ is a countable intersection of dense G_δ subsets of the Baire space X [18, Satz 1.2.2]. □

Proof of Theorem 3.1. Reducing the subspace Y if necessary, we may assume it has a normalized Schauder basis $(e_j)_j$. Let (e_j^*) be its associated sequence in Y^* of coordinate functionals, that is, so that $e_j^*(e_i) = \delta_{i,j}$ for $i, j \in \mathbb{N}$. Let $A(Y, X)$ denote the norm closure (in $L(X, Y)$) of the subspace

$$\left\{ \sum_{j=1}^{n} x_j e_j^*(\cdot) : \ n \in \mathbb{N}, \ x_1, \ldots, x_n \in X \right\}.$$

For each T in $B(X)$, define $L_T : A(Y, X) \to A(Y, X)$ by $L_T V := TV$. We make use of the following lemma, whose proof follows that of Theorem 3.1. Analogous versions of this lemma are proved in [10] for several operator ideals (nuclear, compact, approximable), in a more general context, by using tensor product techniques developed in [24].

Lemma 3.3. *Suppose $\{T_j\}_{j\in\mathbb{N}}$ is a sequence of bounded operators on X that is hereditarily densely universal. Then $\{L_{T_{r_j}}\}_{j\geq 1}$ is a hereditarily densely universal sequence of operators on $A(Y, X)$, for some increasing sequence (r_j) of positive integers.*

Now, notice that by (i) and Lemma 3.3, for each fixed $n \in \mathbb{N}$ there exists a sequence of positive integers $(r_{n,j})_j$ so that the sequence of operators $\{L_{T_{n,r_{n,j}}}\}_{j\in\mathbb{N}}$ is hereditarily densely universal on the Banach space $A(Y, X)$. By Lemma 3.2, there exists V in $A(Y, X)$ that is universal for every sequence $\{L_{T_{n,r_{n,j}}}\}_{j\in\mathbb{N}}$, and hence universal for every $\{L_{T_{n,j}}\}_{j\in\mathbb{N}}$, too ($n \in \mathbb{N}$). Multiplying V by a non-zero scalar if necessary, we may assume that $\|V\| < \frac{1}{2}$. Consider now $X_1 := (i + V)(Y)$, where $i : Y \to X$ is the inclusion. For each $x \in Y$, $\|(i + V)x\| \geq \|x\| - \|Vx\| \geq \frac{1}{2}\|x\|$. So $i + V$ is bounded below and X_1 is closed and of infinite dimension.

Notice that $\{T_{n,j}Vx\}_{j\in\mathbb{N}}$ is dense in X for every $0 \neq x \in Y$ and every $n \in \mathbb{N}$. Indeed, given $\epsilon > 0$, let $z \in X$ be arbitrary, and let S be a finite rank operator in $A(Y, X)$ such that $Sx = z$. By Lemma 3.3, for each n there is some $T_{n,j}$ such that $||T_{n,j}V - S|| < \frac{\epsilon}{\|x\|}$. In particular, $||T_{n,j}Vx - Sx|| = ||T_{n,j}Vx - z|| < \epsilon$. The theorem now follows from condition (ii). □

Proof of Lemma 3.3. Since $\{T_j\}_{j\in\mathbb{N}}$ is hereditarily densely universal on X, it follows from [6, Theorem 2.2] that there exists a dense subspace X_0 of X, an increasing sequence of positive integers (r_j) and (possibly discontinuous) linear mappings $S_j : X_0 \to X$ $(j \in \mathbb{N})$ so that

$$T_{r_j}, S_j, \text{ and } (T_{r_j}S_j - I) \underset{j\to\infty}{\to} 0 \tag{3}$$

pointwise on X_0. Now, consider

$$A_0 := \{V \in A(Y, X) : V(Y) \subset X_0 \text{ and } \dim(V(Y)) < \infty\}.$$

Then A_0 is dense in $A(Y, X)$, and it follows from (3) that

$$L_{T_{r_j}}, L_{S_j}, \text{ and } [L_{T_{r_j}}L_{S_j} - I] \underset{j\to\infty}{\to} 0$$

pointwise on A_0. So $\{L_{T_{r_j}}\}_{j\geq 1}$ is hereditarily densely universal on $A(Y, X)$, by [6, Theorem 2.2]. □

Remark 3.4. *An alternative constructive proof of Theorem 3.1 may be done with the arguments from* [25, Theorem 2.2]. *The proof here is much simpler, and follows arguments from* [10] *and* [11].

