HYPERCYCLIC SUBSPACES IN OMEGA

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ABSTRACT. We show that any countable family of operators of the form $P(B)$, where P is a non-constant polynomial and B is the backward shift operator on the countably infinite product of lines ω, has a common hypercyclic subspace.

The space $\omega = \mathbb{K}^{\mathbb{N}}$ -i.e., the countably infinite product of the (real or complex) scalar field $\mathbb{K}$, endowed with the product topology- is perhaps the most elementary infinite dimensional Fréchet space. Even so, because it does not support a dense subspace with a continuous norm, it sometimes requires to be considered separately when showing hypercyclic properties for all (separable, infinite dimensional) Fréchet spaces, see for example [12, p. 587-8].

A continuous linear operator T acting on a Fréchet space X is said to be *hypercyclic* provided there is some vector z in X whose orbit $\{z, Tz, T^2z, \dots\}$ is dense in X. Such vector z is called a *hypercyclic vector* for T. A *hypercyclic manifold* for T is a dense, invariant subspace of X consisting entirely -except for the origin- of hypercyclic vectors for T. A *hypercyclic subspace* for T is a *closed*, infinite dimensional subspace of X consisting entirely -except for the origin- of hypercyclic vectors for T.

Every separable, infinite dimensional Fréchet space supports a hypercyclic operator; see the works of Ansari [2], Bernal [5], and of Bonet and Peris [12]. It is also well known that once an operator on a Fréchet space has a hypercyclic vector, the smallest manifold invariant for T containing that vector is a hypercyclic manifold; see the works of Bourdon [13], Herrero [19], and Wengenroth [29]. The situation

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for hypercyclic subspaces is different. Consider the backward shift

$$(x_1, x_2, x_3, \dots) \overset{B}{\mapsto} (x_2, x_3, x_4, \dots).$$

While Rolewicz [27] showed that each scalar multiple λB is hypercyclic on ℓ_2 whenever the scalar λ has modulus strictly larger than 1, Montes [24] showed that no such operators have a hypercyclic subspace.

Read [26] and Bernal and Montes [7] constructed the first examples of hypercyclic subspaces. In fact, Read's examples include an operator on ℓ_1 for which *every* non-zero vector in ℓ_1 is hypercyclic. González, León, and Montes [17] showed that if an operator T acting on a Banach space X satisfies that $T \oplus T$ is hypercyclic on $X \times X$, then T has a hypercyclic subspace if and only if there exists a closed, infinite dimensional subspace X_0 of X and integers $1 < n_1 < n_2 < \dots$ so that

$$T^{n_k} x \underset{k\to\infty}{\longrightarrow} 0 \qquad \text{for each } x \in X_0 \tag{1}$$

and, moreover, if and only if the essential spectrum of T meets the closed unit disk. Let us stress here that the condition of $T \oplus T$ being hypercyclic on $X \times X$ is very mild, as all hypercyclic operators that we know seem to have this property; see [9]. In fact, the spectral characterization was used by León and Montes to test the existence of hypercyclic subspaces among a wide variety of classes of hypercyclic operators [22]. They also used this characterization to show that every separable, infinite dimensional Banach space supports an operator with a hypercyclic subspace [21].

Moreover, Condition (1) is sufficient to ensure the existence of a hypercyclic subspace well beyond the Banach space setting, as long as the Frèchet space X supports a continuous norm, see [11, Theorem 3.5] and [17, p. 177]. Indeed, L. Bernal [6, Theorem 2.5] and independently, Petersson [25, Theorem 7], used this fact to show that every separable infinite dimensional Fréchet space with a continuous norm supports a hypercyclic subspace.

On the other hand, Bonet, Martínez-Giménez and Peris [11, Remark 3.6] showed that, in general, Condition (1) is no longer sufficient in the case of Fréchet spaces without a continuous norm: the operator $(x_i)_{i\in\mathbb{Z}} \overset{T}{\mapsto} (2x_{i+1})_{i\in\mathbb{Z}}$ acting on $X = \{(x_i)_{i\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}} : (\mathrm{x_i})_{\mathrm{i}=1}^{\infty} \in \ell_2\}$ satisfies Condition (1) and yet does not have a hypercyclic subspace.

