PSEUDODIFFERENTIAL OPERATORS ON NON QUASIANALYTIC CLASSES OF BEURLING TYPE

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ABSTRACT. In this paper we introduce pseudodifferential operators (of infinite order) in the frame of non quasianalytic classes of Beurling type. We prove that such an operator with (distributional) kernel in a given Beurling class $\mathcal{D}'_{(\omega)}$ is pseudo-local and can be locally decomposed, modulo a smoothing operator, as the composition of a pseudodifferential operator of finite order and an ultradifferential operator with constant coefficients in the sense of Komatsu, both of them with kernel in the same class $\mathcal{D}'_{(\omega)}$. We also develop the corresponding symbolic calculus.

0. INTRODUCTION.

The theory of pseudodifferential operators grew out of the study of singular integral operators, and developed after 1965 with the systematic studies of Kohn-Nirenberg [18], Hörmander [14] and others.

The study of several problems in classes of (non-quasianalytic) ultradifferentiable functions has received also recently much attention. These are intermediate classes between real analytic functions and the class of all C^∞-functions. There are essentially two ways to introduce them, the theory of Komatsu [16], in which one looks at the growth of the derivatives on compact sets, and the theory developed by Björk [2] in 1966, following the ideas previously announced by Beurling, in which one pays attention to the growth of the Fourier transforms. We will work with ultradifferentiable functions as defined by Braun, Meise and Taylor [8]. Their point of view permits a unified treatment of both theories, contains the most relevant cases of Komatsu's theory and it is strictly larger than Beurling-Björk's one.

Pseudodifferential operators (of finite or infinite order) on Gevrey classes have been extensively studied by many authors ([5], [6], [15], [20], [25] among others). We refer to [23] for an excellent introduction to this topic. For more general classes of ultradifferentiable functions, following the approach of Komatsu, we refer to [21]. All of them deal with spaces of Roumieu type.

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These are spaces with a topological structure similar to that of the space of real analytic functions.

The purpose of this paper is to introduce pseudodifferential operators (p.d.o.) in the frame of ultradifferentiable functions of Beurling type, that is, spaces whose topology looks like the one of C^∞. Our aim is to establish the basic theory in order to be able to face in the future topics like for instance hypoellipticity, Fourier integral operators, etc. As in [9] the pseudodifferential operators of (ω)-class are defined as limits of operators with kernel in $\mathcal{E}_{(\omega)}(\Omega \times \Omega)$. With this point of view, it is immediate that the class of pseudodifferential operators is closed under taking adjoints and that every p.d.o. of (ω)-class admits a continuous and linear extension $A : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$. We prove that such an operator shrinks (ω)-singular supports (theorem 2.18). Many operators are pseudodifferential operators according to our definition. In particular, we mention the linear partial differential operators with variable coefficients in a suitable class of functions, the (ω)-smoothing operators and the ultradifferential operators in the sense of Komatsu. The convolution operator with an elementary solution of a given elliptic ultradifferential operator with constant coefficients is also a pseudodifferential operator. However not every convolution operator is a p.d.o.

Since the class of p.d.o. has to be also closed under products of operators, and we need to express this property in terms of the symbols, we develop the symbolic calculus.

The class of pseudodifferential operators of (ω)-class contains the (ω)-smoothing operators, operators of finite order and ultradifferential operators of (ω)-class, and, as a consequence of 2.14 and 3.13, every pseudodifferential operator of (ω)-class can be locally expressed, up to a (ω)-smoothing operator, as the composition of an ultradifferential operator of (ω)-class with constant coefficients and a p.d.o. of (ω)-class and finite order. As far as we know there is no similar result in the Gevrey (Roumieu) setting.

1. Notation and Preliminaries

In this section we introduce the classes of functions, the classes of amplitudes/symbols and we establish some preliminary lemmata.

Definition 1.1. ([8]) *A weight function is an increasing continuous function* $\omega : [0, \infty[\to [0, \infty[$ *with the following properties:*

(α) *there exists* $L \geq 0$ *with* $\omega(2t) \leq L(\omega(t) + 1)$ *for all* $t \geq 0$*,*

(β) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$*,*

(γ) $\log(t) = o(\omega(t))$ *as* t *tends to* ∞*,* *(δ)* $\varphi : t \to \omega(e^t)$ *is convex.*

For $z \in \mathbb{C}^p$ we put $\omega(z) := \omega(|z|)$, where $|z| := \sup |z_k|$.
The *Young conjugate* $\varphi^* : [0, \infty[\to \mathbb{R}$ of φ is given by $\varphi^*(s) := \sup\{st - \varphi(t), t \geq 0\}$. Here φ is related to ω via point (δ) of Definition 1.1.

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Then φ^* has only non-negative values, it is convex, $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$ and $\varphi^{**} = \varphi$. We refer to [8] for properties of φ^*. Moreover, we assume that $\log t \leq \omega(t)$ for all $t > 0$.

Definition 1.2. ([8]) *Let* ω *be a weight function. For an open set* $\Omega \subset \mathbb{R}^p$ *we let* $\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : |f|_{K,\lambda} < \infty \quad \mathit{for} \quad \mathit{every} \quad \lambda > 0,$ *and every* $K \subset \Omega$ *compact*$\}$, *where* $|f|_{K,\lambda} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^p} |f^{(\alpha)}(x)| \exp(-\lambda \varphi^*(\frac{|\alpha|}{\lambda}))$.

$\mathcal{E}_{(\omega)}(\Omega)$ carries the metric locally convex topology given by the sequence of seminorms $|f|_{K_n, \lambda_n}$, where (K_n) is any compact exhaustion of Ω and (λ_n) is any increasing and unbounded sequence of positive numbers.

By $\mathcal{D}_{(\omega)}(K)$, $K \subset \Omega$ compact, we denote the collection of all those $f \in \mathcal{E}_{(\omega)}(\Omega) \cap \mathcal{D}(K)$. For $f \in \mathcal{D}_{(\omega)}(K)$ we put $|f|_\lambda := |f|_{K,\lambda}$. Then $\mathcal{D}_{(\omega)}(\Omega) = \mathrm{ind}_n \mathcal{D}_{(\omega)}(K_n)$, where (K_n) is any compact exhaustion of Ω. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ are called ultradistributions of Beurling type.

The space $\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ is the set of all C^∞-functions f on $\mathbb{R}^p$ such that $\| f \|_{1,n} < \infty$ for each $n \in \mathbb{N}$, where

$$\| f \|_{1,n} := \sup_{\alpha \in \mathbb{N}_0^p} \| f^{(\alpha)} \|_{L_1} \exp(-n \varphi^*(\frac{|\alpha|}{n})).$$

The inclusions $\mathcal{D}_{(\omega)}(\mathbb{R}^p) \subset \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^p)$ are continuous and have dense range.

We start with some elementary, but useful, properties of φ^* that follow from the convexity of φ^* and the fact that $\varphi^*(0) = 0$.

Lemma 1.3. (1) *For every* $\lambda, s, t > 0$ *we have*

$$2\lambda \varphi^*(\frac{s+t}{2\lambda}) \leq \lambda \varphi^*(\frac{s}{\lambda}) + \lambda \varphi^*(\frac{t}{\lambda}) \leq \lambda \varphi^*(\frac{s+t}{\lambda})$$

(2) *Let* $L \in \mathbb{N}$ *be such that* $\omega(et) \leq L(1 + \omega(t))$*. Then*

$$kt + L^k \varphi^*(\frac{t}{L^k}) \leq \varphi^*(t) + \sum_{j=1}^{k} L^j$$

for all $t \geq 0$ *and* $k \in \mathbb{N}$.

Let $L \in \mathbb{N}$ be such that $\omega(et) \leq L(1+\omega(t))$. Then $|\alpha| + nL\varphi^*(\frac{|\alpha|}{nL}) \leq nL + n\varphi^*(\frac{|\alpha|}{n})$. Therefore, if $q_{K,n}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^p} e^{-|\alpha|} |f^{(\alpha)}(x)| \exp(-n\varphi^*(\frac{|\alpha|}{n}))$ then

$$q_{K,n}(f) \leq |f|_{K,n} \leq e^{nL} q_{K,nL}(f)$$

from where it follows that the topology of $\mathcal{E}_{(\omega)}(\Omega)$ can also be described by the system of seminorms $\{q_{K,n}\}$.

Lemma 1.4. *For every $n, k \in \mathbb{N}$ and $t \geq 1$ we have*

(1) $t^k \leq e^{n\varphi^*(\frac{k}{n})}e^{n\omega(t)}$,

(2) $\inf_{j\in\mathbb{N}_0} t^{-j}e^{k\varphi^*(\frac{j}{k})} \leq e^{-k\omega(t)+\log t}$.

The following result permits us to split $\mathbb{R}$ into intervals in which the infimum in 1.4 is attained in a finite set.