Corollary 3.5. *Let T_l $(l \in \mathbb{N})$ be operators acting on a Banach space X. Suppose there exists a closed, infinite dimensional subspace Y of X, increasing sequences $(n_{l,q})_q$ of positive integers, and scalars $c_{l,q}$ so that for $l \in \mathbb{N}$*

i) $\{c_{l,q} T_l^{n_{l,q}}\}_{q \in \mathbb{N}}$ *is hereditarily universal, and*
ii) $\lim_{q \to \infty} \|c_{l,q} T_l^{n_{l,q}} x\| = 0$ *for each* x *in* Y.

Then there exists a closed, infinite dimensional subspace X_1 *of* X *so that* $\{c_{l,q} T_l^{n_{l,q}} x\}_{q \in \mathbb{N}}$ *is dense in* X *for each non-zero* $x \in X_1$ *and each* $l \in \mathbb{N}$. *That is,* X_1 *is a supercyclic subspace for* T_l *for every* $l \in \mathbb{N}$. *Moreover* X_1 *is a hypercyclic subspace for* T_l *for every* $l \in \mathbb{N}$ *if the constants* $c_{l,q}$ *are equal to one.*

In virtue of Theorem 3.1 and Example 2.1 it is natural to ask:

Problem 2. *Let* T_1, T_2 *be two hereditarily hypercyclic operators acting on a Banach space* X, *with a common hypercyclic subspace. Must there exist sequences* $(n_{l,q})_q$ $(l = 1, 2)$ *and a closed infinite dimensional subspace* Y *of* X *so that* $\{T_l^{n_{l,q}}\}_q$ *is hereditarily universal and* $T_l^{n_{l,q}} \underset{q \to \infty}{\to} 0$ *pointwise on* Y $(l = 1, 2)$?

4. An Application to Countable Families of Operators

We now apply Theorem 3.1 to show the following extension of [22, Theorem 4.1] to countable families of operators.

Theorem 4.1. *Let* $\mathcal{F} = \{T_l = U_l + K_l : l \in \mathbb{N}\}$ *be a family of operators acting on a common Banach space* X. *Suppose that for each* $l \in \mathbb{N}$

a) $\|U_l\| \leq 1$, K_l *is compact, and*
b) $\{T_l^{n_{l,q}}\}_{q \geq 1}$ *is hereditarily universal, for some increasing sequence* $(n_{l,q})_{q \geq 1}$ *of positive integers.*

Then the operators in $\mathcal{F}$ *have a common hypercyclic subspace.*

To show Theorem 4.1, we make use of the three lemmas below. Lemma 4.2 and Lemma 4.3 follow from slight modifications of a proof by Mazur [14, p 38-39] and of a proof by Bernal-González and Calderón-Moreno [7, Theorem 3.1], respectively. Lemma 4.4 is proved at the end of this section.

Lemma 4.2. *Let (X_n) be a sequence of closed, finite-codimensional subspaces of X, with $X_n \supseteq X_{n+1}$ $(n \geq 1)$. Then there exists a normalized basic sequence (e_n) so that e_n belongs to X_n for all $n \geq 1$.*

Lemma 4.3. *Let $T_{l,j}$ $(l, j \in \mathbb{N})$ be bounded operators on a Banach space X so that for each $l \in \mathbb{N}$ the family $\{T_{l,j}\}_j$ is hereditarily densely universal. Then there exists a dense manifold X_0 of X and, for each $l \in \mathbb{N}$, an increasing sequence of positive integers $(r_{l,q})_q$ so that*

$$\lim_{q\to\infty} \|T_{l,r_{l,q}} x\| = 0 \qquad (x \in X_0).$$

Moreover, X_0 may be chosen so that each non-zero vector of X_0 is universal for $\{T_{l,j}\}_{j\geq 1}$, for each $l \in \mathbb{N}$.