We show in this note that ω supports operators with a hypercyclic subspace too, even that ω is known not to have dense subspaces with a continuous norm [23, Corollary 1]. Indeed, we show the following.

Theorem 1. *Let $(P_k)_{k=1}^{\infty}$ be any sequence of non-constant polynomials, and let B be the backward shift acting on ω. Then the operators $P_k(B)$ $(k \in \mathbb{N})$ have a common hypercyclic subspace. That is, there exists a closed infinite dimensional subspace S of ω satisfying that*

$$\{x, P_k(B)x, P_k^2(B)x, \dots\}$$

is dense in ω for each $0 \neq x \in S$ and each $k \in \mathbb{N}$.

Theorem 1 also improves a result by Herzog and Lemmert [20, Bemerkungen 1], who showed that each operator on ω of the form $P(B)$, where P is a non-constant polynomial and B the backward shift, has a hypercyclic vector.

For more on hypercyclicity results we refer to the surveys by Grosse-Erdmann [15, 16] and by Bonet, Martìnez-Gimènez and Peris [10]. For work on common hypercyclic vectors and common hypercyclic subspaces, we refer to the articles of Abakumov and Gordon [1], Bayart [4], Costakis and Sambarino [14], and by Aron et al. [3].

Before proving Theorem 1 we first show two lemmas. For each $m \in \mathbb{N}$, we let Π_m denote the standard projection of ω onto $\mathbb{K}^m$; that is, $\Pi_m x = (x_1, \dots, x_m)$ for each $x = (x_i)_{i=1}^{\infty}$ in ω.

Lemma 2. *Let $T = P(B)$, where B is the backward shift on ω and $P(t) = a_1 + a_2 t + \cdots + a_{d+1} t^d$ is any polynomial of degree $d \geq 1$. Then for each l, $m \in \mathbb{N}$, $(y_1, y_2, \dots, y_l) \in \mathbb{K}^l$ and $(x_1, x_2, \dots, x_{md}) \in \mathbb{K}^{md}$, there exists a unique $(z_1, z_2, \dots, z_l) \in \mathbb{K}^l$ so that*

$$\Pi_l T^m (x_1, x_2, \dots, x_{md}, z_1, z_2, \dots, z_l, h_1, h_2, \dots) = (y_1, y_2, \dots, y_l)$$

for each $h_1, h_2, \dots$ in $\mathbb{K}$.

Proof. Notice that each $x = (x_i)_{i=1}^{\infty} \in \omega$ we have

$$Tx = ((a_1 x_j + a_2 x_{j+1} + \cdots + a_d x_{j+d-1}) + a_{d+1} x_{j+d})_{j=1}^{\infty},$$

and in general, for each $m \in \mathbb{N}$ the m^{th} iterate of T is of the form

$$T^m x = (\varphi_{m,j}(x_1, x_2, \dots, x_{j+md-1}) + (a_{d+1})^m x_{j+md})_{j=1}^{\infty},$$

for some linear functions $\varphi_{m,j} : \mathbb{K}^{\mathrm{md+j-1}} \to \mathbb{K}$ $(j \in \mathbb{N})$ that are independent of x. Thus the Lemma follows, since $a_{d+1} \neq 0$. $\square$

Lemma 3. *Let $[f_{i,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ be an infinite matrix with coefficients in $\mathbb{K}$ and no row of zeroes. For each row $f_n = (f_{n,1}, f_{n,2}, \dots)$, let $a_n := \min\{j \in \mathbb{N} : f_{n,j} \neq 0\}$. Then if $(a_m)_{m=1}^{\infty}$ is strictly increasing*

i) *$\{f_1, f_2, \dots\}$ is linearly independent, and*
ii) $\overline{\mathrm{span}\{f_1, f_2, \dots\}}^{\omega} = \{\sum_{n=1}^{\infty} \alpha_n f_n : (\alpha_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}\}$.