Lemma 1.5. *Fix $k, N \in \mathbb{N}$ and assume $\frac{k}{N}\varphi^*(\frac{N}{k}) \leq \log t < \frac{k}{N+1}\varphi^*(\frac{N+1}{k})$. Then*

(1) $\min_{0\leq j\leq N} t^{-j}e^{k\varphi^*(\frac{j}{k})} \leq e^{-k\omega(t)+\log t}$,

(2) $t^{-N}e^{2k\varphi^*(\frac{N}{2k})} \leq e^{-k\omega(t)+\log t}$.

Proof: (1) Since $\frac{\varphi^*(t)}{t}$ is an increasing function, we have that $\log t < \frac{k}{j}\varphi^*(\frac{j}{k})$ (and, consequently, $t^{-j}e^{k\varphi^*(\frac{j}{k})} > 1$) for every $j \geq N+1$. Now the conclusion follows from lemma 1.4.
(2) We already know that $t^{-(N-l)}e^{k\varphi^*(\frac{N-l}{k})} \leq e^{-k\omega(t)+\log t}$ for some $l = 0, 1, \ldots, N$ (1.4). Then, using that $\frac{k}{l}\varphi^*(\frac{l}{k}) \leq \log t$, we obtain

$$\begin{aligned} t^{-N}e^{2k\varphi^*(\frac{N}{2k})} &\leq t^{-(N-l)}t^{-l}e^{k\varphi^*(\frac{N-l}{k})}e^{k\varphi^*(\frac{l}{k})} \\ &\leq e^{-k\omega(t)+\log t}. \end{aligned}$$

■

The definition of symbol in [13, 23, 25] motivates our next definition. As we will check in 2.11, for the limit case $\omega(t) = \log(1+t)$ we recover the symbols in [13] whereas for the Gevrey weights $\omega(t) = t^d$, $0 < d < 1$, our definition is what can be reasonably expected if one translates to the Beurling setting the definition of symbol of Gevrey class (Roumieu setting) which is in [23, 25]. The introduction of symbols of type (ρ, δ) perhaps makes 'uncomfortable' the reading of the paper but it seems convenient for the construction of parametrices of hypoelliptic operators. See proposition 2.12.

Definition 1.6. *Let Ω be an open set in $\mathbb{R}^p$, $0 \leq \delta < \rho \leq 1$, $d := \rho - \delta$ and let us assume that $\omega(t) = o(t^d)$ as $t \to \infty$. An amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$ is a function $a(x, y, \xi)$ in $C^\infty(\Omega \times \Omega \times \mathbb{R}^p)$ such that for every compact set $Q \subset \Omega\times\Omega$ there are $R \geq 1$ and a sequence $C_n > 0$, $n \in \mathbb{N}$, with the property*

$$|D^\alpha_x D^\gamma_y D^\beta_\xi a(x, y, \xi)| \leq C_n e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\beta+\gamma|}{n})}e^{m\omega(\xi)}(1 + |\xi|)^{|\alpha+\gamma|\delta-|\beta|\rho} \quad (*)$$

for every $n \in \mathbb{N}$, $(x, y) \in Q$, $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$.

For $|\beta| = 0$ the estimate holds for every $\xi \in \mathbb{R}^p$.

In the case $a(x, y, \xi) = p(x, \xi)$, the function $p(x, \xi)$ is usually called a symbol.

Since φ^* is convex, we may take the estimate $e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|}{n})+(\rho-\delta)n\varphi^*(\frac{|\beta|}{n})}$ instead of $e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\beta+\gamma|}{n})}$ in $(*)$ (see lemma 1.3).

Some examples will be given in 2.11.

Remark 1.7. We would like to make some comments on the requirement $\omega(t) = o(t^d)$ in the definition of amplitude.

(1) If $\rho = 1$ and $\delta = 0$, this does not mean any restriction on the weight function ω. For other values of ρ and/or δ, this extra assumption means that $\mathcal{E}_{(\omega)}(\Omega)$ contains, as a continuously and densely embedded subspace, the Gevrey class $\Gamma^{\{\frac{1}{d}\}}(\Omega)$ and it ensures that $j! = O(e^{(\rho-\delta)n\varphi^*(\frac{j}{n})})$ as j goes to infinity for every $n \in \mathbb{N}$. As it is shown in [22], if ω is a weight function for which an extension of the classical Borel's theorem holds, then $\mathcal{E}_{(\omega)}(\Omega)$ contains $\Gamma^{\{s\}}(\Omega)$ for some $s > 1$.

(2) If $\omega(t) = (\log(1+t))^s$, $s > 1$, for t big enough then $\omega(t) = o(t^d)$ for every $0 < d < 1$, whereas if $\omega(t) = t(\log t)^{-a}$ $(a > 1)$, then $\mathcal{E}_{(\omega)}(\Omega)$ does not contain any Gevrey class.

Lemma 1.8. *Let $a(x,y,\xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$. Then for every compact set $Q \subset \Omega \times \Omega$ there exists a sequence $C_n > 0$, $n \in \mathbb{N}$, such that*

$$|D^\alpha_x D^\gamma_y a(x,y,\xi)| \le C_n e^{(\rho-\delta)n\varphi^*(\frac{|\alpha|}{n})} e^{(\rho-\delta)n\varphi^*(\frac{|\gamma|}{n})} |\xi|^{\delta|\alpha+\gamma|} e^{m\omega(\xi)}$$

for every $(x,y) \in Q$ and $|\xi| \ge 1$.

Proof: We put $B := (2^\delta)^{(\frac{1}{\rho-\delta})}$ and we take $k \in \mathbb{N}$ with $B \le e^k$ and $L \in \mathbb{N}$ such that $\omega(et) \le L(1+\omega(t))$ for all $t \ge 0$. Finally we fix $n \in \mathbb{N}$ and we take $\ell := 2nL^k$. According to definition 1.6 there is $C > 0$ such that, for all $(x,y) \in Q$ and $|\xi| \ge 1$ we have

$$|D^\alpha_x D^\gamma_y a(x,y,\xi)| \le C e^{(\rho-\delta)\ell\varphi^*(\frac{|\alpha+\gamma|}{\ell})} (2|\xi|)^{\delta|\alpha+\gamma|} e^{m\omega(\xi)}.$$

An application of lemma 1.3 gives $k|\alpha+\gamma| + \ell\varphi^*(\frac{|\alpha+\gamma|}{\ell}) \le A + n\varphi^*(\frac{|\alpha|}{n}) + n\varphi^*(\frac{|\gamma|}{n})$ for some constant $A > 0$. Since $2^{\delta|\alpha+\gamma|} \le e^{k(\rho-\delta)|\alpha+\gamma|}$, we conclude that

$$|D^\alpha_x D^\gamma_y a(x,y,\xi)| \le C e^A e^{(\rho-\delta)n\varphi^*(\frac{|\alpha|}{n})} e^{(\rho-\delta)n\varphi^*(\frac{|\gamma|}{n})} |\xi|^{\delta|\alpha+\gamma|} e^{m\omega(\xi)}.$$

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Proposition 1.9. *Let $a(x,y,\xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$ and let $f \in \mathcal{D}_{(\omega)}(\Omega)$ be given. For every compact set $K \subset \Omega$ and $n, \lambda \in \mathbb{N}$ there is a constant $C > 0$ such that*

$$\left|\int D^\alpha_x a(x,y,\xi) f(y) e^{-iy\xi} dy\right| \le C e^{-\lambda\omega(\xi)} e^{n\varphi^*(\frac{|\alpha|}{n})}$$

for every $x \in K$, $\xi \in \mathbb{R}^p$. Moreover, a similar estimate holds if we replace the amplitude $a(x,y,\xi)$ by the function $b(x,y,\xi) := a(x,y,\xi)e^{ix\xi}$.

Proof: For every $s, n \in \mathbb{N}$ there is $C > 0$ such that

$$|D_x^\alpha D_y^\gamma a(x,y,\xi)| \le Ce^{(\rho-\delta)sn\varphi^*(\frac{|\alpha|}{sn})}e^{(\rho-\delta)sn\varphi^*(\frac{|\gamma|}{sn})}|\xi|^{\delta|\alpha+\gamma|}e^{m\omega(\xi)}$$

for every $x \in K$, $y \in \text{supp}f$ and $|\xi| \ge 1$.
Since $\frac{\varphi^*(t)}{t}$ is an increasing function we get $(\rho-\delta)sn\varphi^*(\frac{|\alpha|}{sn}) \le (\rho-\delta)n\varphi^*(\frac{|\alpha|}{n})$.
From 1.4 we deduce $|\xi|^{\delta|\alpha|}e^{-\delta n\varphi^*(\frac{|\alpha|}{n})} \le e^{n\delta\omega(\xi)}$, from where it follows

$$|D_x^\alpha D_y^\gamma a(x,y,\xi)| \le Ce^{\rho n\varphi^*(\frac{|\alpha|}{n})}e^{(\rho-\delta)sn\varphi^*(\frac{|\gamma|}{sn})}|\xi|^{\delta|\gamma|}e^{(m+n\delta)\omega(\xi)}$$

for every $x \in K$, $y \in \text{supp}f$ and $|\xi| \ge 1$.

