Linear transitivity criteria

José A. Conejero and Alfredo Peris*

Abstract

We show that different transitivity criteria for strongly continuous semigroups of operators are equivalent. We also give new results concerning the equivalence of transitivity criteria in the case of iterations of a single operator. [1]

Transitivity was a concept introduced by G.D. Birkhoff in the 20's for dynamical systems. This property is reflected in the most important notions of topological chaos. We refer the reader to the survey of Kolyada and Snoha [16] for a detailed discussion with many examples and related notions. Our interest here is the transitivity for special dynamical systems: infinite dimensional linear systems. We study discrete and continuous dynamical systems and discuss different sufficient conditions for transitivity.

The first section is devoted to the discrete case. We recall some known transitivity criteria, introduce a new one in terms of the existence of suitable backward orbits, and show their equivalence with the property of weakly mixing. We also characterize weakly mixing invertible operators with an easy "computable" condition.

In the second section we introduce new transitivity criteria for semigroups (semiflows) of operators inspired by the corresponding discrete versions and show their equivalence with weakly mixing.

1 Transitivity for linear operators

Given a metric space X, a continuous map $T : X \to X$, and $x \in X$, the *orbit* of x under T is $Orb(x, T) := \{x, Tx, T^2x, \dots\}$.

*Supported by MCYT and FEDER, Proy. BFM 2001-2670 and AVCIT Grupo 03/50.

[1] 2000 Mathematics Subject Classification: 37C10, 37D45, 47A16, 47D06, 54H20.

Key Words: hypercyclic vectors, topological transitivity, semigroups of operators, weakly mixing, chaos.

T is said to be *transitive* if for every pair of non-empty open sets $U, V \subseteq X$ there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. In other words, there is a point $x \in U$ whose orbit intersects V.

If X is separable, Baire and perfect (that is, without isolated points), then T is transitive if and only if T admits a dense orbit, i.e., there exists $x \in X$ such that $\overline{Orb(x,T)} = X$.

Our framework will be complete and metrizable topological vector spaces ($\mathcal{F}$-spaces) X, which are also separable, and linear and continuous maps (operators) $T : X \to X$. Within this context, T is *hypercyclic* if there exists $x \in X$ such that $\overline{Orb(x,T)} = X$. (Equivalently, T is transitive).

No finite dimensional X admits a hypercyclic operator [15], and every hypercyclic operator has the so called "sensitive dependence on initial conditions" [11].

The first example of a hypercyclic operator was given by Birkhoff [4] who showed that the translation operator $T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $f(z) \mapsto f(z+a)$, $(a \neq 0)$ is hypercyclic on the $\mathcal{F}$-space $\mathcal{H}(\mathbb{C})$ of entire functions endowed with the compact-open topology.

In the study of the transitive dynamics of linear operators, there is a couple of topological concepts which appear frequently.

Definition 1.1 *T is said to be weakly mixing if $T \oplus T$ is transitive on $X \oplus X$, and T is mixing if for every pair of non-empty open sets $U, V \subseteq X$ there exists some $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, for all $n \geq n_0$.*

These notions are related to the concept of hereditarily hypercyclic.

Definition 1.2 *T is said to be hereditarily hypercyclic with respect to a strictly increasing sequence (n_k) of natural numbers if, for any subsequence (n_{k_j}) of (n_k), there is $x \in X$ such that $\{T^{n_{k_j}} x \ , \ j \in \mathbb{N}\}$ is dense in X.*

Proposition 1.3 *(1) T is weakly mixing if and only if T is hereditarily hypercyclic with respect to some sequence (n_k).*

(2) T is mixing if and only if T is hereditarily hypercyclic with respect to the sequence of all natural numbers.

The equivalence (1) was proved in [3]. The mixing case (2) follows essentially straightforward from the definition.

In the study of the dynamics of continuous functions on compact spaces, there are rare examples on weakly mixing maps which are not mixing. In our context the examples are easier.