Proof. Notice that since $(a_m)_{m=1}^{\infty}$ is strictly increasing, for each $s \in \mathbb{N}$ we have $f_{s,a_s} \neq 0$ and $f_{n,j} = 0$ for each $(n,j) \in (s,\infty) \times [1, a_s]$. Hence (i) follows, and $\sum_{n=1}^{\infty} \alpha_n f_n$ converges in ω for any $(\alpha_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$. Now, let $g \in \overline{\mathrm{span}\{f_1, f_2, \dots\}}^{\omega}$. There exist integers $1 < r_1 < r_2 < \dots$ and sequences $(\alpha_{n,1})_{n=1}^{\infty}$, $(\alpha_{n,2})_{n=1}^{\infty}, \dots$ in $\mathbb{K}$ so that

$$P_n := (\alpha_{n,1} f_1 + \alpha_{n,2} f_2 + \dots + \alpha_{n,r_n} f_{r_n}) \underset{n\to\infty}{\to} g. \tag{2}$$

It remains to show that there exists a strictly increasing sequence $(\alpha_s)_{s=1}^{\infty}$ in $\mathbb{N}$ so that

$$\Pi_{a_s}(\alpha_1 f_1 + \dots + \alpha_s f_s) = \Pi_{a_s}(g) \quad (s \in \mathbb{N}). \tag{3}$$

Now, for $n > a_1$ $\alpha_{n,1}(f_{1,1}, f_{1,2}, \dots, f_{1,a_1}) = \Pi_{a_1}(P_n)$, and so by (2) $\alpha_{n,1} \underset{n\to\infty}{\to} \alpha_1$ and $\Pi_{a_1}(g) = \Pi_{a_1}(\alpha_1 f_1)$, where $\alpha_1 = \frac{g_{a_1}}{f_{1,a_1}}$. Inductively, suppose that we found $\alpha_j \in \mathbb{K}$ $(1 \leq j \leq s-1)$ so that

$$\alpha_{n,j} \underset{n\to\infty}{\to} \alpha_j \quad \text{and} \quad \Pi_{a_j}(g) = \Pi_{a_j}(\alpha_1 f_1 + \dots + \alpha_j f_j) \tag{4}$$

for each $(1 \leq j \leq s-1)$. Again, since $(a_m)_{m=1}^{\infty}$ is strictly increasing, $\Pi_{a_s}(\alpha_{n,1} f_1 + \dots + \alpha_{n,s} f_s) = \Pi_{a_s}(P_n)$ for each $n > s$ and so by (4) and (2) we have $\alpha_{n,s} \underset{n\to\infty}{\to} \alpha_s$ and $\Pi_{a_s}(g) = \Pi_{a_s}(\alpha_1 f_1 + \dots + \alpha_s f_s)$, where $\alpha_s = \frac{g_{a_s} - (\alpha_1 f_{1,a_s} + \dots + \alpha_{s-1} f_{s-1,a_s})}{f_{s,a_s}}$. So (3) follows. $\square$

Proof of Theorem 1: Let $\{r_l : \; l \in \mathbb{N}\}$ be a countable dense set in ω so that each $r_l = (r_{l,j})_{j=1}^\infty$ satisfies $r_{l,j} \neq 0$ if and only if $1 \leq j \leq l$. For each $k \in \mathbb{N}$, let $T_k := P_k(B)$ and $d_k := \text{degree}(P_k)$. We make use of the following claim.

