Compactness of time-frequency localization operators on $L^2(\mathbb{R}^d)$

Carmen Fernández and Antonio Galbis

Departamento de Análisis Matemático

Universidad de Valencia

Doctor Moliner 50

46100 *Burjasot (Valencia), Spain*

Carmen.Fdez-Rosell@uv.es ; Antonio.Galbis@uv.es

Author to whom mail should be sent:

Antonio Galbis (e-mail: antonio.galbis@uv.es)

Departamento de Análisis Matemático

Universidad de Valencia

C/Dr. Moliner, 50; 46100-Burjasot (Valencia) Spain

Abstract

In this paper, we consider localization operators on $L^2(\mathbb{R}^d)$ defined by symbols in a subclass of the modulation space $M^\infty(\mathbb{R}^{2d})$. We show that these operators are compact and that this subclass is "optimal" for compactness.

Key words: Localization operator, compact operator, Short-time Fourier transform, Modulation space

2000 MSC: 47G30, 47B10, 46F05

[1] The research of the authors was partially supported by MEC and FEDER, Project MTM2004-02262, by MCYT and MURST-MIUR Acción Integrada HI 2003-0066 and by AVCIT Grupos 03/050.

1 Introduction.

The localization operators were introduced by Daubechies [8] in 1988 to localize a signal both in time and frequency. The localization operators are also known in some cases as Toeplitz operators or anti-Wick operators. If $f(t)$ represents a signal, its Fourier transform $\hat{f}(\omega)$ represents the distribution of frequencies ω in the signal, but it does not give any information about "when" these frequencies appear. To overcome this difficulty, following an idea due to Gabor, one can consider a cut-off, or window, function $\varphi(t)$ localized around the origin and to analyze the frequencies around a fixed time x, that is, we consider the Fourier transform of $\overline{\varphi(t-x)}f(t)$ thus obtaining a function both of time and frequency

$$V_\varphi f(x,\omega) = \int f(t)\overline{\varphi(t-x)}e^{-2\pi\imath\omega t}dt,$$

which is called *short time Fourier transform* of f (or STFT) with respect to the window φ.

The signal f can be reconstructed from its STFT by the formula

$$f(t) = \int\int V_\varphi f(x,\omega)\varphi(t-x)e^{2\pi\imath\omega t}dxd\omega$$

provided that $||\varphi||_{L^2} = 1$. It is often convenient, before reconstructing the signal, to modify $V_\varphi f$ by multiplying by a suitable function F. In this way, what we recover is a filtered version of the original signal f,

$$L^F_\varphi f(t) = \int\int F(x,\omega)V_\varphi f(x,\omega)\varphi(t-x)e^{2\pi\imath\omega t}dxd\omega,$$

where the integral is interpreted in a weak sense.

Operators as above are called localization operators with symbol F and window φ. These operators were traditionally defined on L^2 or on some Sobolev spaces, but they can be defined on more general spaces, the so-called modulation spaces. Boundedness and Schatten-class conditions of localization operators have been investigated in [6,7] where it is shown that the sufficient conditions obtained there are, in some sense, optimal. Concerning compactness, as far as we know, only sufficient conditions are given, usually as a consequence of the fact that functions (or distributions) with compact support define compact localization operators (see for instance [2]).

In this paper, we consider localization operators on $L^2(\mathbb{R}^d)$ defined by symbols in a subclass of the modulation space $M^\infty(\mathbb{R}^{2d})$. We show that these operators are compact and that this subclass is "optimal" for compactness. We also recover the necessary conditions in [7] although our methods are completely different from those in [7].

2 Notation and Preliminaries

We use brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$ and parenthesis (f, g) to denote the bilinear form which defines the dual pair $\big(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)\big)$. That is, $\langle f, g\rangle = (f, \overline{g})$.

The modulation and translation operators are defined by

$$M_\omega f(t) = e^{2\pi \imath \omega t} f(t) \text{ and } T_x f(t) = f(t - x).$$

For a non-zero $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, the short-time Fourier transform (STFT) of f with respect to the window φ, is given by

$$V_\varphi f(x,\omega) = \langle f, M_\omega T_x \varphi \rangle \, .$$

Clearly, $V_\varphi f$ is a continuous function on $\mathbb{R}^{2d}$. In what follows, all the windows used in the STFT and in the localization operators are assumed to be in $\mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

Modulation space norms are measures of the time-frequency concentration of a function or distribution. The modulation spaces are defined as follows: Given a non-zero window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$, the space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\varphi f \in L^{p,q}(\mathbb{R}^{2d})$ and it is endowed with the norm

$$||f||_{M^{p,q}} := ||V_\varphi f||_{L^{p,q}} = (\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} |V_\varphi f(x,\omega)|^p dx)^{q/p} d\omega)^{1/q}.$$

For $p = q$, we simply write M^p.

$M^{p,q}(\mathbb{R}^d)$ is a Banach space and its definition is independent of the choice of the window φ . M^2 is exactly L^2, and weighted versions produce, among others, the Sobolev spaces (see [6,9]). In the case $1 \leq p, q < \infty$, the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ and its dual can be identified with $M^{p',q'}(\mathbb{R}^d)$ for $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. We refer to [9] for the necessary background on modulation spaces.