Remark 1.4 *There are weakly mixing operators which are not mixing.*

For instance, within the framework of weighted backward shift operators $B_w(x_1, x_2, \dots) := (w_2 x_2, w_3 x_3, \dots)$ on the space l_2 of 2-summable sequences, we characterized in [3, Prop. 3.3] the property of being hereditarily hypercyclic with respect to a fixed increasing sequence (n_k) of natural numbers. As a consequence, e.g., the weighted shift given by the sequence of weights

$$w = (2, 1/2, 1/2, 2, 2, 2, 1/2, 1/2, 1/2, 1/2, 2, 2, 2, 2, 2, \dots)$$

is weakly mixing but not mixing. See also [5] for a complete study of mixing weighted shifts.

We recall some usual transitivity criteria for linear operators and show their equivalence. The first one was given by Kitai [15] in her unpublished thesis, latter rediscovered by Gethner and Shapiro [10], who reformulated it. The second one appeals to the concept of backward orbit. We say that a sequence (x_n) is a *backward orbit* of x under T if $Tx_1 = x$, $Tx_{n+1} = x_n$. Finally, the third one, constitutes what today is known as the Hypercyclicity Criterion for operators. It is a reformulation of Grosse-Erdmann [13] of the criterion given by Bès (see, e.g., [3]), in terms of what Grosse-Erdmann calls "intertwining collapse with blow-up". The following result improves known equivalences of hypercyclicity criteria for linear operators, and clarifies the weakly mixing property of operators with computable conditions.

Theorem 1.5 *Let T be an operator on a separable $\mathcal{F}$-space. The following statements are equivalent:*

(1) [Kitai/Gethner-Shapiro's Criterion] There exist dense subspaces $Y, Z \subset X$, a strictly increasing sequence (n_k) in $\mathbb{N}$, and a linear map $S : Z \to Z$ such that

(i) $\forall y \in Y, \quad \lim_k T^{n_k} y = 0,$

(ii) $\forall z \in Z, \quad \lim_k S^{n_k} z = 0$ *and* $TSz = z.$

(2) There exist dense subsets $Y, Z \subset X$, and a strictly increasing sequence (n_k) in $\mathbb{N}$ such that

(i) $\forall y \in Y, \quad \lim_k T^{n_k} y = 0,$

(ii) $\forall z \in Z$, *there exists a backward orbit (x_n) of z:* $\lim_k x_{n_k} = 0.$

(3) [Bès/Grosse-Erdmann's Hypercyclicity Criterion] There exist dense subsets $Y, Z \subset X$, and a strictly increasing sequence (n_k) in $\mathbb{N}$ such that

(i) $\forall y \in Y, \quad \lim_k T^{n_k} y = 0,$

(ii) $\forall z \in Z, \quad \exists (x_k) \subset X : \quad \lim_k x_k = 0 \quad$ *and* $\quad \lim_k T^{n_k} x_k = z.$

(4) T is weakly mixing.

Proof.

In [3] it was shown that the Hypercyclicity Criterion (criterion (3)) is equivalent to the property of weakly mixing. In [19], using a Mittag-Leffler procedure, we showed the equivalence of a version of criterion (1) -with weaker conditions- with the property of weakly mixing. More precisely, in the hypothesis of [19, Thm. 1.1], Y, Z were just dense *subsets* and S was only a map (not necessarily linear). A closer look at the proof of Theorem 2.3 in [19] gives the desired hypothesis in the present version. Indeed, if T is weakly mixing, we proved in [19, Thm. 2.3] that there are $z \in X$, a backward orbit (z_n) of z, a dense subset $Y \subset X$, and a strictly increasing sequence (n_k) in $\mathbb{N}$ such that

$$B := \{z_n \ : \ n \geq 1\}$$

is dense in X, z is hypercyclic for T, $\lim_k z_{n+n_k} = 0$ for all n, and $\lim_k T^{n_k} y = 0$ for all $y \in Y$. If we select the linear span of Y (call it Y again!), we get condition (i) of criterion (1). We define now Z as the linear span of B. The countable set B is (dense and) linearly independent. Otherwise we would get the existence of $m \in \mathbb{N}$ and scalars α_k, $k = 1, \ldots, m$, not all zero such that

$$\sum_{k=1}^{m} \alpha_k z_k = 0.$$

By applying T^m to the above equality, we arrive at

$$\sum_{k=1} \alpha_k T^{m-k} z = 0,$$

which is impossible since the orbit of any hypercyclic vector is linearly independent. We can set $S z_n := z_{n+1}$ for each n, and extend the map by linearity to Z. Condition $\lim_k z_{n+n_k} = 0$ for each n implies that $\lim_k S^{n_k} z = 0$ for all $z \in Z$. The definition of S clearly implies that $TSz = z$ for each $z \in Z$.