Claim 4. *There exists an infinite, upper triangular matrix $F = [f_{i,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ satisfying*

a) *No row $f_n = (f_{n,1}, f_{n,2}, \dots)$ is zero.*
b) *The sequence $(a_n)_{n=1}^\infty$ given by $a_n := \min\{j \in \mathbb{N} : \; f_{n,j} \neq 0\}$ is strictly increasing.*
c) *For each $(k, i, l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $k < i + l$, there exists a positive integer $m_{k,i,l}$ so that*

$$\Pi_l \, T_k^{m_{k,i,l}} f_n = \begin{cases} (r_{l,1}, r_{l,2}, \dots, r_{l,l}) & \text{if } n = i \\ (\;0,\; 0, \dots,\; 0\;) & \text{if } n \neq i. \end{cases}$$

Suppose the Claim holds. We show now that $S := \overline{\text{span}\{f_1, f_2, \dots\}}^{\omega}$ is a hypercyclic subspace for each T_k $(k \in \mathbb{N})$.

By (a), (b), and Lemma 3(i), the closed subspace S is infinite dimensional. Let $0 \neq f \in S$. We show that f is hypercyclic for T_k, $k \in \mathbb{N}$. By Lemma 3, $f = \sum_{n=1}^\infty \alpha_n f_n$ for some sequence of scalars $(\alpha_n)_{n=1}^\infty$. Multiplying f by a nonzero scalar if necessary, we may assume without loss of generality that $\alpha_i = 1$ for some $i \in \mathbb{N}$. But by (c), for each $l > \max\{k - i, 1\}$

$$\begin{aligned} \Pi_l \, T_k^{m_{k,i,l}} \, f \;&=\; \sum_{n=1}^\infty \alpha_n \, \Pi_l \, T_k^{m_{k,i,l}} \, f_n \\ &= \Pi_l \, T_k^{m_{k,i,l}} \, f_i \\ &= (r_{l,1}, r_{l,2}, \dots, r_{l,l}). \end{aligned}$$

It follows that f is hypercyclic for T_k. We finish the proof of Theorem 1 by showing the Claim.

Proof of Claim. Let $M_{0,0} := 1$. Inductively, for each $N \in \mathbb{N}$ define

$$\begin{cases} M_N & := d_N \, M_{(N-1),(N-1)^2} \\ M_{N,i} & := 2^{N+i} M_N \quad (1 \leq i \leq N^2) \\ M_{(N-1),(N-1)^2+1} & := M_{N,1}. \end{cases}$$

Also, for each $(k, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $1 \le k \le (i + l) - 1$, let

$$m_{k,i,l} := \frac{M_{(i+l-1),((k-1)(i+l-1)+i)}}{d_k}.$$

Finally, let $f_{n,j} = 0$ for each $(n, j) \in \mathbb{N} \times [1, M_{1,1}]$. We complete the definition of the matrix $F = [f_{n,j}]$ inductively. At each step N we define $f_{n,j}$ for all $(n, j) \in \mathbb{N} \times (M_{N,1}, M_{N+1,1}]$.

Step $N = 1$. We define $f_{n,j}$ for all $(n, j) \in \mathbb{N} \times (M_{1,1}, M_{2,1}]$ so that

$$\Pi_1\, T_1^{m_{1,1,1}}(f_{n,1}, f_{n,2}, \dots, f_{n,M_{2,1}}, *, *, \dots) = \begin{cases} r_{1,1} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases} \tag{5}$$

By Lemma 2 (letting $l = 1$, $m = m_{1,1,1}$, $T = T_1$, $d = d_1$, $y_1 = r_{1,1}$ and $x_j = f_{1,j}$ $(1 \le j \le M_{1,1})$), there exists a unique $z \in \mathbb{K}$ so that

$$\Pi_1\, T_1^{m_{1,1,1}}(f_{1,1}, f_{1,2}, \dots, f_{1,M_{1,1}}, z, *, *, \dots) = r_{1,1}.$$

So (5) is satisfied if we define $f_{1,M_{1,1}+1} := z$, and $f_{n,j} = 0$ for each $(1, M_{1,1} + 1) \neq (n, j) \in \mathbb{N} \times (M_{1,1}, M_{2,1}]$.

Step N $(N \ge 2)$.