Lemma 4.4. *Let X and Z be Banach spaces, and let $K_{l,n} : X \to Z$ be compact operators $(l, n \geq 1)$. Given $\epsilon > 0$, there exist closed linear subspaces X_n of finite codimension in X $(n \geq 1)$ so that*

i) $X_n \supseteq X_{n+1}$
ii) $\|K_{l,n}x\| \leq \epsilon \|x\| \quad (x \in X_n,\ 1 \leq l \leq n)$

Proof of Theorem 4.1. Notice that for each $l \in \mathbb{N}$, $\{T_l^{n_{l,q}}\}_{q\geq 1}$ must be hereditarily densely universal [8, Lemma 2.5]. Hence, by Theorem 3.1 it suffices to get a closed, infinite dimensional subspace Y of X and subsequences $(m_{l,q})_q$ of $(n_{l,q})_q$ so that

$$\lim_{q\to\infty} \|T_l^{m_{l,q}} x\| = 0 \qquad (x \in Y,\ l \in \mathbb{N}).$$

For each pair of positive integers n and l, let $K_{l,n}$ be the compact operators defined by $T_l^n = (U_l + K_l)^n = U_l^n + K_{l,n}$. Apply Lemma 4.4

to get closed, finite codimensional subspaces X_n of X satisfying

$$(4) \qquad \begin{cases} a) & X_n \supseteq X_{n+1} \\ b) & \|K_{l,n}x\| \leq \|x\| \quad (x \in X_n, 1 \leq l \leq n). \end{cases}$$

By Lemma 4.2, we can pick a normalized basic sequence (e_n) in X so that $e_n \in X_n$ $(n \in \mathbb{N})$. Let $K > 0$ be the basis constant of (e_n), and pick a decreasing sequence of positive scalars, (ϵ_m), so that $\sum_{n=1}^{\infty} \epsilon_n < \frac{1}{2K}$. By Lemma 4.3 (applied to the operators $T_{l,j} = T_l^{n_{l,j}}$ $l, j \in \mathbb{N}$), there exist subsequences $(\tilde{n}_{l,q})_q$ of $(n_{l,q})_q$ and a dense subspace X_0 of X so that

$$(5) \qquad \lim_{q \to \infty} \|T_l^{\tilde{n}_{l,q}} x\| = 0 \qquad (x \in X_0).$$

Pick a sequence (z_m) in X_0 so that

$$(6) \qquad \|e_n - z_n\| < \frac{\epsilon_n}{\max\{\, \|T_l^i\| : \ l, i \leq n. \,\}}.$$

Notice that $\|e_n - z_n\| < \epsilon_n \quad (n \geq 1)$ and, because (e_n) is normalized, $|e_n^*(x)| \leq 2K\|x\| \quad (n \geq 1)$ for all x in $Y_0 = \overline{\text{span}\{e_1, e_2, \dots\}}$, where (e_n^*) is the sequence of functional coefficients associated with the Schauder basis (e_n) of Y_0. Hence $\sum_{n=1}^{\infty} \|e_n^*\| \, \|e_n - z_n\| < 2K \sum_{n=1}^{\infty} \epsilon_n < 1$, and so any subsequence (z_{n_k}) of (z_m) is equivalent to the corresponding basic sequence (e_{n_k}) [14, p 46]. We let $Y := \overline{\text{span}\{z_{n_k} : k \geq 1\}}$, where $(z_{n_k}) \subseteq (z_n)$ is defined as follows. Let $n_0 := 1$. For $l \in \mathbb{N}$, choose $m_{l,1}$ in $(\tilde{n}_{l,q})$ so that $\|T_l^{m_{l,1}} z_{n_0}\| < \frac{\epsilon_{n_0}}{2}$. Also, let $n_1 := m_{1,1}$. Next, for each $l \in \mathbb{N}$, since $z_{n_0},\ z_{n_1} \in X_0$, we may apply (5) to get $m_{l,2} \in (\tilde{n}_{l,q})_q$ which satisfies the following conditions.

$$\begin{cases} m_{l,2} & > \max\{2, n_1, m_{l,1}\} \\ \|T_l^{m_{l,2}} z_{n_i}\| & < \frac{\epsilon_{n_i}}{2^2} \qquad i = 0, 1. \end{cases}$$