The following spaces are defined and studied en [1]: We denote by $M^{0,0}(\mathbb{R}^d)$ or simply $M^0(\mathbb{R}^d)$ the closed subspace of $M^\infty(\mathbb{R}^d)$ consisting of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\varphi f$ vanishes at infinity, and we denote by $M^{0,q}(\mathbb{R}^d)$ and $M^{p,0}(\mathbb{R}^d)$, for $p, q < \infty$ the closed normed subspaces of $M^{\infty,q}(\mathbb{R}^d)$ and $M^{p,\infty}(\mathbb{R}^d)$ respectively defined by

$$M^{0,q}(\mathbb{R}^d) := M^{0,0}(\mathbb{R}^d) \cap M^{\infty,q}(\mathbb{R}^d)$$

and

$$M^{p,0}(\mathbb{R}^d) := M^{0,0}(\mathbb{R}^d) \cap M^{p,\infty}(\mathbb{R}^d).$$

The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^0(\mathbb{R}^d)$, $M^{0,q}(\mathbb{R}^d)$ and $M^{p,0}(\mathbb{R}^d)$ and the following dualities hold for $1 \leq p, q < \infty$:

$$(M^{p,0}(\mathbb{R}^d))^* = M^{p',1},\ (M^{0,q}(\mathbb{R}^d))^* = M^{1,q'}(\mathbb{R}^d) \text{ and } (M^{0,0}(\mathbb{R}^d))^* = M^{1,1}(\mathbb{R}^d).$$

Given $F \in \mathcal{S}(\mathbb{R}^{2d})$ and $\varphi, \psi, g \in \mathcal{S}(\mathbb{R}^d)$ $(\varphi, \psi \neq 0)$ we define the localization operator $L^F_{\varphi,\psi}$ as

$$(L^F_{\varphi,\psi})(g)(t) := \int_{\mathbb{R}^{2d}} F(x,\omega) V_\varphi g(x,\omega) e^{2\pi i t\omega} \psi(t-x)\, dx d\omega.$$

Then, for every $f \in \mathcal{S}(\mathbb{R}^d)$ we obtain

$$\int (L^F_{\varphi,\psi})(g)(t)\overline{f(t)}dt = \int_{\mathbb{R}^{2d}} F(x,\omega) V_\varphi g(x,\omega)\overline{V_\psi f(x,\omega)}\, dx d\omega,$$

that is,

$$\left\langle (L^F_{\varphi,\psi})(g), f \right\rangle = \left(F, V_\varphi g \overline{V_\psi f} \right).$$

The identity above permits to consider localization operators defined by symbols $F \in \mathcal{S}'(\mathbb{R}^{2d})$ and windows $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Then $L^F_{\varphi,\psi}$ is a continuous and linear operator $L^F_{\varphi,\psi} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.

For $F \equiv 1$ and provided that $\langle \varphi, \psi \rangle = 1$ the localization operator is just the identity [9, 3.2.3]. In general, localization operators act as follows: given a signal f, with $V_\varphi f$ one identifies the frequencies of the signal present in small intervals, and, after filtering through the symbol F, reconstruct a filtered signal. In case φ and ψ are equal to the gaussian $2^{\frac{d}{4}} \mathrm{e}^{-\pi x^2}$, the operator $L^F_{\varphi,\psi}$ is called anti-Wick operator. Localization operators are special pseudodifferential operators. In fact, $L^F_{\varphi,\psi}$ can also be interpreted as a Weyl operator L_σ with Weyl symbol $\sigma = F * W(\psi,\varphi)$ where $W(\psi,\varphi)(x,\omega) = \int \psi(x+\frac{t}{2})\overline{\varphi(x-\frac{t}{2})}e^{-2\pi\imath\omega t}dt$ is the Wigner transform of $\psi,\varphi \in \mathcal{S}(\mathbb{R}^d)$, which belongs to $\mathcal{S}(\mathbb{R}^{2d})$ and $L_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is given by ([4])

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g,f)\rangle \,.$$

By $L(L^2(\mathbb{R}^d)) = L(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ we denote the space of all continuous linear operators on the Hilbert space $L^2(\mathbb{R}^d)$ and $K(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)) = K(L^2(\mathbb{R}^d))$ is the space of all compact linear operators on $L^2(\mathbb{R}^d)$. For every $T \in K(L^2(\mathbb{R}^d))$ and all $n \in \mathbb{N}_0$ the singular numbers $s_n(T)$ of T have the following interpretation (see for instance [12, 16.5])

$$s_n(T) = \inf\{||T-B|| : B \in L(L^2(\mathbb{R}^d)) \text{ and } \dim \operatorname{Im}(B) \leq n\}.$$

The sequence $(s_n(T))_n$ is a null sequence. For $1 \leq p < \infty$, the Schatten class S_p is the space of compact operators with singular values in ℓ_p. For $p = 1$ we get the space of trace class or nuclear operators on $L^2(\mathbb{R}^d)$.

The trace of a nuclear operator $T \in L(L^2(\mathbb{R}^d))$ representable as $T = \sum_{n=1}^{\infty} \langle \cdot, g_n\rangle f_n$, where $\sum_{n=1}^{\infty} ||f_n||_2 ||g_n||_2 < +\infty$, is given by

$$\text{Trace } T := \sum_{n=1}^{\infty} \langle f_n, g_n\rangle \,.$$

The dual of the space $K(L^2, L^2)$ of compact operators can be identified with the space S_1 of nuclear operators on L^2 via trace duality:

$$S_1 \to \left(K(L^2, L^2)\right)^*, \quad R \mapsto (T \mapsto \text{Trace}(T \circ R))\,.$$

Also the dual of S_1 is identified with the space of all bounded operators. In general, $(S_p)^*$ $(1 < p < \infty)$ [11, 20.2.6] is isomorphic to $S_{p'}$, where p and p' are conjugate, via trace duality. We refer to [12] for standard notation concerning Banach spaces.