We divide this step into N^2 substeps; one for each $(k, i) \in [1, N] \times [1, N]$. We start with substep $N.1.1$, and follow with the "lexicographic" order given by the relation $(k', i') < (k, i)$ if and only if either $k' < k$ or both $k' = k$ and $i' < i$.

At each substep $N.k.i$ we define the coordinates $f_{n,j}$ for all indexes (n, j) in $\mathbb{N} \times (M_{N,(k-1)N+i}, M_{N,(k-1)N+i+1}]$, so that

$$\Pi_l\, T_k^{m_{k,i,l}} g_n = \begin{cases} (r_{l,1}, \dots, r_{l,l}) & \text{if } n = i \\ (\ 0, \dots,\ 0\ ) & \text{if } n \neq i, \end{cases} \tag{6}$$

for any g_n of the form $g_n = (f_{n,1}, \dots, f_{n,M_{N,(k-1)N+i+1}}, *, *, \dots)$ and $l = N + 1 - i$.

Substep $N.1.1$.

Applying N times Lemma 2 (Taking, for each $1 \le n \le N$: $l = N$, $m = m_{1,1,N}$, $T = T_1$, $d = d_1$, $x_j^{(n)} = f_{n,j}$ $(1 \le j \le M_{N,1})$, and $(y_1^{(n)}, \dots y_N^{(n)}) = (r_{N,1}, \dots, r_{N,N})$ if $n = 1$ and $(y_1^{(n)}, \dots, y_N^{(n)}) = (0, \dots, 0) \in \mathbb{K}^N$ if $n \neq 1$), we get $(z_1^{(n)}, z_2^{(n)}, \dots, z_N^{(n)}) \in$

$\mathbb{K}^N$ $(1 \le n \le N)$ so that

$$\Pi_N\, T_1^{m_{1,1,N}} g_n = \begin{cases} (r_{N,1}, \dots, r_{N,N}) & \text{if } n = 1 \\ (0, \dots, 0) & \text{if } n \neq 1. \end{cases} \tag{7}$$

for any g_n of the form $g_n = (f_{n,1}, \dots, f_{n,M_{N,1}}, z_1^{(n)}, \dots, z_N^{(n)}, *, *, \dots)$. Hence (6) is satisfied for $(k, i) = (1, 1)$ if we define

$$(f_{n,M_{N,1}+1}, \dots, f_{n,M_{N,1}+N}) = (z_1^{(n)}, \dots, z_N^{(n)}) \quad (1 \le n \le N)$$

and $f_{n,j} = 0$ for each (n, j) in either $(\mathbb{N} \setminus \{1, \dots, N\}) \times (M_{N,1}, M_{N,2}]$ or in $\mathbb{N} \times (M_{N,1} + N + 1, M_{N,2}]$.

<u>Substep *N.k.i*</u>.

We have already defined $f_{n,j}$ for each $(n, j) \in \mathbb{N} \times [1, M_{N,(k-1)N+i}]$, so that equation (6) holds for each $(1, 1) \le (k', i') < (k, i)$. That is, so that

$$\Pi_l\, T_{k'}^{m_{k',i',l}} g_n = \begin{cases} (r_{l,1}, \dots, r_{l,l}) & \text{if } n = i' \\ (0, \dots, 0) & \text{if } n \neq i', \end{cases} \tag{8}$$

for any $g_n \in \omega$ of the form $g_n = (f_{n,1}, \dots, f_{n,M_{N,(k'-1)N+i'+1}}, *, *, \dots)$ and $l = N + 1 - i'$.