Finally, it is easy to see that the hypothesis of the criterion (2) are weaker than the ones of criterion (1), but stronger than the ones of criterion (3). ■

Other criteria given by Grosse-Erdmann [12] have been shown to be equivalent to weakly mixing [2]. Chaotic operators constitute also an important class of hypercyclic operators. We recall that T is said to be *chaotic* in the sense of Devaney if T is hypercyclic and X has a dense subset of T-periodic points. In [3, Prop. 2.14] it was shown that chaotic operators are weakly mixing.

It is worth mentioning that for invertible operators T (i.e., bijective with continuous inverse), the backward orbit of each element is unique, and then the criteria (1) and (2) acquire a nice formulation.

Corollary 1.6 *An invertible operator T is weakly mixing if and only if there exist dense subspaces $Y, Z \subset X$ and a strictly increasing sequence $(n_k) \subset \mathbb{N}$, such that $\lim_k T^{n_k} y = 0$, and $\lim_k (T^{-1})^{n_k} z = 0$ for each $y \in Y$ and $z \in Z$.*

There is still an (probably, the most) important open question about hypercyclicity posed by Herrero [14]:

Open problem (Herrero, 1992):
If T hypercyclic, should be T necessarily weakly mixing?
Equivalently, due to the above mentioned equivalence with the Hypercyclicity Criterion:
If T is hypercyclic, does T satisfy necessarily the Hypercyclicity Criterion?

2 Transitivity criteria for semigroups of operators

A family $\{T_t : X \to X \ , \ t \geq 0\}$ of operators is called a *strongly continuous semigroup* (or C_0-semigroup) if:
(a) $T_0 = I$, the identity operator.
(b) $T_{s+t} = T_s T_t$ for all $t, s \geq 0$.
(c) The map $t \mapsto T_t x$ is continuous on $\mathbb{R}_+$ for each $x \in X$.

A C_0-semigroup is *transitive* when, given any pair $U, V \subset X$ of non-empty open sets, there is $t > 0$ such that $T_t(U) \cap V \neq \emptyset$. In our context, transitivity is equivalent to hypercyclicity, i.e., to the existence of $x \in X$ such that its orbit $\{T_t x \ , \ t \geq 0\}$ is dense in X. The semigroup is *weakly mixing* if $\{T_t \oplus T_t \ , \ t \geq 0\}$ is transitive on $X \oplus X$.

Transitivity of strongly continuous semigroups has been studied by several authors (see, e.g., [17, 20, 8, 9, 6, 1]). Transitivity criteria for semigroups were given in [8] (see also [7] for the case of unbounded operators). Our purpose in this section is to give some transitivity criteria, inspired by the discrete case, and to characterize them via the property of weakly mixing.

Given $x \in X$ and strongly continuous semigroup of operators $\{T_t \ , \ t \geq 0\}$ on X, a *backward orbit* of x under $\{T_t \ , \ t \geq 0\}$ is a family $\{x_t \ , \ t \geq 0\}$ of vectors in X such that $x_0 = x$ and $T_t x_s = x_{s-t}$ for all $s \geq t \geq 0$. The criteria (1) and (2) in the following result involve the existence of suitable backward orbits. Inspired by Grosse-Erdmann's formulation of the hypercyclicity criterion for single operators, we can give a collapse/blow-up criterion (3) for hypercyclicity of semigroups.

Theorem 2.1 *Let $\{T_t\ ,\ t \geq 0\}$ be a strongly continuous semigroup of operators on a separable $\mathcal{F}$-space X. The following are equivalent:*

(1) There exist $(t_k) \subset \mathbb{R}_+$ strictly increasing and tending to infinity, dense subspaces $Y, Z \subset X$, and a family $\{S_t : Z \to X\ ,\ t \geq 0\}$ of linear maps satisfying

(i) $\forall y \in Y, \quad \lim_k T_{t_k} y = 0$,

(ii) $\forall z \in Z, \quad \lim_k S_{t_k} z = 0$ and $T_t S_t z = z$, for each $t \geq 0$.