3 Results

Sufficient conditions for compactness of localization operators or pseudodifferential operators are given for instance in [1–3,5], but to our knowledge no characterization of compact localization operators is available in the literature. In this section we introduce a subclass of symbols and we show that each localization operator with symbol in this class and windows in $\mathcal{S}(\mathbb{R}^d)$ belongs to $K(L^2(\mathbb{R}^d))$. Conversely, we will show that this is the "optimal" class for compactness of the localization operator on $L^2(\mathbb{R}^d)$.

In what follows, we fix two windows $\varphi, \psi, \in \mathcal{S}(\mathbb{R}^d)$ and we assume $F \in M^\infty(\mathbb{R}^{2d})$.

The following result is essentially known ([2]), but we include a proof for the sake of completeness.

Proposition 3.1 *Let E and F be two Banach spaces and let $T : E \to F$ be a continuous linear map admitting the following factorization $T = Q \circ S_1$ where $Q : \mathcal{S}(\mathbb{R}^d) \to F$ and $S_1 : E \to \mathcal{S}(\mathbb{R}^d)$ are continuous and linear. Then T is a compact operator.*

Proof: Let us denote by B_E the closed unit ball of E. Then $S_1(B_E)$ is bounded in $\mathcal{S}(\mathbb{R}^d)$, hence it is relatively compact, since $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet-Montel space ([14, p.235]). Now, the continuity of Q shows that $T(B_E)$ is relatively compact. ■

Corollary 3.2 *If $F \in \mathcal{S}(\mathbb{R}^{2d})$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, then the operator $L^F_{\varphi,\psi}$ is compact.*

Proof: It is clear that $L^F_{\varphi,\psi}(f) \in \mathcal{S}(\mathbb{R}^d)$ for each $f \in L^2(\mathbb{R}^d)$. Thus, it can be factorized as above. ∎

We recall the following result, which explains why $M^\infty(\mathbb{R}^{2d})$ is the optimal class for boundedness of the localization operators on $L^2(\mathbb{R}^d)$.

Theorem 3.3 ([6]) *If $F \in M^\infty(\mathbb{R}^{2d})$ then $L^F_{\varphi,\psi}$ can be extended to a bounded operator*

$$L^F_{\varphi,\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

with norm $||L^F_{\varphi,\psi}|| \leq C||F||_{M^\infty}||\varphi||_{M^1}||\psi||_{M^1}$ and for some constant $C > 0$.

Conversely, if $F \in \mathcal{S}'(\mathbb{R}^{2d})$ and $L^F_{\varphi,\psi}$ is a bounded operator on $L^2(\mathbb{R}^d)$ for each pair of windows $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ with norm estimate

$$||L^F_{\varphi,\psi}|| \leq C||\varphi||_{M^1}||\psi||_{M^1}$$

for some constant C depending on F but not on the windows, then $F \in M^\infty(\mathbb{R}^{2d})$.

Then, fixing the windows φ, ψ, the previous theorem gives a continuous linear map

$$T_{\varphi,\psi} : M^\infty(\mathbb{R}^{2d}) \to L(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$$

$$F \to L^F_{\varphi,\psi}$$

The basic idea to get compactness is as follows: Localization operators with symbols and also windows in the Schwartz class are compact and, since compact operators are a closed subspace of the space of all bounded operators and the Schwartz class is dense in M^0, localization operators with symbols in M^0 are also compact. However, if we take $F = \delta$ then the corresponding localization operator is compact (it is a rank one operator) although F is not in M^0 since for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have $V_\varphi \delta(x,\omega) = \varphi(x)$, which does not tend to zero when x is fixed and $|\omega|$ goes to infinity. Note that the previous statement that $\delta \notin M^0$ does not contradicts the fact that δ is the weak limit in $\mathcal{E}'(\mathbb{R}^d)$ of a sequence of test functions, since $\mathcal{E}'(\mathbb{R}^d)$ is not contained in M^∞ (see for instance the calculations in [6, Proposition 3.6]).

It follows that the operators given by symbols in M^0 do not exhaust the class of compact localization operators. The example just mentioned suggest the condition in our next results.

Lemma 3.4 *Let $F \in M^\infty(\mathbb{R}^d)$ and $g_0 \in \mathcal{S}(\mathbb{R}^d)$ be given with the property that $\lim_{|x|\to\infty} \sup_{|\xi|\le R} |V_{g_0}F(x,\xi)| = 0$ for every $R > 0$. Then $F * H \in M^{0,1}(\mathbb{R}^d)$ for any $H \in \mathcal{S}(\mathbb{R}^d)$.*

To show this lemma we recall the following convolution relation

Lemma 3.5 [6, 2.4] *Let $g_0 \in \mathcal{S}(\mathbb{R}^d)$ and $g := g_0 * g_0$ be given. Let us assume $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{r}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{s}$. Then, for any $f \in M^{p_1,q_1}(\mathbb{R}^d)$ and $h \in M^{p_2,q_2}(\mathbb{R}^d)$, we have $f * h \in M^{r,s}(\mathbb{R}^d)$ and*

$$V_g(f * h)(\cdot,\omega) = V_{g_0} f(\cdot,\omega) * V_{g_0} h(\cdot,\omega).$$