We apply N times Lemma 2 (taking, for each $1 \le n \le N$: $l = N + 1 - i$, $m = m_{k,i,l}$, $T = T_k$, $d = d_k$, $x_j^{(n)} = f_{n,j}$ $(1 \le j \le M_{N,(k-1)N+i})$, and $(y_1^{(n)}, \dots y_l^{(n)}) = (r_{l,1}, \dots, r_{l,l})$ if $n = i$ and $(y_1^{(n)}, \dots, y_l^{(n)}) = (0, \dots, 0) \in \mathbb{K}^l$ if $n \neq i$), to obtain $(z_1^{(n)}, \dots, z_l^{(n)}) \in \mathbb{K}^l$ $(1 \le n \le N)$, so that

$$\Pi_l\, T_k^{m_{k,i,l}} g_n = \begin{cases} (r_{l,1}, \dots, r_{l,l}) & \text{if } n = i \\ (\,0, \dots,\ 0\,) & \text{if } n \neq i, \end{cases} \tag{9}$$

for any $g_n \in \omega$ of the form $g_n = (f_{n,1}, \dots, f_{n,M_{N,(k-1)N+i}}, z_1^{(n)}, \dots, z_l^{(n)}, *, *, \dots)$ and $l = N+1-i$. So equation (6) is satisfied if we define $f_{n,M_{N,(k-1)N+i}+s} = z_s^{(n)}$ when $(n, s) \in [1, N] \times [1, l]$, and $f_{n,j} = 0$ for all indexes (n, j) in either $(\mathbb{N} \setminus \{1, \dots, N\}) \times (M_{N,(k-1)N+i}, M_{N,(k-1)N+i+1}]$ or in $\{1, \dots, N\} \times (M_{N,(k-1)N+i}+l, M_{N,(k-1)N+i+1}]$.

We have now completely defined the matrix $[f_{n,j}] \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$. Notice that for each $N \in \mathbb{N}$, $f_{N,j} = 0$ for $1 \le j \le M_{N,N}$, and (as defined on substep $N.1.N$ of step N) $f_{N,M_{N,N}+1} \neq 0$. So $a_N = \min\{j \in \mathbb{N} :\ f_{N,j} \neq 0\} = M_{N,N} + 1$, and (a) and

(b) of the Claim hold. Finally, given any $(k, i, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $k < i + l$, our definitions on substep $N.k.i$ of step $N = i + l - 1$ given by (6) ensure that

$$\Pi_l \, T_k^{m_{k,i,l}} f_n = \begin{cases} (r_{l,1}, \dots, r_{l,l}) & \text{if } n = i \\ (\ 0\ , \dots,\ 0\ ) & \text{if } n \neq i. \end{cases}$$

So Part (c) of the Claim holds, and the proof of Theorem 1 is now complete.

□

Corollary 5. *The set of operators on ω that have a hypercyclic subspace is dense, with respect to the Strong Operator Topology (S. O. T.), in the algebra $L(\omega)$ of all continuous linear operators on ω.*

Proof. By a result of Hadwin, Nordgreen, Radjavi and Rosenthal [18] (cf. [8, Corollary 6]), the set of operators on ω having a hypercyclic subspace, which is invariant under conjugations, must be either empty or S.O.T.-dense in $L(\omega)$. Theorem 1 then gives the desired conclusion. □

Remark 6. *A simple modification to Lemma 2 allows to generalize Theorem 1 to backward shifts B_b with non-zero weights. Namely, if $(b_n)_{n=2}^{\infty}$ is a sequence of nonzero weights and $(x_1, x_2, x_3, \dots) \overset{B_b}{\mapsto} (b_2x_2, b_3x_3, b_4x_4, \dots)$ is its associated weighted shift on ω, then any countable collection of operators of the form $P(B_b)$, where P is a non-constant polynomial, has a common hypercyclic subspace in ω.*

Solving a problem by Salas [28], Abakumov and Gordon [1] showed that the family $\{\lambda B : |\lambda| > 1\}$ of all scalar multiples of the backward shift B on ℓ_2 (with the scalars of modulus strictly larger than 1) have a common hypercyclic vector. Hence (cf. also [14, Remark 8.3]) it is natural to ask

Problem 7. *Let $\mathcal{F}$ be the collection of all operators on ω of the form $P(B)$, where P is a non-constant polynomial and B is the backward shift. Do the operators in $\mathcal{F}$ have a common hypercyclic vector in ω? Do they share a common hypercyclic subspace?*

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