We now fix $\xi \in \mathbb{R}^p$, $|\xi| \ge 1$, and we take $1 \le k \le p$ with $|\xi| = |\xi_k|$. For every $j \in \mathbb{N}$ we have, after integrating by parts,

$$\int D_x^\alpha a(x,y,\xi)f(y)e^{-iy\xi}dy = \frac{1}{\xi_k^j}\int D_{y_k}^j(D_x^\alpha a(x,y,\xi)f(y))e^{-iy\xi}dy.$$

Hence, for every $j, s \in \mathbb{N}$, we have, for some constant C which only depends on n, s and on the Lebesgue measure of the support of f,

$$|\textstyle\int D_x^\alpha a(x,y,\xi)f(y)e^{-iy\xi}dy| \le$$

$$C|f|_{sn}e^{\rho n\varphi^*(\frac{|\alpha|}{n})}e^{(m+n\delta)\omega(\xi)}\textstyle\sum_{l=0}^j\binom{j}{l}\frac{e^{(\rho-\delta)sn\varphi^*(\frac{l}{sn})}}{|\xi|^{l-\delta l}}\frac{e^{sn\varphi^*(\frac{j-l}{sn})}}{|\xi|^{j-l}}.$$

Let us consider the natural number N such that

$$\frac{sn}{N}\varphi^*(\frac{N}{sn}) \le \log(\frac{|\xi|}{2^{1/(\rho-\delta)}}) < \frac{sn}{N+1}\varphi^*(\frac{N+1}{sn}).$$

Then, for every $l < j \le N$, we have $e^{\frac{sn}{j-l}\varphi^*(\frac{j-l}{sn})} \le e^{\frac{sn}{N}\varphi^*(\frac{N}{sn})} \le |\xi|$, hence $\frac{e^{sn\varphi^*(\frac{j-l}{sn})}}{|\xi|^{j-l}} \le 1$, which implies, since $0 < \rho-\delta \le 1$, $\frac{e^{sn\varphi^*(\frac{j-l}{sn})}}{|\xi|^{j-l}} \le \left(\frac{e^{sn\varphi^*(\frac{j-l}{sn})}}{|\xi|^{j-l}}\right)^{\rho-\delta}$ and

$$\frac{e^{(\rho-\delta)sn\varphi^*(\frac{l}{sn})}}{|\xi|^{(\rho-\delta)l}}\frac{e^{sn\varphi^*(\frac{j-l}{sn})}}{|\xi|^{j-l}} \le (\frac{e^{sn\varphi^*(\frac{j}{sn})}}{|\xi|^j})^{(\rho-\delta)}$$

(see lemma 1.3). We finally deduce

$$|\textstyle\int D_x^\alpha a(x,y,\xi)f(y)e^{-iy\xi}dy| \le$$

$$C|f|_{sn}e^{(\delta n+m)\omega(\xi)}e^{n\varphi^*(\frac{|\alpha|}{n})}\textstyle\sum_{l=0}^j\binom{j}{l}\Big(\frac{e^{sn\varphi^*(\frac{j}{sn})}}{|\xi|^j}\Big)^{(\rho-\delta)}$$

$$\le C|f|_{sn}e^{(\delta n+m)\omega(\xi)}e^{n\varphi^*(\frac{|\alpha|}{n})}\Big(\frac{e^{sn\varphi^*(\frac{j}{sn})}}{(\frac{|\xi|}{2^{1/(\rho-\delta)}})^j}\Big)^{(\rho-\delta)}$$

for every $j \le N$. It follows from lemma 1.5 that

$$|\textstyle\int D_x^\alpha a(x,y,\xi)f(y)e^{-iy\xi}dy| \le$$

$$C|f|_{sn}e^{(\delta n+m)\omega(\xi)}e^{n\varphi^*(\frac{|\alpha|}{n})}e^{-sn(\rho-\delta)\omega(\frac{|\xi|}{2^{1/(\rho-\delta)}})+\log(\frac{|\xi|}{2^{1/(\rho-\delta)}})}.$$

Now it suffices to choose s large enough. The corresponding estimate for the function $b(x,y,\xi)$ can be deduced with a similar argument. ∎

2. Pseudodifferential operators

In this section we define pseudodifferential operators on non-quasianalytic classes of Beurling type. Our approach is as in [9], that is, pseudodifferential operators on $\mathcal{D}_{(\omega)}(\Omega)$ are obtained as limits of operators with kernels in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$. We examine several examples showing that the class of pseudodifferential operators contains enough elements and we show that they are pseudolocal.

It is easy to see from the definition of amplitude that $\{a(.,.,\xi); |\xi|\leq T\}$ is a bounded set in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$ for every $T>0$, from where we easily deduce the following

Lemma 2.1. *Let $a(x,y,\xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$ and let $\Psi\in\mathcal{D}_{(\omega)}(\mathbb{R}^p)$ be given. Then*

1. *$K(x,y) := \int a(x,y,\xi)e^{i(x-y)\xi}\Psi(\xi)d\xi$ belongs to $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$,*
2. *$B:\mathcal{D}_{(\omega)}(\Omega)\to\mathcal{E}_{(\omega)}(\Omega)$, $B(f)(x) := \int K(x,y)f(y)dy$, is a continuous and linear operator.*

Let $\Psi\in\mathcal{D}_{(\omega)}(\mathbb{R}^p)$ be a test function such that $\Psi(\xi)=1$ for $|\xi|\leq 1$ and $\Psi(\xi)=0$ for $|\xi|\geq 2$. We put

$$(A_\delta f)(x) := \int\int a(x,y,\xi)e^{i(x-y)\xi}f(y)\Psi(\delta\xi)dyd\xi.$$

Theorem 2.2. *Let $a(x,y,\xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$. Then*

1. *For every $f\in\mathcal{D}_{(\omega)}(\Omega)$ there exists $A(f) := \mathcal{E}_{(\omega)}(\Omega)-\lim_{\delta\to 0+}A_\delta(f)$ and*
$$A:\mathcal{D}_{(\omega)}(\Omega)\to\mathcal{E}_{(\omega)}(\Omega)$$
is a continuous and linear operator,
2. *$(Af)(x) = \int\big(\int a(x,y,\xi)e^{i(x-y)\xi}f(y)dy\big)d\xi$.*

Proof: (1) We fix a compact set $K\subset\Omega$ and $n\in\mathbb{N}$. We put

$$I(x,\xi) := \int a(x,y,\xi)f(y)e^{i(x-y)\xi}dy$$

and we apply proposition 1.9 to get a constant $C>0$ such that

$$|D^\alpha_x I(x,\xi)|e^{-n\varphi^*(\frac{|\alpha|}{n})}\leq Ce^{-\omega(\xi)}$$

for every $x\in K$, $\alpha\in\mathbb{N}^p_0$ and $\xi\in\mathbb{R}^p$.

Then, for $0<\delta_2<\delta_1<1$ we can estimate

$$q_{K,n}(A_{\delta_1}f-A_{\delta_2}f)\ \leq C\int_{|\xi|\geq\frac{1}{\delta_1}}e^{-\omega(\xi)}|\Psi(\delta_1\xi)-\Psi(\delta_2\xi)|d\xi$$

from where it follows that there exists the limit $A(f) := \mathcal{E}_{(\omega)}(\Omega)-\lim_{\delta\to 0+}A_\delta(f)$. An application of the uniform boundedness principle gives the continuity of $A:\mathcal{D}_{(\omega)}(\Omega)\to\mathcal{E}_{(\omega)}(\Omega)$.

(2) We observe that

$$(Af)(x) = \lim_{n\to\infty} \int \Big(\int a(x,y,\xi) f(y) e^{-iy\xi} dy \Big) e^{ix\xi} \Psi(\frac{\xi}{n}) d\xi.$$

Since for every $k \in \mathbb{N}$ there is $C > 0$ such that $|\int a(x,y,\xi) f(y) e^{-iy\xi} dy| \leq Ce^{-k\omega(\xi)}$ for every $\xi \in \mathbb{R}^p$, we can apply the Dominated Convergence Theorem to conclude that $(Af)(x) = \int \big(\int a(x,y,\xi) e^{i(x-y)\xi} f(y) dy \big) d\xi$. ∎

Definition 2.3. *The operator* $A : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ *introduced in the theorem 2.2 is called pseudodifferential operator of* (ω)*-class associated to the amplitude* $a(x,y,\xi)$.

In the case $a(x,y,\xi) = p(x,\xi)$ the pseudodifferential operator A is denoted by $P(x,D)$ and we have

$$P(x,D)f = \int p(x,\xi) e^{ix\xi} \hat{f}(\xi) d\xi$$

for every $f \in \mathcal{D}_{(\omega)}(\Omega)$. It is clear that the expression above makes sense for $f \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$, and even for a wider class of functions.