Also, let $n_2 := \max_{1\le l\le 2}\{m_{l,2}\}$. Continuing this process we get, for each $l \in \mathbb{N}$, an integer $m_{l,s}$ in $(\tilde{n}_{l,q})_q$ so that

$$\text{(7)}\qquad \begin{cases} i) \qquad m_{l,s} > \max\{s, n_{s-1}, m_{l,s-1}\} \\ ii) \ \|T_l^{m_{l,s}} z_{n_i}\| < \frac{\epsilon_{n_i}}{2^s} \qquad i = 0, \dots, s-1, \end{cases}$$

where $n_r = \max_{1\le l\le r}\{m_{l,r}\}$ for each $r \in \mathbb{N}$. It suffices to show that $T_l^{m_{l,s}} \underset{s\to\infty}{\to} 0$ pointwise on Y $(l \in \mathbb{N})$. Let $0 \neq z = \sum_{j=1}^{\infty} \alpha_j z_{n_j}$ in Y, $l \in \mathbb{N}$ be fixed, and $s \ge l$ be arbitrary. Then

$$\text{(8)}\quad T_l^{m_{l,s}} z = \sum_{j=1}^{s-1} \alpha_j T_l^{m_{l,s}} z_{n_j} + \sum_{j=s}^{\infty} \alpha_j\, T_l^{m_{l,s}}(z_{n_j} - e_{n_j}) + T_l^{m_{l,s}}(\sum_{j=s}^{\infty} \alpha_j\, e_{n_j}).$$

Notice that $|\alpha_j| \le 2L\|z\| \quad (1 \le j)$, where L is the basis constant of (z_{n_k}). By (7.ii),

$$\text{(9)}\qquad \|\sum_{j=1}^{s-1} \alpha_j T_l^{m_{l,s}} z_{n_j}\| < \sum_{j=1}^{s-1} |\alpha_j| \frac{\epsilon_{n_j}}{2^s} \le \frac{L\|z\|}{2^{s-1}} \sum_{j=1}^{s-1} \epsilon_{n_j}.$$

Also, by (7i) and (6)

$$\text{(10)}\qquad \|\sum_{j=s}^{\infty} \alpha_j\, T_l^{m_{l,s}}(z_{n_j} - e_{n_j})\| \le 2L\|z\| \sum_{j=s}^{\infty} \epsilon_{n_j}.$$

Finally, since $X_{n_s} \subseteq X_{m_{l,s}}$ and $\|U_l\| \le 1$, by (4b)

$$\text{(11)}\qquad \begin{aligned} \|T_l^{m_{l,s}} \sum_{j=s}^{\infty} \alpha_j\, e_{n_j}\| &= \|(U_l^{m_{l,s}} + K_{l,m_{l,s}})(\sum_{j=s}^{\infty} \alpha_j\, e_{n_j})\ \| \\ &\le\ 2\ \|\sum_{j=s}^{\infty} \alpha_j\, e_{n_j}\| \qquad (s \ge l). \end{aligned}$$

So by (8), (9), (10), and (11), $\lim_{s\to\infty} \|T_l^{m_{l,s}} z\| = 0$. We finish the proof of Theorem 4.1 by showing Lemma 4.4.

Proof of Lemma 4.4. Let $n \ge 1$ and $\epsilon > 0$ be fixed. Because each ${K_{l,n}}^* : Z^* \to X^*$ is compact, there exist $x^*_{l,n,1}, \dots, x^*_{l,n,k_{l,n}}$ in X^* so that

$$\text{(12)}\qquad K^*_{l,n}(B_{Z^*}) \subseteq \cup_{i=1}^{k_{l,n}}\ B(x^*_{l,n,i}, \epsilon).$$

For each positive integer s, let $X_s := \cap_{n=1}^{s} \cap_{l=1}^{n} \cap_{i=1}^{k_{l,n}} \mathrm{Ker}(x^*_{l,n,i})$. So each X_s is closed and of finite codimension in X, and $X_s \supseteq X_{s+1}$ $(s \geq 1)$. Now, let $x \in X_n$, and let $1 \leq l \leq n$ be fixed. By the Hahn-Banach theorem, there is a functional z^* of norm one so that $\|K_{l,n}x\| = \langle K_{l,n}x, z^* \rangle$. By (12), we may choose $1 \leq j \leq k_{l,n}$ so that $\| K^*_{l,n}z^* - x^*_{l,n,j}\| < \epsilon$. Hence, because x is in $X_n \subseteq \mathrm{Ker}(x^*_{l,n,j})$, $\|K_{l,n}x\| = \langle x, K^*_{l,n}z^* - x^*_{l,n,j}\rangle \leq \epsilon \, \|x\|$. □

The proof of Theorem 4.1 is now complete. □

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