(2) There exist $(t_k) \subset \mathbb{R}_+$ strictly increasing and tending to infinity, and dense subsets $Y, Z \subset X$, satisfying

(i) $\forall y \in Y, \quad \lim_k T_{t_k} y = 0$,

(ii) every $z \in Z$ admits a backward orbit $\{z_t\ ,\ t \geq 0\}$ such that $\lim_k z_{t_k} = 0$.

(3) There exist $(t_k) \subset \mathbb{R}_+$ strictly increasing and tending to infinity, and dense subsets $Y, Z \subset X$, satisfying

(i) $\forall y \in Y, \quad \lim_k T_{t_k} y = 0$,

(ii) for each $z \in Z$ there is a sequence (z_k) in X with $\lim_k z_k = 0$ and $\lim_k T_{t_k} z_k = z$.

(4) $\{T_t\ ,\ t \geq 0\}$ is weakly mixing.

Proof.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4): The hypercyclicity criterion easily passes to the direct sum semigroup $\{T_t \oplus T_t\ ,\ t \geq 0\}$ on $X \oplus X$. The equivalence (4) is now a consequence of [12, Thm. 2] or [3, 2.6(3)].

(4) $\Rightarrow$ (1): If the semigroup $\{T_t \oplus T_t\ ,\ t \geq 0\}$ is transitive then, by a result of Oxtoby and Ulam [18, Thm. 6, Sect. 11], there is $t_0 > 0$ (in fact, there is a dense G_δ-subset of $\mathbb{R}_+$ of such indexes t_0) so that the operator $T_{t_0} \oplus T_{t_0} : X \oplus X \to X \oplus X$ is transitive. Then, by [3, Thm. 2.3], the operator T_{t_0} satisfies the hypercyclicity criterion. Therefore, due to [19, Thm. 2.3], we find $Y_0, Z_0 \subset X$ dense subsets, $(n_k) \subset \mathbb{N}$ strictly increasing, and a map $S : Z_0 \to Z_0$ such that

(i)' $\lim_k T_{n_k t_0} y = \lim_k T_{t_0}^{n_k} y = 0$ for each $y \in Y_0$,

(ii)' $\lim_k S^{n_k} z = 0$ for each $z \in Z_0$, and

(iii)' $T_{t_0} S z = z$, for each $z \in Z_0$.

Moreover, as we saw in the proof of Theorem 1.5, the set Z_0 can be chosen to be a countable linearly independent set of the form $Z_0 = \{w_m\ , m \in \mathbb{N}\}$, where $Sw_m = w_{m+1}$, $\forall m \in \mathbb{N}$, and $z := T_{t_0} w_1$ is a hypercyclic vector for T_{t_0}. We then set Y and Z as the linear spans of Y_0 and Z_0, respectively. If we define $z_0 := z$, $z_{nt_0} := w_n$, and $z_t := T_{nt_0 - t} z_{nt_0}$ for $(n-1)t_0 < t < nt_0$, $n \in \mathbb{N}$, then the family $\{z_t \ , \ t \geq 0\}$ is a backward orbit of z under $\{T_t \ , \ t \geq 0\}$.

We then set $S_t z_{nt_0} := z_{t+nt_0}$, $n \in \mathbb{N}$, and extend it by linearity to Z, for each $t \geq 0$. Thus

$$T_t S_t z_{nt_0} = T_t z_{t+nt_0} = z_{nt_0}, \quad n \in \mathbb{N}, \quad t \geq 0,$$

which implies that $T_t S_t z' = z'$ for every $z' \in Z$.