Proposition 2.4. *The operator* $P(x,D)$ *associated to a symbol* $p(x,\xi)$ *in* $S^{m,\omega}_{\rho,\delta}(\Omega)$ *can be extended to* $\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ *and the extension is linear and continuous taking values in* $\mathcal{E}_{(\omega)}(\Omega)$.

Proof: Given $f \in \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ and $k \in \mathbb{N}$, its Fourier transform satisfies

$$\sup_{\xi\in\mathbb{R}^p} |\hat{f}(\xi)| e^{k\omega(\xi)} \leq C \parallel f \parallel_{1,k+1}$$

for some constant C depending only on the weight ω (see [12, 1.1.23]). Hence, the integral above is convergent also for $f \in \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$. Thus $P(x,D)$ can be extended to $\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$, and the extension is linear.

Given a compact subset K of Ω and $\alpha \in \mathbb{N}_0^p$,

$$|D^\alpha_x(P(x,D)f)| \leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \int |\widehat{D^\beta f}(\xi)| |D^{\alpha-\beta}_x p(x,\xi)| d\xi.$$

As in 1.9, there is a sequence of constants (C_n) such that

$$|D^\gamma_x p(x,\xi)| \leq C_n e^{n\varphi^*(\frac{|\gamma|}{n})} e^{(m+n\delta)\omega(\xi)}$$

for every multi-index γ, each $x \in K$ and $\xi \in \mathbb{R}^p$. Moreover, we deduce from the properties of φ^* that

$$\sup_{\xi\in\mathbb{R}^p} |\widehat{D^\beta f}(\xi)| e^{(n+m)\omega(\xi)} \leq C \parallel D^\beta f \parallel_{1,n+m+1} \leq C \parallel f \parallel_{1,2n+2m+2} e^{n\varphi^*(\frac{|\beta|}{n})}.$$

Therefore,

$$q_{K,n}(P(x,D)f) \leq CC_n \parallel f \parallel_{1,2n+2m+2} \int e^{-\omega(\xi)} d\xi,$$

for n large enough, which finishes the proof. ∎

Theorem 2.5. *The pseudodifferential operator A associated to an amplitude $a(x,y,\xi)$ in $S^{m,\omega}_{\rho,\delta}(\Omega)$ admits a continuous and linear extension $\mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$.*

Proof: We consider $b(x,y,\xi) := a(y,x,-\xi)$, which is an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$, and we denote by $B : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ the associated pseudodifferential operator. To conclude we only have to show that the transposed operator B^t of B is the desired extension of A. To do this, we put $(B_\delta f)(x) := \int\int b(x,y,\xi)e^{i(x-y)\xi}f(y)\Psi(\delta\xi)dyd\xi$. Then it is easy to prove that $\int \varphi(B_\delta\Phi) = \int (A_\delta\varphi)\Phi$ and an application of Theorem 2.2(i) gives the conclusion. ∎

Corollary 2.6. *Let $A : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ be the pseudodifferential operator with amplitude $a(x,y,\xi)$. Then $A^t|_{\mathcal{D}_{(\omega)}(\Omega)} : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ is a pseudodifferential operator with amplitude $a(y,x,-\xi)$.*

Theorem 2.7. *The extension $\overline{P}(x,D) : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ of the pseudodifferential operator $P(x,D)$ is given by*

$$< \overline{P}(x,D)\mu, \psi >= \int \hat{\mu}(\xi)\Big(\int e^{ix\xi}p(x,\xi)\psi(x)dx\Big)d\xi.$$

Proof: Since $< P(x,D)\varphi, \psi >= \int \hat{\varphi}(\xi)\big(\int e^{ix\xi}p(x,\xi)\psi(x)dx\big)d\xi$ for every $\varphi, \psi \in \mathcal{D}_{(\omega)}(\Omega)$, we only have to prove that $\overline{P}(x,D) : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ is a well defined, continuous and linear operator. To do this we first fix $\mu \in \mathcal{E}'_{(\omega)}(\Omega)$ and we let B be a bounded set in $\mathcal{D}_{(\omega)}(\Omega)$. By the Paley-Wiener theorem ([8, 7.4]) there are constants $A > 0$ and $D > 0$ such that $|\hat{\mu}(\xi)| \le De^{A\omega(\xi)}$ for every $\xi \in \mathbb{R}^p$. Now we take $k := A + 1$ and we apply (the proof of) Proposition 1.9 to find a constant $C > 0$ such that $|\int e^{ix\xi}p(x,\xi)\psi(x)dx| \le Ce^{-k\omega(\xi)}$ for every $\psi \in B$. Then we have

$$|\hat{\mu}(\xi)\Big(\int e^{ix\xi}p(x,\xi)\psi(x)dx\Big)| \le CDe^{-\omega(\xi)},$$

from where it follows that $\overline{P}(x,D)\mu : \mathcal{D}_{(\omega)}(\Omega) \to \mathbb{C}$ is bounded on bounded sets and, consequently, $\overline{P}(x,D)\mu \in \mathcal{D}'_{(\omega)}(\Omega)$. Moreover the estimates just obtained also show that $\overline{P}(x,D) : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ transforms bounded sets of $\mathcal{E}'_{(\omega)}(\Omega)$ into weakly bounded sets in $\mathcal{D}'_{(\omega)}(\Omega)$ and we conclude that $\overline{P}(x,D)$ is continuous. ∎

The correspondence between amplitudes and operators is not one-to-one, that is, two different amplitudes may define the same operator (see example 2.11(4) below). The situation is much better for symbols.

Corollary 2.8. *Let $p(x,\xi)$ and $q(x,\xi)$ be symbols in $S^{m,\omega}_{\rho,\delta}(\mathbb{R}^p)$ defining the same pseudodifferential operator. Then $p(x,\xi) = q(x,\xi)$.*

Proof: We put $r(x,\xi) := p(x,\xi) - q(x,\xi)$ and we will show that $r(x,\xi) = 0$. By hypothesis $R(x,D)$ is the null operator and hence $< \overline{R}(x,D)\delta_y, \psi >= 0$ for every $y \in \mathbb{R}^p$ and $\psi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$. For a fixed $\psi \in \mathcal{D}_{(\omega)}(\Omega)$ we take $I(\xi) := \int e^{ix\xi} r(x,\xi)\psi(x)dx$ and we deduce from Theorem 2.7 that the Fourier transform of the continuous function $I(\xi)$ vanishes, from where it follows that $\int e^{ix\xi} r(x,\xi)\psi(x)dx = 0$ for every $\psi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$ and $r(x,\xi) = 0$ everywhere. ■

In some cases it is possible to recover the symbol from the pseudodifferential operator it defines.

Proposition 2.9. *Let $p(x,\xi)$ a symbol in $S^{m,\omega}_{\rho,\delta}(\mathbb{R}^p)$ and let us assume that $p(x,D)$ admits a continuous and linear extension $A : \mathcal{E}_{(\omega)}(\mathbb{R}^p) \to \mathcal{E}_{(\omega)}(\mathbb{R}^p)$. Then*

$$p(x,\xi) = \frac{1}{(2\pi)^p} e^{-ix\xi} A(e^{i(.)\xi})(x).$$

Proof: Since $A^t : \mathcal{E}'_{(\omega)}(\mathbb{R}^p) \to \mathcal{E}'_{(\omega)}(\mathbb{R}^p)$ we deduce from 2.6 that $A^t|_{\mathcal{D}_{(\omega)}(\mathbb{R}^p)} : \mathcal{D}_{(\omega)}(\mathbb{R}^p) \to \mathcal{D}_{(\omega)}(\mathbb{R}^p)$ is a pseudodifferential operator defined by $b(x,y,\xi) := p(y,-\xi)$. Then, for every $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$ we obtain

$$(A^t\varphi)(x) = \int e^{ix\xi} \Big(\int p(y,-\xi)\varphi(y) e^{-iy\xi} dy \Big) d\xi.$$

We put $I(\xi) := \int p(y,-\xi)\varphi(y)e^{-iy\xi}dy$. Then $I \in L_1$ and moreover $\hat{I}(-x) = (A^t\varphi)(x)$ which implies, in particular, that $\hat{I} \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$. Hence

$$\begin{aligned} < A(e^{i(.)\xi}), \varphi > &= \int e^{ix\xi} \hat{I}(-x)dx \\ &= (2\pi)^p I(-\xi) = (2\pi)^p \int p(x,\xi)\varphi(x)e^{ix\xi}dx \end{aligned}$$

which finishes the proof. ■

In most of the forthcoming results we will need stronger conditions on the amplitude.