Proof of Lemma 3.4: Since $V_{g_0}H \in \mathcal{S}(\mathbb{R}^{2d})$ then for every $N \in \mathbb{N}$ there is $C_N > 0$ such that

$$|V_{g_0}H(t,\omega)| \le C_N (1 + |\omega|)^{-2N} (1 + |t|)^{-N}.$$

Consequently, for $g = g_0 * g_0$, we obtain

$$|V_g(F * H)(x,\omega)| = |\left(V_{g_0}F(\cdot,\omega) * V_{g_0}H(\cdot,\omega)\right)(x)|$$
$$\leq C_N(1+|\omega|)^{-2N} \int_{\mathbb{R}^d} |V_{g_0}F(x-t,\omega)| \frac{1}{(1+|t|)^N} dt.$$

We take N large enough so that $\int_{\mathbb{R}^d} \frac{dt}{(1+|t|)^N} < +\infty$ (for instance, $N = d+1$).

Now, given $\epsilon > 0$ there is $R_1 > 0$ such that $|\omega| \geq R_1$ implies

$$(1+|\omega|)^{-N} C_N ||F||_{M^\infty} \int_{\mathbb{R}^d} \frac{dt}{(1+|t|)^N} \leq \epsilon.$$

Then

$$(1+|\omega|)^N |V_g(F * H)(x,\omega)| \leq \epsilon$$

for all $x \in \mathbb{R}^d$ and $|\omega| \geq R_1$. We now choose $\delta > 0$ small enough so that

$$\delta + C_N \delta \int_{\mathbb{R}^d} \frac{dt}{(1+|t|)^N} \leq \epsilon.$$

Since $\lim_{|x|\to\infty} \sup_{|\xi|\leq R_1} |V_{g_0}F(x,\xi)| = 0$ then there is $R_2 > 0$ such that

$$\int_{|t|>R_2} \frac{dt}{(1+|t|)^N} \leq \frac{\delta}{C_N ||F||_{M^\infty}}$$

and $|V_{g_0}F(x,\omega)| \leq \delta$ whenever $|x| \geq R_2$ and $|\omega| \leq R_1$. Consequently, for $|x| \geq 2R_2$ and $|\omega| \leq R_1$, we obtain that

$$(1+|\omega|)^N |V_g(F * H)(x,\omega)|$$

is less than or equal to

$$\delta + C_N (1+|\omega|)^{-N} \int_{|t|\leq R_2} |V_{g_0}F(x-t,\omega)| \frac{1}{(1+|t|)^N} dt$$

$$\leq \delta + C_N \delta \int\limits_{\mathbb{R}^d} \frac{dt}{(1+|t|)^N}$$

$$\leq \epsilon.$$

Hence, $|x| \geq 2R_2$ gives

$$(1+|\omega|)^N |V_g(F*H)(x,\omega)| \leq \epsilon$$

for all $\omega \in \mathbb{R}^d$. That is,

$$\int\limits_{\mathbb{R}^d} |V_g(F*H)(x,\omega)| d\omega \leq \epsilon \int\limits_{\mathbb{R}^d} \frac{d\omega}{(1+|\omega|)^N}$$

for all $|x| \geq 2R_2$. We conclude that

$$\lim_{|x|\to\infty} ||V_g(F*H)(x,\cdot)||_{L^1} = 0$$

which implies that $F*H \in M^{\infty,1}(\mathbb{R}^d)$. Moreover, the previous estimates give that $V_g(F*H)$ vanishes at infinity and $F*H \in M^{0,1}$ ([1, def. 2.1]). ■

We observe that lemma 3.4 does not hold for arbitrary $F \in M^\infty$. In fact, if we consider $F \equiv 1 \in M^\infty$ and $H \in S(\mathbb{R}^d)$ with $\int H = 1$ then $F*H$ is constant equal to 1 and

$$V_g(F*H)(x,\omega) = e^{-2\pi i \omega x}\hat{\overline{g}}(\omega).$$

That is, for a fixed ω, the function of x, $|V_g(F*H)(x,\omega)|$ is a constant different from zero and, consequently, $F*H \notin M^0$.

The sufficient conditions in [3, 4.6,4.7] in the case $p = q = 2$ are particular cases of our following result. In the proof we make use of the representation of localization operators as Weyl operators.

Proposition 3.6 *Let $g \in \mathcal{S}(\mathbb{R}^{2d})$ be given. If $\lim\limits_{|x|\to\infty} \sup\limits_{|\xi|\leq R} |V_g F(x,\xi)| = 0$ for* *every $R > 0$, then $L^F_{\varphi,\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator.*

Proof: We put $H := W(\psi, \varphi) \in \mathcal{S}(\mathbb{R}^{2d})$. Then $\sigma := F * H \in M^{0,1}(\mathbb{R}^{2d})$ and $L^F_{\varphi,\psi}$ is compact since it coincides with the Weyl operator L_σ and we use [1, 2.3]. ■

Theorem 3.7 [6, 3.4] *The map $T_{\varphi,\psi}$ restricted to $M^1(\mathbb{R}^{2d})$ takes values in the space of trace class operators and*

$$T_{\varphi,\psi} : M^1(\mathbb{R}^{2d}) \to S_1 \ , F \mapsto L^F_{\varphi,\psi},$$

is continuous.