Finally, by setting $t_k := n_k t_0$, $k \in \mathbb{N}$, we get condition (i) of (1) and also

$$\lim_k S_{t_k} z_{nt_0} = \lim_k z_{n_k t_0 + nt_0} = \lim_k S^{n_k} w_n = 0, \quad n \in \mathbb{N},$$

which concludes condition (ii) in (1). ■

Remark 2.2 *We should compare the hypothesis of criterion (1) in Theorem 2.1 with those of its discrete version in Theorem 1.5: The domain and the range of the linear maps S_t do not coincide in the criterion for semigroups. When trying to get coincidence of domain and range of these maps, one essentially faces the problem of whether the orbit $\{T_t z \ , \ t \geq 0\}$ of a vector z hypercyclic under $\{T_t \ , \ t \geq 0\}$ is linearly independent. We do not know if this is true in general.*

References

[1] T. Bermúdez, A. Bonilla and A. Martinón, On the existence of chaotic and hypercyclic semigroups in Banach spaces, Proc. Amer. Math. Soc. **131** (2003), 2435-2441.

[2] T. Bermúdez, A. Bonilla and A. Peris, On hypercyclicity and supercyclicity criteria, Bull. Austral. Math. Soc. **70** (2004), 45-54.

[3] J. Bès and A. Peris, Hereditarily hypercyclic operators, J. Funct. Anal. **167** (1999), no. 1, 94-112.

[4] G.D. Birkhoff, Démonstration d'un théoréme elémentaire sur les fonctions entiéres, C. R. Acad. Sci. Paris **189** (1929), 473-475.

[5] G. Costakis and M. Sambarino, Topologically mixing hypercyclic operators, Proc. Amer. Math. Soc. **132** (2004), 385-389.

[6] R. DeLaubenfels, H. Emamirad and V. Protopopescu, Linear chaos and approximation, J. Approx. Theory **105** (2000), 176-187.

[7] R. DeLaubenfels, H. Emamirad and K.G. Grosse-Erdmann, Chaos for semigroups of unbounded operators, Math. Nachr. **261-262** (2003), 47-59.

[8] W. Desch, W. Schappacher and G. F. Webb, Hypercyclic and chaotic semigroups of linear operators, Ergodic Theory Dynam. Systems **17** (1997), 793-819.

[9] H. Emamirad, Hypercyclicity in the scattering theory for linear transport equation, Trans. Amer. Math. Soc. **350** (1998), 3707-3716.

[10] R.M. Gethner and J.H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, Proc. Amer. Math. Soc. **100** (1987), 281-288.

[11] G. Godefroy and J.H. Shapiro, Operators with dense invariant cyclic vector manifolds, J. Funct. Anal. **98** (1991), 229-269.

[12] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. **36** (1999), 345-381.

[13] K.G. Grosse-Erdmann, Recent developments in hypercyclicity, Rev. R. Acad. Cienc. Serie A. Mat. **97** (2003), 273–286.

[14] D.A. Herrero, Hypercyclic operators and chaos, J. Operator Theory **28** (1992), 93-103.

[15] C. Kitai, *Invariant Closed Sets for Linear Operators*, Ph.D. thesis, University of Toronto, 1982.

[16] S. Kolyada and L. Snoha, Some aspects of topological transitivity-A survey, in Collection: Iteration Theory (ECIT 94) (Opava) 3-35. Series: Grazer Math. Ber., 334. Karl Franzens Univ. Graz.

[17] C.R. MacCluer, Chaos in linear distributed systems, J. Dynam. Systems Measurement Control **114** (1992), 322-324.

[18] J.C. Oxtoby and S.M. Ulam, Measure-preserving homeomorphisms and metrical transitivity, Ann. Math. **42** (1941), 874-920.

[19] A. Peris, Hypercyclicity criteria and the Mittag-Leffler theorem, Bull. Soc. Roy. Sci. Liège **70** (2001), 365-371.

[20] V. Protopopescu and Y. Azmy, Topological chaos for a class of linear models, Math. Models Methods Appl. Sci. **2** (1992), 79-90.

Authors' Addresses:
José A. Conejero
Facultat d'Informática,
Departament de Matemàtica Aplicada,
Universitat Politècnica de València, E-46022 València, SPAIN
E-mail: aconejero@mat.upv.es

Alfredo Peris
E.T.S. Arquitectura,
Departament de Matemàtica Aplicada,
Universitat Politècnica de València, E-46022 València, SPAIN
E-mail: aperis@mat.upv.es