Definition 2.10. *Let Ω be an open set in $\mathbb{R}^p$, $0 \le \delta < \rho \le 1$, $d := \rho - \delta$ and let us assume $\omega(t) = o(t^d)$ as $t \to \infty$. An amplitude in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ is a function $a(x,y,\xi)$ in $C^\infty(\Omega \times \Omega \times \mathbb{R}^p)$ such that for every compact set $Q \subset \Omega \times \Omega$ there are $R \ge 1$, $B \ge 1$ and a sequence $C_n > 0$, $n \in \mathbb{N}$, with the property*

$$|D^\alpha_x D^\gamma_y D^\beta_\xi a(x,y,\xi)| \le C_n B^\beta \beta! e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|}{n})} e^{m\omega(\xi)} (1+|\xi|)^{|\alpha+\gamma|\delta - |\beta|\rho}$$

for every $n \in \mathbb{N}$, $(x,y) \in Q$, $\log(\frac{|\xi|}{R}) \ge \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$.

An amplitude in $AS^{m,\omega}_{\rho,\delta}(\Omega)$ is said to be of finite order if it satisfies the inequalities above with $(1+|\xi|)^m$ instead of $e^{m\omega(\xi)}$.

It follows from the Stirling formula and the condition $\omega(t) = o(t^d)$, $d := \rho - \delta$, that $AS^{m,\omega}_{\rho,\delta}(\Omega) \subset S^{m,\omega}_{\rho,\delta}(\Omega)$. We observe that given two weight functions $\sigma = O(\omega)$ each amplitude of finite order with respect to ω is also an amplitude with respect to σ, thus the corresponding pseudodifferential operator admits a continuous linear extension to $\mathcal{D}_{(\sigma)}(\Omega)$ which takes values in $\mathcal{E}_{(\sigma)}(\Omega)$.

Now we give examples of amplitudes and of pseudodifferential operators.

Example 2.11. (1) $\omega(t) := \log(1+t)$. This is a limit case that we are not considering, since ω does not satisfy property γ in 1.1. Then $\mathcal{E}_{(\omega)}(\Omega) = C^\infty(\Omega)$ and $\varphi^*(t) = +\infty$ for every $t > 1$. It follows that $a(x,y,\xi)$ is an amplitude of (ω)-class in $S^{m,\omega}_{\rho,\delta}(\Omega)$ according to definition 1.6 if, and only if, for every compact set $K \subset \Omega$ and for every $\alpha, \beta, \gamma \in \mathbb{N}_0^p$ there is a constant $C > 0$ such that

$$|D_x^\alpha D_y^\gamma D_\xi^\beta a(x,y,\xi)| \le C(1+|\xi|)^{m+\delta(|\alpha+\gamma|)-\rho|\beta|}$$

for $x, y \in K$. This means that $a(x,y,\xi)$ is a symbol in the sense of Grigis and Sjöstrand [13].

(2) $\omega(t) := t^d$, $0 < d < 1$. Then $\mathcal{E}_{(\omega)}(\Omega)$ is a Gevrey class of Beurling type. In this case $n\varphi^*(\frac{t}{n}) = \frac{t}{d}\log(\frac{t}{nd}) - \frac{t}{d}$ and it follows from Stirling formula that for every $n \in \mathbb{N}$ there are positive constants A_n and B_n such that for every $\alpha \in \mathbb{N}_0^p$,

$$A_n(\alpha!)^{\frac{1}{d}}(\frac{1}{nd})^{\frac{|\alpha|}{d}} \le e^{n\varphi^*(\frac{|\alpha|}{n})} \le B_n(\alpha!)^{\frac{1}{d}}(\frac{2p}{nd})^{\frac{|\alpha|}{d}}.$$

Then, $a(x,y,\xi)$ is an amplitude in $AS^{m,\omega}_{1,0}(\Omega)$ if, and only if, for every compact set $K \subset \Omega$ there are $R \ge 1$, $B \ge 1$ such that for every $\lambda > 0$ there is $C > 0$ satisfying

$$|D_x^\alpha D_y^\gamma D_\xi^\beta a(x,y,\xi)| \le CB^\beta \beta!(\alpha!)^{\frac{1}{d}}(\gamma!)^{\frac{1}{d}}\lambda^{|\alpha+\gamma|}e^{m\omega(\xi)}(1+|\xi|)^{-|\beta|}$$

for every $(x,y) \in K$ and $|\xi| \ge R(\beta\lambda)^{\frac{1}{d}}$.
This should be compared with the definition of amplitude of Gevrey class (of Roumieu type) which can be viewed for instance in Rodino [23].

(3) Linear partial differential operators with coefficients in $\mathcal{E}_{(\omega)}(\Omega)$ are examples of pseudodifferential operators defined by symbols of finite order.

(4) Let $K(x,y) \in \mathcal{E}_{(\omega)}(\Omega \times \Omega)$ be given. The integral operator with kernel K, $(A\varphi)(x) = \int K(x,y)\varphi(y)dy$, is a pseudodifferential operator. In fact, given any $\chi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$ with $\int \chi = 1$, $a(x,y,\xi) := K(x,y)e^{-i(x-y)\xi}\chi(\xi)$ is an amplitude in $AS^{0,\omega}_{1,0}(\Omega)$. This can be easily deduced from the compactness of the support of χ and the fact that $K \in \mathcal{E}_{(\omega)}(\Omega \times \Omega)$.

The operators $T : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ admitting a continuous and linear extension $\tilde{T} : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ are called (ω)-*smoothing*. These are exactly the integral operators defined by kernels in $\mathcal{E}_{(\omega)}(\Omega \times \Omega)$.

(5) Every ultradifferential operator in the sense of Komatsu [17, p. 42] defines a pseudodifferential operator. We recall that an ultradifferential

operator of (ω)-class in the sense of Komatsu is an operator $G(x,D) := \sum_{\alpha\in\mathbb{N}_0^p} a_\alpha(x)D^\alpha$ such that $a_\alpha \in \mathcal{E}_{(\omega)}(\Omega)$ and that satisfies the following condition: there exists $m \in \mathbb{N}$ such that for every compact set $K \subset \Omega$ and for every $n \in \mathbb{N}$ there is $C_n > 0$ with

$$\sup_{x\in K} |D^\beta a_\alpha(x)| \le C_n e^{n\varphi^*(\frac{|\beta|}{n})} e^{-m\varphi^*(\frac{|\alpha|}{m})}$$

for every $\alpha, \beta \in \mathbb{N}_0^p$.

It is easy to prove that $p(x,\xi) := \frac{1}{(2\pi)^p}\sum_{\alpha\in\mathbb{N}_0^p} a_\alpha(x)\xi^\alpha$ is a symbol in $AS^{k,\omega}_{1,0}(\Omega)$ for some $k \ge m$. Moreover, for every $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$ we have

$$G(x,D)\varphi = \frac{1}{(2\pi)^p}\sum_{\alpha\in\mathbb{N}_0^p} a_\alpha(x)\int e^{ix\xi}\xi^\alpha\hat{\varphi}(\xi)d\xi = P(x,D)\varphi.$$

Examples of ultradifferential operators in the sense of Komatsu are the partial differential operators with coefficients in the class $\mathcal{E}_{(\omega)}(\Omega)$ as well as the ultradifferential operators with constant coefficients ([7]). In this case $G(z) := \sum_{\alpha\in\mathbb{N}_0^p} a_\alpha z^\alpha$ is an entire function satisfying $\log|G(z)| = O(\omega(z))$. Therefore an application of the Paley-Wiener's theorem ([8, 7.3]) gives the existence of an element $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^p)$ with support $\{0\}$ such that $G(D)\varphi = \mu * \varphi$ for every $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$.

(6) If $f \in \mathcal{E}_{(\omega)}(\Omega)$ then $a(x,y,\xi) := (2\pi)^{-p}f(x) \in AS^{0,\omega}_{1,0}(\Omega)$. Thus the operator $\varphi \to f\varphi$ is a pseudodifferential operator.

(7) Let $f \in \mathcal{D}_{(\omega)}(\mathbb{R})$ be a test function with $\mathrm{supp} f = [-1,1]$ and $f(0) \ne 0$. We put $\varphi := f\chi_{[0,+\infty[}$ which is an ultradistribution with compact support and $\mathrm{sing}_{(\omega)}\,\mathrm{supp}\varphi = \{0\}$. Then $\hat{\varphi}(\xi)$ is a symbol and the pseudodifferential operator associated to it is the operator $\psi \to 2\pi(\varphi * \psi)$. (See the comment after Theorem 2.18.)

We first observe that, for every $N \in \mathbb{N}$ we have

$$\hat{\varphi}(\xi) = \sum_{k=0}^N \frac{f^{(k)}(0)}{(i\xi)^{k+1}} + \frac{1}{(i\xi)^{N+1}}\int_0^1 f^{(N+1)}(t)e^{-it\xi}dt.$$

This follows after integrating by parts N times in $\hat{\varphi}(\xi) = \int_0^1 f(t)e^{-it\xi}dt$, using the fact that f and all its derivatives vanish at point $t = 1$.