Now, we look at the transpose map $T^t_{\varphi,\psi} : L\left(L^2(\mathbb{R}^d)\right) \to M^\infty(\mathbb{R}^{2d})$, which assigns to every bounded linear operator on $L^2(\mathbb{R}^d)$ a symbol in $M^\infty(\mathbb{R}^{2d})$. We are particularly interested in the action of $T^t_{\varphi,\psi}$ on compact operators and on localization operators.

Since compact linear operators on $L^2(\mathbb{R}^d)$ can be approximated by finite rank operators, we get

Lemma 3.8 *$T^t_{\varphi,\psi}\left(K(L^2, L^2)\right)$ is contained in $M^0(\mathbb{R}^{2d})$.*

Proof: Since $\left(M^1(\mathbb{R}^{2d})\right)^* = M^\infty(\mathbb{R}^{2d})$ and M^0 is a closed subspace of M^∞ we only have to prove that $T^t_{\varphi,\psi}(S) \in M^0$ for any finite rank operator $S : L^2 \to L^2$. To do this, we fix a finite rank operator

$$S := \sum_{k=1}^{N} \langle \cdot, g_k \rangle f_k$$

where $f_k, g_k \in L^2$. Then, for any $F \in M^1(\mathbb{R}^{2d})$ we have

$$\left(T^t_{\varphi,\psi}(S), F\right) = (S, T_{\varphi,\psi}(F)) = \mathrm{Trace}(S \circ L^F_{\varphi,\psi}).$$

Since the adjoint operator of $L^F_{\varphi,\psi}$ is $L^{\overline{F}}_{\psi,\varphi}$, then

$$S \circ L^F_{\varphi,\psi} = \sum_{k=1}^{N} \left\langle \cdot, L^{\overline{F}}_{\psi,\varphi} g_k \right\rangle f_k$$

and

$$\left(T^t_{\varphi,\psi}(S), F\right) = \sum_{k=1}^{N} \left\langle f_k, L^{\overline{F}}_{\psi,\varphi} g_k \right\rangle = \sum_{k=1}^{N} \int_{\mathbb{R}^d} f_k \overline{L^{\overline{F}}_{\psi,\varphi} g_k}.$$

On the other hand, for $F \in \mathcal{S}(\mathbb{R}^{2d})$ we have

$$\int_{\mathbb{R}^d} f_k \overline{L^{\overline{F}}_{\psi,\varphi} g_k} = \int_{\mathbb{R}^{2d}} F(x,\omega) \overline{V_\psi g_k(x,\omega)} V_\varphi f_k(x,\omega) dx d\omega.$$

Hence

$$T^t_{\varphi,\psi}(S) = \sum_{k=1}^{N} \overline{V_\psi g_k} \cdot V_\varphi f_k,$$

which is an element of $L^1(\mathbb{R}^{2d}) \subset M^0(\mathbb{R}^{2d})$. The proof is finished. ∎

Our aim now is to find the relation between the symbol $F \in M^\infty(\mathbb{R}^{2d})$ of the localization operator $L^F_{\varphi,\psi}$ and the symbol we obtain as the image of $L^F_{\varphi,\psi}$ by $T^t_{\varphi,\psi}$. In the next lemma, (x_j, ω_j) are points in the partition we get when we divide the cube $[a,b]^{2d}$ into k^{2d} equal cubes.

Lemma 3.9 *Let $F \in \mathcal{D}(\mathbb{R}^{2d})$ be a test function with support contained in $[a,b]^{2d}$. Then $L^F_{\varphi,\psi}$ is the limit in the space $L\left(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)\right)$ of the sequence of finite rank operators*

$$S_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

defined by

$$S_k := (\frac{b-a}{k})^{2d} \sum_{j=1}^{k^{2d}} F(x_j, \omega_j) \left\langle \cdot, M_{\omega_j} T_{x_j} \varphi \right\rangle M_{\omega_j} T_{x_j} \psi.$$

Proof: In fact, since for every $f \in L^2(\mathbb{R}^d)$ the map $\mathbb{R}^{2d} \to L^2(\mathbb{R}^d)$ given by

$(x,\omega) \mapsto M_\omega T_x f$ is continuous, we conclude that

$$(x,\omega) \mapsto F(x,\omega) \langle \cdot, M_\omega T_x \varphi \rangle M_\omega T_x \psi$$

is a uniformly continuous map

$$\Phi : [a,b]^{2d} \to L_b \left(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d) \right).$$

Then $L^F_{\varphi,\psi} : L^2 \to L^2$ is the vector integral

$$L^F_{\varphi,\psi} = \int_{[a,b]^{2d}} \Phi(x,\omega) dx d\omega,$$

and the conclusion follows since each S_k is a Riemann sum of the integral. ∎

Proposition 3.10 *For every $F \in M^\infty(\mathbb{R}^{2d})$ we have*

$$T^t_{\varphi,\psi} \left(L^F_{\varphi,\psi} \right) = F * \left(\overline{V_\psi \varphi} V_\varphi \psi \right).$$

Proof: In fact, let us first assume that $F \in \mathcal{D}(\mathbb{R}^{2d})$ is a test function with support contained in $[a,b]^{2d}$. Then

$$T^t_{\varphi,\psi}(L^F_{\varphi,\psi}) = \lim_{k\to\infty} T^t_{\varphi,\psi}(S_k).$$