Hence

$$\begin{aligned}\hat{\varphi}^{(N)}(\xi) &= \textstyle\sum_{k=0}^N (-1)^N \frac{f^{(k)}(0)}{(i\xi)^{k+1}} N!\binom{N+k}{k}\frac{1}{\xi^N} \\ &\quad + \textstyle\frac{1}{\xi^N i^{N+1}}\sum_{k=0}^N \binom{N}{k}\frac{(N+k)!}{N!}(-1)^k\frac{1}{\xi^{k+1}}\int_0^1 (-it)^{N-k}f^{(N+1)}(t)e^{-it\xi}dt.\end{aligned}$$

For every $n \in \mathbb{N}$ we put $C_n := |f|_n$ and let us assume that $\log|\xi| \ge \frac{n}{N}\varphi^*(\frac{N}{n})$. Then, for every $k \le N$ we have $\frac{|f^{(k)}(0)|}{|\xi|^k} \le C_n$ and

$$|\sum_{k=0}^N (-1)^N \frac{f^{(k)}(0)}{(i\xi)^{k+1}} N!\binom{N+k}{k}\frac{1}{\xi^N}| \le \frac{C_n 4^N N!}{|\xi|^{N+1}}.$$

Moreover, since $|\xi|^{k+1} \geq e^{n\varphi^*(\frac{k}{n})} \geq \epsilon k!$ for some $\epsilon > 0$, there is $D_n > 0$ with

$$|\sum_{k=0}^{N} \binom{N}{k} \frac{(N+k)!}{N!} (-1)^k \frac{1}{\xi^{k+1}} \int_0^1 (-it)^{N-k} f^{(N+1)}(t) e^{-it\xi}| \leq D_n 8^N e^{n\varphi^*(\frac{N+1}{n})}.$$

We conclude that

$$|\hat{\varphi}^{(N)}(\xi)| \leq A_n B^N |\xi|^{-N} e^{n\varphi^*(\frac{N}{n})}$$

for some $A_n > 0$, $B > 0$ and for every $\log|\xi| \geq \frac{n}{N}\varphi^*(\frac{N}{n})$.

Our next example is less obvious. It shows that the class of operators under consideration contains not only ultradifferential operators but also parametrices under some extra assumptions.

Proposition 2.12. *Let ω be a weight, $\omega(t) = o(t^d), d < 1$, and let $G(D)$ be an (ω)-ultradifferential operator with constant coefficients such that $G(\xi)$ does not vanish on $\mathbb{R}^p$. If one of the two following conditions is satisfied:*

(1) *$G(D)$ is elliptic,*

or

(2) *$G(D) : \mathcal{E}_{(\omega)}(\mathbb{R}^p) \to \mathcal{E}_{(\omega)}(\mathbb{R}^p)$ is surjective and it is $\{t^d\}$-hypoelliptic,*

then, there exists a pseudodifferential operator of (ω)-class $P : \mathcal{D}_{(\omega)}(\mathbb{R}^p) \to \mathcal{E}_{(\omega)}(\mathbb{R}^p)$ such that $G(D) \circ P$ gives the identity on $\mathcal{D}_{(\omega)}(\mathbb{R}^p)$.

Proof: (1) We know from [11, Thms 3,4] and [3, 2.1] that there exists a constant $A > 0$ such that the entire function G has no zeros in $\{z \in \mathbb{C}^p : |\text{Im}z| \leq A|\text{Re}z|\}$ and

$$|G(\xi)| \geq Ae^{-\frac{1}{A}\omega(\xi)}, \quad \xi \in \mathbb{R}^p.$$

Applying the minimum-modulus theorem of Chou [10, II.2.1] as in [3, 2.6, 2.8], we may find a new constant $C > 0$ such that

$$|G(z)|^{-1} \leq Ce^{C\omega(z)}$$

for $z \in \mathbb{C}^p$ with $|\text{Im}z| \leq |\text{Re}z|/C$. Since $1/G$ is holomorphic in $\{z \in \mathbb{C}^p : |\text{Im}z| < |\text{Re}z|/C\}$, we conclude from the Cauchy inequalities that $((2\pi)^p G)^{-1} \in AS^{m,\omega}_{1,0}(\mathbb{R}^p)$, for some m. The pseudodifferential operator P defined by this symbol is the convolution operator defined by a fundamental solution of $G(D)$, hence it satisfies $G(D) \circ P\varphi = \varphi$, for every $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$.

To prove (2), we may proceed as before taking into account that there is a constant $A > 0$ such that $G(\xi) \neq 0$ whenever $|\text{Im}z| \leq A|\text{Re}z|^d$ [3, 3.3]. In this case, the operator P is defined by a symbol in $AS^{m,\omega}_{d,0}(\mathbb{R}^p)$. ∎

We recall that a continuous linear operator $T : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ is properly supported if the support of its kernel is a proper set in $\Omega \times \Omega$. As for C^∞, this implies that T can be extended as a continuous linear map from $\mathcal{E}_{(\omega)}(\Omega)$ to $\mathcal{E}_{(\omega)}(\Omega)$.

Remark 2.13. *Observe that the solution operator P in* 2.12 *does not admit a continuous and linear extension $\tilde{P}:\mathcal{E}_{(\omega)}(\mathbb{R}^p)\to\mathcal{D}'_{(\omega)}(\mathbb{R}^p)$* ([4, Prop. 8]), *therefore it is not properly supported.*

We already know that partial differential operators with coefficients in $\mathcal{E}_{(\omega)}(\Omega)$ and ultradifferential operators are examples of pseudodifferential operators. Next we see that in most cases pseudodifferential operators of (ω)-class can be expressed as the composition of an ultradifferential operator of (ω)-class and a finite order pseudodifferential operator. The argument depends on the possibility of constructing ultradifferential operators on Beurling spaces which are elliptic in a very strong sense. See for instance [7, 19].

Proposition 2.14. *Let $P(x,D)$ be the pseudodifferential operator associated to $p(x,\xi)\in AS^{m,\omega}_{\rho,\delta}(\Omega)$. Then we may find an ultradifferential operator $G(D)$ of (ω)-class and a symbol $q(x,\xi)\in AS^{m,\omega}_{\rho,\delta}(\Omega)$ of finite order such that if $Q(x,D)$ is the corresponding pseudodifferential operator, we have that $P(x,D)=Q(x,D)\circ G(D)$.*

Proof: We take $D>0$ such that $D\omega(\frac{\xi}{2})>m\omega(\xi)$. Let G be an even entire function satisfying $\log|G(z)|=O(\omega(z))$ as $|z|$ tends to infinity and $|G(z)|\geq e^{D\omega(z)}$ whenever $|\mathrm{Im}z|\leq|\mathrm{Re}z|/D$ (the existence of such a function follows from [19, Corollary 1.4]). Then, $1/G$ is a symbol as in 2.10. Indeed, it is clear that $|1/G(\xi)|\leq e^{-D\omega(\xi)}$ for $\xi\in\mathbb{R}^p$, and since it is holomorphic in $\{z\in\mathbb{C}^p:|\mathrm{Im}z|<|\mathrm{Re}z|/D\}$, we conclude from the Cauchy integral formula that, for some $C>0$ and ξ large enough

$$|(\frac{1}{G(\xi)})^{(\beta)}|\leq C^{|\beta|}\beta!\frac{e^{-D\omega(\xi/2)}}{|\xi|^{|\beta|}}\leq C^{|\beta|}\beta!\frac{e^{-D\omega(\xi/2)}}{|\xi|^{\rho|\beta|}}.$$

We define $q(x,\xi)=\frac{p(x,\xi)}{G(\xi)}$. It is easy to see that q is a symbol of finite order. Moreover, for every $\varphi\in\mathcal{D}_{(\omega)}(\Omega)$ we have

$$\begin{aligned}(Q(x,D)\circ G(D))(\varphi)&=\int q(x,\xi)e^{ix\xi}\widehat{G(D)(\varphi)}(\xi)d\xi\\&=\int q(x,\xi)e^{ix\xi}G(\xi)\widehat{\varphi}(\xi)d\xi=P(x,D)(\varphi).\end{aligned}$$ ∎

Remark 2.15. *The ultradifferential operator $G(D):\mathcal{D}'_{(\omega)}(\Omega)\to\mathcal{D}'_{(\omega)}(\Omega)$ in the proposition above satisfies that $G(D)f\in\mathcal{E}_{(\omega)}(\Omega)$ if, and only if, $f\in\mathcal{E}_{(\omega)}(\Omega)$* [3, 2.1]. *Hence the decomposition given in Proposition 2.14 could be useful in order to study hypoellipticity.*

We observe that each ultradifferential operator of (ω)-class acts continuously from $\mathcal{D}_{(\sigma)}(\Omega)$ into $\mathcal{E}_{(\sigma)}(\Omega)$ for any weight $\sigma\geq\omega$, whereas each pseudodifferential operator of (ω)-class and finite order is also a pseudodifferential operator of (τ)-class for $\tau \leq \omega$. However one cannot expect that pseudodifferential operators of infinite order of (ω)-class be pseudodifferential operators of a different class. Roughly speaking, the p.d.o. (of infinite order) of (ω)-class are strongly tied to the Beurling space $\mathcal{D}_{(\omega)}(\Omega)$.