According to lemma 3.8 and after applying ([9, 3.1.3]) we get

$$T^t_{\varphi,\psi}(S_k) = (\frac{b-a}{k})^{2d} \sum_{j=1}^{k^{2d}} F(x_j,\omega_j) \overline{V_\psi \left(M_{\omega_j} T_{x_j} \varphi \right)} V_\varphi \left(M_{\omega_j} T_{x_j} \psi \right)$$

$$= (\frac{b-a}{k})^{2d} \sum_{j=1}^{k^{2d}} F(x_j,\omega_j) \left(\overline{V_\psi \varphi} V_\varphi \psi \right) (x - x_j, \omega - \omega_j).$$

Consequently $T^t_{\varphi,\psi}(L^F_{\varphi,\psi})(x,\omega)$ is given by

$$\lim_{k\to\infty}(\frac{b-a}{k})^{2d}\sum_{j=1}^{k^{2d}} F(x_j,\omega_j)\left(\overline{V_\psi\varphi}V_\varphi\psi\right)(x-x_j,\omega-\omega_j)$$

$$= \int_{\mathbb{R}^{2d}} F(y,\varsigma)\left(\overline{V_\psi\varphi}V_\varphi\psi\right)(x-y,\omega-\varsigma)dyd\varsigma$$

$$= F*\left(\overline{V_\psi\varphi}V_\varphi\psi\right)(x,\omega).$$

To finish we consider an arbitrary $F\in M^\infty(\mathbb{R}^{2d})$. Since $M^\infty(\mathbb{R}^{2d})$ is the bidual of $M^0(\mathbb{R}^{2d})$ and $\mathcal{D}(\mathbb{R}^{2d})$ is a dense subspace of $M^0(\mathbb{R}^{2d})$ we can find a net (F_j) consisting of test functions which converges to F in the weak $*$ topology $\sigma(M^\infty, M^1)$.

We now show that $L^F_{\varphi,\psi}$ is the $\sigma(L(L^2,L^2),S_1)$-limit of the net $L^{F_j}_{\varphi,\psi}$. To do this we fix a nuclear operator

$$S := \sum_{n=1}^{\infty}\langle\cdot,g_n\rangle f_n$$

where $\sum_{n=1}^\infty ||f_n||_{L^2}||g_n||_{L^2}<+\infty$. Then

$$\left(L^F_{\varphi,\psi}-L^{F_j}_{\varphi,\psi},S\right) = \text{Trace}\left(L^{F-F_j}_{\varphi,\psi}\circ S\right)$$

$$= \sum_{n=1}^{\infty}\left\langle F-F_j,\overline{V_\psi g_n}V_\varphi f_n\right\rangle.$$

Since the series $h := \sum_{n=1}^{\infty}\overline{V_\psi g_n}V_\varphi f_n$ is absolutely converging in $M^1(\mathbb{R}^{2d})$ ([6, 5.2]) we conclude that

$$\left(L^F_{\varphi,\psi} - L^{F_j}_{\varphi,\psi}, S\right) = \langle F - F_j, h \rangle \,,$$

hence

$$\lim_j \left(L^F_{\varphi,\psi} - L^{F_j}_{\varphi,\psi}, S\right) = 0.$$

Consequently

$$T^t_{\varphi,\psi}\left(L^F_{\varphi,\psi}\right) = \sigma(M^\infty, M^1) - \lim_j T^t_{\varphi,\psi}\left(L^{F_j}_{\varphi,\psi}\right)$$

$$= \sigma(M^\infty, M^1) - \lim_j \left(F_j * \left(\overline{V_\psi \varphi} V_\varphi \psi\right)\right).$$

In particular, for every $g \in \mathcal{S}(\mathbb{R}^{2d})$ we have

$$\left(T^t_{\varphi,\psi}\left(L^F_{\varphi,\psi}\right), g\right) = \lim_j \left(F_j * \left(\overline{V_\psi \varphi} V_\varphi \psi\right), g\right)$$

$$= \left(F * \left(\overline{V_\psi \varphi} V_\varphi \psi\right), g\right)$$

since $\left(\overline{V_\psi \varphi} V_\varphi \psi\right)\check{} * g \in \mathcal{S}(\mathbb{R}^{2d})$ and (F_j) converges to F weakly in $\mathcal{S}'(\mathbb{R}^{2d})$. ∎

Proposition 3.11 *Let $F \in M^\infty(\mathbb{R}^{2d})$ be a symbol such that*

$$L^F_{\varphi,\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is a compact operator. Then

$$F * \left(\overline{V_\psi \varphi} V_\varphi \psi\right) \in M^0(\mathbb{R}^{2d}).$$

This necessary condition for compactness gives some information as the following example shows.

Example 3.12 *Let $F \in M^\infty(\mathbb{R}^{2d})$, $F \neq 0$, be a periodic distribution. Then $L^F_{\varphi,\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is not a compact operator.*

To obtain a complete characterization of compact localization operators in terms of the symbol we need the following result on collectively compact sets of operators.

Theorem 3.13 [13] *Let E, F be Banach spaces and $\mathcal{K} \subset K(E,F)$ a set of compact operators. Then, $\mathcal{K}$ is relatively compact if, and only if, the following two conditions are satisfied:*

(i) $\bigcup (R(B_E) : R \in \mathcal{K})$ is relatively compact in F.