Next, we analyze the behavior of the pseudodifferential operator when it is defined by an amplitude which does not depend on the x-variable. A combination of the next result and proposition 2.4 will be useful to study the composition of p.d.o.'s in theorem 3.18.

Proposition 2.16. *Let $b(y,\xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$, and let B be the pseudodifferential operator associated to it. Then $Bf \in \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ for every $f \in \mathcal{D}_{(\omega)}(\Omega)$ and $B : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ is continuous.*

Proof: Clearly $(Bf)(x) = \int \big(\int b(y,\xi) e^{i(x-y)\xi} f(y) dy \big) d\xi$ can be defined for each $x \in \mathbb{R}^p$ and $B : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\mathbb{R}^p)$ is linear and continuous.

To show that $Bf \in \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ it is enough to check that $Bf \in L_1$ and that $G(D)(Bf)$ is in L_1 for each ultradifferential operator of (ω)-class ([1, 2.11]).

We denote by $I(\xi) = \int b(y,\xi) f(y) e^{-iy\xi} dy$, so that $(Bf)(x) = \int I(\xi) e^{ix\xi} d\xi$. Then for each $\alpha \in \mathbb{N}_0^p$,

$$D^\alpha_\xi I(\xi) = \int f(y) D^\alpha_\xi (b(y,\xi) e^{-iy\xi}) dy$$

hence, using Leibniz formula and 1.9 applied to the function $y^\beta f(y)$, $\beta \leq \alpha$ we find for every $k > 0$ and each multi-index α a constant $C_{\alpha,k}$ such that

$$|D^\alpha_\xi I(\xi)| \leq C_{\alpha,k} e^{-k\omega(\xi)}.$$

Therefore $D^\alpha_\xi I \in L_1$ and

$$x^\alpha (Bf)(x) = \int I(\xi) D^\alpha_\xi (e^{ix\xi}) d\xi = (-1)^{|\alpha|} \int D^\alpha_\xi I(\xi) e^{ix\xi} d\xi.$$

Consequently, $x^\alpha (Bf)$ is bounded for each α, thus Bf is integrable. Moreover, from 1.9, $D^\alpha_x (e^{ix\xi} I(\xi)) = \xi^\alpha I(\xi) e^{ix\xi} \in L_1$, and hence, $D^\alpha_x (Bf)(x) = \int \xi^\alpha I(\xi) e^{ix\xi} d\xi$.

Let $G(D) = \sum_\alpha a_\alpha D^\alpha$ be an ultradifferential operator of (ω)-class. There exist $m \in \mathbb{N}$ and $C > 0$ such that $|a_\alpha| \leq C e^{-m\varphi^*(\frac{|\alpha|}{m})}$. From the estimates above and lemma 1.4 we conclude that

$$G(D)(Bf)(x) = \int \Big(\int b(y,\xi) G(\xi) e^{i(x-y)\xi} f(y) dy \Big) d\xi.$$

Since $G(z)$ is an entire function and $\log |G(z)| = O(\omega(z))$ we get that $b(y,\xi) G(\xi)$ is an amplitude in some $S^{k,\omega}_{\rho,\delta}(\Omega)$. Proceeding as before, we have $G(D)(Bf) \in L_1$. The continuity of $B : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ follows from the closed graph theorem. ∎

We close this section showing that pseudodifferential operators are pseudolocal, that is, they shrink singular supports. We recall that the (ω)-singular support of an ultradistribution $T \in \mathcal{D}'_{(\omega)}(\Omega)$ is the complement of the largest open subset U with the property that $T \in \mathcal{E}_{(\omega)}(U)$.

Let $a(x,y,\xi)$ be an amplitude in $S^{m,\omega}_{\rho,\delta}(\Omega)$ with associated pseudodifferential operator $A : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ and let $K \in \mathcal{D}'_{(\omega)}(\Omega\times\Omega)$ be the kernel of A. We consider a test function $\Psi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$ such that $\Psi(\xi) = 1$ for $|\xi| \le 1$ and $\Psi(\xi) = 0$ for $|\xi| \ge 2$ and we put $K_n(x,y) := \int a(x,y,\xi)e^{i(x-y)\xi}\Psi(\frac{\xi}{2^n})d\xi$. It follows from Lemma 2.1 and Theorem 2.2 that $K_n \in \mathcal{E}_{(\omega)}(\Omega\times\Omega)$ and $K = \mathcal{D}'_{(\omega)}(\Omega\times\Omega) - \lim_{n\to\infty} K_n$.

Theorem 2.17. *The (ω)-singular support of the kernel K of a pseudodifferential operator A is contained in $\Delta := \{(x,y) \in \Omega\times\Omega; x = y\}$.*

Proof: Given $(x_0,y_0) \in (\Omega\times\Omega)\setminus\Delta$ we take a relatively compact open neighbourhood Q of (x_0,y_0), disjoint with Δ. We show that (K_n) is a Cauchy sequence in $\mathcal{E}_{(\omega)}(Q)$. Without loss of generality we may take $1 \le l \le p$ and $c_0 > 0$ such that $|x_l - y_l| \ge c_0$ for every $(x,y) \in Q$. Let $R \ge \frac{3}{c_0}$ and a sequence of positive constants (C_k) such that

$$|D^\alpha_x D^\gamma_y D^\beta_\xi a(x,y,\xi)| \le C_k e^{(\rho-\delta)k\varphi^*(\frac{|\alpha+\gamma|}{k}) + (\rho-\delta)k\varphi^*(\frac{|\beta|}{k})}(1+|\xi|)^{|\alpha+\gamma|\delta - |\beta|\rho}e^{m\omega(\xi)}$$

for every $k \in \mathbb{N}$, $(x,y) \in Q$ and $\log(\frac{|\xi|}{R}) \ge \frac{k}{|\beta|}\varphi^*(\frac{|\beta|}{k})$.
We fix $\overline{k} \in \mathbb{N}$ and we take $k > \overline{k}$ to be chosen later. For every $N \in \mathbb{N}$ we have, after integrating by parts N times,

$$D^\alpha_x D^\gamma_y\big(K_n(x,y) - K_{n+1}(x,y)\big) = \sum_{\beta\le\alpha}\sum_{\mu\le\gamma}\binom{\alpha}{\beta}\binom{\gamma}{\mu}\frac{(-1)^{N+\mu}}{(x_l-y_l)^N}\int e^{i(x-y)\xi}\lambda_{N,\alpha,\beta,\gamma,\mu}d\xi$$

where

$$\lambda_{N,\alpha,\beta,\gamma,\mu} = D^N_{\xi_l}\{\xi^{\mu+\beta}D^{\alpha-\beta}_x D^{\gamma-\mu}_y a(x,y,\xi)\big(\Psi(\tfrac{\xi}{2^n}) - \Psi(\tfrac{\xi}{2^{n+1}})\big)\} =$$

$$\sum \frac{N!}{N_1!N_2!N_3!}\frac{(\mu_l+\beta_l)!}{(\mu_l+\beta_l-N_1)!}\xi^{\mu+\beta-N_1e_l}D^{N_2}_{\xi_l}\big(\Psi(\tfrac{\xi}{2^n}) - \Psi(\tfrac{\xi}{2^{n+1}})\big)D^{N_3}_{\xi_l}D^{\alpha-\beta}_x D^{\gamma-\mu}_y a(x,y,\xi),$$

and the last sum extends over all N_1, N_2, N_3 such that $N_1 + N_2 + N_3 = N$ and $N_1 \le \mu_l + \beta_l$. Here e_l is the multi-index with 1 in the l-th position and 0 elsewhere.
The support of $\Psi(\frac{\xi}{2^n}) - \Psi(\frac{\xi}{2^{n+1}})$ is contained in $2^n \le |\xi| \le 2^{n+2}$ and this inequality implies that $|\xi^{\mu+\beta-N_1e_l}| \le |\xi|^{|\mu+\beta|-N_1} \le \frac{(2^{n+2})^{|\mu+\beta|}}{(2^n)^{N_1}}$. We also have that $|D^{N_2}_{\xi_l}\big(\Psi(\frac{\xi}{2^n}) - \Psi(\frac{\xi}{2^{n+1}})\big)| \le 2|\Psi|_k e^{k\varphi^*(\frac{N_2}{k})}\frac{1}{(2^n)^{N_2}}$.