(ii) For every $u \in F'$, the set $\{u \circ R : R \in \mathcal{K}\}$ is relatively compact in E^*.

Lemma 3.14 *Let $F \in M^\infty(\mathbb{R}^{2d})$ be a symbol and let us assume that $L^F_{\varphi,\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a compact operator for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Then, for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ and $R > 0$ the set*

$$\mathcal{K} := \{L^{M_\omega F}_{\varphi,\psi} : |\omega| \leq R\}$$

is relatively compact in $K(L^2, L^2)$.

Proof: According to [6, p.126], for $\omega = (\omega_1, \omega_2) \in \mathbb{R}^{2d}$ and $f \in L^2(\mathbb{R}^d)$ we have

$$M_\omega V_\varphi f = V_{M_{\omega_1} T_{\omega_2} \varphi} \left(M_{\omega_1} T_{-\omega_2} f \right).$$

Hence

$$L^{M_\omega F}_{\varphi,\psi} f = L^F_{M_{\omega_1} T_{\omega_2} \varphi, \psi} \left(M_{\omega_1} T_{-\omega_2} f \right).$$

In particular, each $L^{M_\omega F}_{\varphi,\psi}$ is a compact operator. Since the map

$$\mathbb{R}^{2d} \to \mathcal{S}(\mathbb{R}^d) \ , \ \omega \mapsto M_{\omega_1} T_{\omega_2} \varphi,$$

is continuous ([9, 11.2.2]) and $\mathcal{S}(\mathbb{R}^d)$ is contained in $M^1(\mathbb{R}^d)$ with continuous inclusion, we can conclude, after applying [6, 3.2], that the set

$$\mathcal{K}_1 := \{L^F_{M_{\omega_1}T_{\omega_2}\varphi,\psi} \ : |\omega| \leq R\}$$

is a compact subset of $K(L^2, L^2)$. We can now apply 3.13 to show that $\mathcal{K}$ is relatively compact in $K(L^2, L^2)$. In fact, (i) since M_{ω_1} and $T_{-\omega_2}$ are isometries on $L^2(\mathbb{R}^d)$ we get that

$$\{M_{\omega_1}T_{-\omega_2}f : ||f||_{L^2} \leq 1 \ , |\omega| \leq R\}$$

is a subset of the unit ball of $L^2(\mathbb{R}^d)$. Since $\mathcal{K}_1$ is a compact set of compact operators we deduce that

$$\mathcal{K} = \{L^F_{M_{\omega_1}T_{\omega_2}\varphi,\psi} \circ (M_{\omega_1}T_{-\omega_2})\}$$

transforms the unit ball into a relatively compact subset of $L^2(\mathbb{R}^d)$. Moreover,

(ii) the adjoint map of $L^{M_\omega F}_{\varphi,\psi}$ is given by $L^{M_{-\omega}\overline{F}}_{\psi,\varphi}$ and, for any $f \in L^2(\mathbb{R}^d)$, we can proceed as in (i) to conclude that the set

$$\{L^{M_{-\omega}\overline{F}}_{\psi,\varphi} f \ : \ |\omega| \leq R\}$$

is relatively compact in $L^2(\mathbb{R}^d)$. ■

Theorem 3.15 *Let $F \in M^\infty(\mathbb{R}^{2d})$ and $g_0 \in \mathcal{S}(\mathbb{R}^{2d})$ be given. Then, the following conditions are equivalent:*

(a) The localization operator $L^F_{\varphi,\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is compact for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$.

(b) For every $R > 0$ we have

$$\lim_{|x|\to\infty} \sup_{|\xi|\leq R} |V_{g_0}F(x,\xi)| = 0.$$

Proof: (b) implies (a) is the content of proposition 3.6. We proceed to prove that (a) implies (b). To do this, we fix a non zero and even window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and we take $\psi = \varphi$ and $\Phi := \overline{V_\psi \varphi} V_\varphi \psi = |V_\varphi \varphi|^2 \in \mathcal{S}(\mathbb{R}^{2d})$. Since

$$V_{\Phi * \Phi} F(x, \omega) = \langle F, M_\omega T_x(\Phi * \Phi) \rangle = \langle M_{-\omega} F, \Phi * T_x \Phi \rangle$$

$$= \langle M_{-\omega} F * \Phi, T_x \Phi \rangle = V_\Phi \left(M_{-\omega} F * \Phi \right)(x, 0)$$

and

$$\mathcal{K} := \{ L_{\varphi,\varphi}^{M_{-\omega} F} : |\omega| \leq R \}$$

is a relatively compact subset of compact operators then we conclude that

$$T_{\varphi,\varphi}^t(\mathcal{K}) = \{ M_{-\omega} F * \Phi : |\omega| \leq R \}$$

is a relatively compact subset of $M^0(\mathbb{R}^{2d})$. Since $\{ V_\Phi \left(M_{-\omega} F * \Phi \right) : |\omega| \leq R \}$ is relatively compact in the Banach space $C_0(\mathbb{R}^{2d})$ consisting of those continuous functions vanishing at infinity, then

$$\lim_{|x| \to \infty} \sup_{|\omega| \leq R} |V_{\Phi * \Phi} F(x, \omega)| = \lim_{|x| \to \infty} \sup_{|\omega| \leq R} |V_\Phi \left(M_{-\omega} F * \Phi \right)(x, 0)| = 0.$$

Finally, it follows from the inequality [9, 11.3.3]

$$|V_{g_0} F(x, \omega)| \leq ||\gamma||_{L^2}^{-1} \left(|V_\gamma F| * |V_{g_0} \gamma| \right)(x, \omega)$$

for $\gamma = \Phi * \Phi$, that condition (b) holds. ∎

Remark 3.16 Our methods permit to recover the necessary Schatten class conditions in [7] as follows. Given two windows φ, ψ in the Schwartz class on $\mathbb{R}^d$ and for $1 < p < \infty$ or $p = 0$ the map

$$T_{\varphi,\psi}^p : M^p(\mathbb{R}^{2d}) \to S_p \ , F \mapsto L_{\varphi,\psi}^F,$$

(here $S_0 := K(L^2, L^2)$) is well-defined and continuous, and in fact we have the norm estimate

$$||L^F_{\varphi,\psi}||_{S_p} \leq B||F||_{M^p}||\varphi||_{M^1}||\psi||_{M^1}$$

For $p \neq 0$ this result is part of [7, Th 1]. Clearly $T^p_{\varphi,\psi}|_{M^1(\mathbb{R}^{2d})}$ coincides with $T_{\varphi,\psi}$ in Theorem 3.7. Now we use that, for $1 < p < \infty$, $(S_p)^*$ is isomorphic to $S_{p'}$ via trace duality, therefore

$$(T^p_{\varphi,\psi})^t : S_{p'} \to M^{p'}(\mathbb{R}^{2d})$$

is also linear and continuous and since finite rank operators are dense in $S_{p'}$ and the Schwartz class is dense in $M^{p'}(\mathbb{R}^{2d})$ we can easily conclude that $(T^p_{\varphi,\psi})^t = T^t_{\varphi,\psi}|S_{p'}$. Consequently, if $M \in M^\infty(\mathbb{R}^{2d})$ and the corresponding localization operator $L^M_{\varphi,\psi} \in S_{p'}$ we have $F * \Phi \in M^{p'}(\mathbb{R}^{2d})$ for $\Phi = \overline{V_\psi\varphi}V_\varphi\psi$. Let us now assume that $L^F_{\varphi,\psi} \in S_{p'}$ for every pair of windows φ, ψ in the Schwartz class with norm estimate

$$||L^F_{\varphi,\psi}||_{S_p} \leq B||F||_{M^p}||\varphi||_{M^1}||\psi||_{M^1}$$

where the constant B may depend on F but it is independent on the windows. Then, from $L^{M_\omega F}_{\varphi,\psi} f = L^F_{M_{\omega_1}T_{\omega_2}\varphi,\psi}(M_{\omega_1}T_{-\omega_2}f)$ we deduce that $\{L^{M_\omega F}_{\varphi,\psi} : \omega \in \mathbb{R}^{2d}\}$ is a bounded subset of $S_{p'}$, hence $\{M_\omega F * \Phi : \omega \in \mathbb{R}^{2d}\}$ is bounded in $M^{p'}(\mathbb{R}^{2d})$ and a fortiori in $M^{p',\infty}(\mathbb{R}^{2d})$. Now, we may proceed as in the previous theorem to conclude that $F \in M^{p',\infty}(\mathbb{R}^{2d})$.

Acknowledgement. The authors are indebted to Paolo Boggiatto for bringing their attention to this topic.

References

[1] A. Bényi, K. Gröchenig, C. Heil, K. Okoudjou; *Modulation spaces and a class of multilinear pseudodifferential operators*, J. Operator Theory, to appear.

[2] P. Boggiatto; *Localization operators with L^p symbols on modulation spaces*, Advances in Pseudodifferential Operators, Operator Theory: Advances and Appl., Birkhäuser, Proc. ISAAC Congress, 2003.

[3] P. Boggiatto; *Boundedness and compactness of localization operators on modulation spaces*, Quaderni del Departamento di matematica, N. 15 (2004).

[4] P. Boggiatto, E. Cordero, K. Gröchenig; *Generalized Anti-Wick operators with symbols in distributional Sobolev spaces*, Integr. equ. oper. theory **48** (2004), 427-442.

[5] P. Boggiatto, A. Oliaro, M. W. Wong; *Some results of L^p boundedness and compactness for localization operators*, Preprint (2004).

[6] E. Cordero, K. Gröchenig; *Time-Frequency analysis of localization operators*, Journal of Functional Analysis, **205** (2003), 107-131.

[7] E. Cordero, K. Gröchenig; *Necessary conditions for Schatten class localization operators*, PAMS (To appear).

[8] I. Daubechies; *Time-frequency Localization Operators: a Geometric Phase Space Approach*, IEEE Trans. Inform. Theory, **34**(4) (1988), 605-612.

[9] K. Gröchenig; Foundations of Time-Frequency Analysis, *Birkhäuser* (2001).

[10] K. Gröchenig, C. Heil; *Modulation spaces and pseudodifferential operators*, Integral Equations Operator Theory, **34** (1999), 439-457.

[11] H. Jarchow; Locally convex spaces, *B. G. Teubner* (1981).

[12] R. Meise, D. Vogt; Introduction to Functional Analysis, *Clarendon Press, Oxford* (1997).

[13] W. Ruess; *Compactness and collective compactness in spaces of compact operators*, J. Math. Anal. Appl. **84** (2) (1981), 400-417.

[14] L. Schwartz; Théorie des distributions, *Hermann Paris* (1978).