Let $N \in \mathbb{N}$ be such that $\frac{k}{N}\varphi^*(\frac{N}{k}) \le \log(\frac{2^n}{R^{1/(\rho-\delta)}})$. Then $\log(\frac{|\xi|}{R}) \ge \frac{k}{N_3}\varphi^*(\frac{N_3}{k})$ and consequently, using that $\frac{\varphi^*(t)}{t}$ is increasing and lemma 1.4(1), we have

$$|D_x^{\alpha-\beta}D_y^{\gamma-\mu}D_{\xi_l}^{N_3}a(x,y,\xi)| \leq$$

$$C_k e^{(\rho-\delta)k\varphi^*(\frac{|\alpha-\beta+\gamma-\mu|}{k})+(\rho-\delta)k\varphi^*(\frac{N_3}{k})}(1+|\xi|)^{|\alpha-\beta+\gamma-\mu|\delta-N_3\rho}e^{m\omega(\xi)} \leq$$

$$C_k e^{\overline{k}\varphi^*(\frac{|\alpha-\beta+\gamma-\mu|}{\overline{k}})}\Big(\frac{e^{k\varphi^*(\frac{N_3}{k})}}{(2^n)^{N_3}}\Big)^{\rho-\delta}e^{(m+\overline{k})\omega(2^{n+3})}.$$

Using $N_1! \leq E_k e^{k\varphi^*(\frac{N_1}{k})}$, we deduce

$$\begin{aligned}|\lambda_{N,\alpha,\beta,\gamma,\mu}| &\leq \sum \tfrac{N!}{N_1!N_2!N_3!}\tfrac{(\mu_l+\beta_l)!}{(\mu_l+\beta_l-N_1)!N_1!}\times \\ &\quad 2C_kE_k|\Psi|_k \frac{e^{k\varphi^*(\frac{N_1}{k})}}{(2^n)^{N_1}}\frac{e^{k\varphi^*(\frac{N_2}{k})}}{(2^n)^{N_2}}\Big(\frac{e^{k\varphi^*(\frac{N_3}{k})}}{(2^n)^{N_3}}\Big)^{(\rho-\delta)}\times \\ &\quad e^{\overline{k}\varphi^*(\frac{|\alpha-\beta+\gamma-\mu|}{\overline{k}})}(2^{n+2})^{|\mu+\beta|}e^{(m+\overline{k})\omega(2^{n+3})}.\end{aligned}$$

Since $\log(\frac{2^n}{R}) \geq \frac{k}{N}\varphi^*(\frac{N}{k}) \geq \frac{k}{N_i}\varphi^*(\frac{N_i}{k})$ $(i=1,2)$ we have $\frac{e^{k\varphi^*(\frac{N_i}{k})}}{(2^n)^{N_i}} \leq 1$. On the other hand $\varphi^*(\frac{N_1}{k})+\varphi^*(\frac{N_2}{k})+\varphi^*(\frac{N_3}{k}) \leq \varphi^*(\frac{N}{k})$,

$$\textstyle\sum_{N_1+N_2+N_3=N}\frac{N!}{N_1!N_2!N_3!} = 3^N, \qquad \sum_{N_1\leq\mu_l+\beta_l}\frac{(\mu_l+\beta_l)!}{(\mu_l+\beta_l-N_1)!N_1!} \leq 2^{|\mu+\beta|}$$

and $(2^{n+3})^{|\mu+\beta|} \leq e^{\overline{k}\omega(2^{n+3})+\overline{k}\varphi^*(\frac{|\mu+\beta|}{\overline{k}})}$. Hence

$$\begin{aligned}|\lambda_{N,\alpha,\beta,\gamma,\mu}| &\leq 2C_kE_k|\Psi|_k 3^N\Big(\frac{e^{k\varphi^*(\frac{N}{k})}}{(2^n)^N}\Big)^{(\rho-\delta)}e^{\overline{k}\varphi^*(\frac{|\alpha+\gamma|}{\overline{k}})}e^{(m+2\overline{k})\omega(2^{n+3})} \\ &:= I_{N,\alpha,\gamma}.\end{aligned}$$

Since the support of $\lambda_{N,\alpha,\beta,\gamma,\mu}(x,y,.)$ is contained in the set $2^n \leq |\xi| \leq 2^{n+2}$, which has measure $(2^{n+1})^p(4^p-1)$, we finally obtain

$$|D_x^\alpha D_y^\gamma\big(K_n(x,y)-K_{n+1}(x,y)\big)| \leq 2^{|\alpha+\gamma|}(2^{n+1})^p(4^p-1)\frac{1}{c_0^N}I_{N,\alpha,\gamma}.$$

We put $R^* := R^{1/(\rho-\delta)}$. Then, using that $\frac{3}{c_0} \leq R$, we deduce

$$\begin{aligned}&q_{Q,\overline{k}}(K_n-K_{n+1}) \leq \\ &2E_kC_k|\Psi|_k(2^{n+1})^p(4^p-1)\Big(\frac{(R^*)^Ne^{k\varphi^*(\frac{N}{k})}}{(2^n)^N}\Big)^{(\rho-\delta)}e^{(m+2\overline{k})\omega(2^{n+3})}\end{aligned}$$

whenever $\log(\frac{2^n}{R^*}) \geq \frac{k}{N}\varphi^*(\frac{N}{k})$. Observe that the estimates just obtained also hold if we replace N by $j \leq N$. Now, selecting k large enough, and taking N such that $\frac{k}{N}\varphi^*(\frac{N}{k}) \leq \log(\frac{2^n}{R^*}) < \frac{k}{N+1}\varphi^*(\frac{N+1}{k})$ an application of Lemma 1.5 gives

$$q_{Q,\overline{k}}(K_n-K_{n+1}) \leq 2C_kE_k|\Psi|_k(2^{n+1})^p(4^p-1)e^{-\omega(2^n)}$$

from where it easily follows that (K_n) is a Cauchy sequence in $\mathcal{E}_{(\omega)}(Q)$. Thus (x_0,y_0) does not belong to the (ω)-singular support of K. ∎

As in [25] we can conclude that the pseudodifferential operators are pseudolocal.

Theorem 2.18. *Let $A : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ be the pseudodifferential operator associated to an amplitude $a(x, y, \xi)$ in $S^{m,\omega}_{\rho,\delta}(\Omega)$. Then*

$$\text{sing}_{(\omega)} \text{ supp}(A\mu) \subset \text{sing}_{(\omega)} \text{ supp}(\mu)$$

for every $\mu \in \mathcal{E}'_{(\omega)}(\Omega)$.

If a convolution operator $\psi \to \psi * S$, $\psi \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$, $S \in \mathcal{D}'_{(\omega)}(\mathbb{R}^p)$, is a pseudodifferential operator, by 2.18 the (ω)-singular support of S reduces to $\{0\}$. Therefore, not every convolution operator is a pseudodifferential operator.

3. Symbolic calculus

One of the problems one has to face is how to determine the class of symbols in order that the theory for the operators can be converted into an algebraic theory for the corresponding symbols. Moreover, the class of operators should be closed under products of operators. This leads to the necessity of developing in our setting the classical symbolic calculus. The definitions below are motivated by [23, 25].

Definition 3.1. *We denote by $FAS^{m,\omega}_{\rho,\delta}(\Omega)$ the set of all formal sums*

$$\sum_{j\in\mathbb{N}_0} a_j(x, y, \xi),$$

such that $a_j(x, y, \xi) \in C^\infty(\Omega\times\Omega\times\mathbb{R}^p)$ and for every compact set $Q \subset \Omega\times\Omega$ there are $R \geq 1$, $B \geq 1$ and a sequence $C_n > 0$, $n \in \mathbb{N}$, with the property

$$|D^\alpha_x D^\gamma_y D^\beta_\xi a_j(x, y, \xi)| \leq C_n B^{|\beta|}\beta! e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|+j}{n})} e^{m\omega(\xi)} (1+|\xi|)^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)j}$$

for every $j \in \mathbb{N}_0$, $(x, y) \in Q$ and $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|+j}\varphi^*(\frac{|\beta|+j}{n})$.

Let $a \in AS^{m,\omega}_{\rho,\delta}(\Omega)$ be given and we put $a_0 := a$, $a_j := 0$ for $j \neq 0$. Then we can identify a with the formal sum $\sum a_j$.

Example 3.2. Let $a \in AS^{m,\omega}_{\rho,\delta}(\Omega)$ be given, then the series $\sum_{j=0}^\infty p_j(x, \xi)$, where $p_j(x, \xi) := \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha_\xi \partial^\alpha_y a(x, y, \xi)_{|y=x}$, is a formal sum in $FAS^{m,\omega}_{\rho,\delta}(\Omega)$.

Definition 3.3. *Two formal sums $\sum a_j$ and $\sum b_j$ in $FAS^{m,\omega}_{\rho,\delta}(\Omega)$ are said to be equivalent if for every compact set $Q \subset \Omega \times \Omega$ there are $R \geq 1$, $B \geq 1$ and two sequences $C_n > 0$ and N_n ($n \in \mathbb{N}$) with the property*

$$|D^\alpha_x D^\gamma_y D^\beta_\xi \textstyle\sum_{j<N}(a_j - b_j)| \leq$$
$$C_n B^{|\beta|}\beta! e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|+N}{n})} e^{m\omega(\xi)} (1 + |\xi|)^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)N}$$

for every $(x, y) \in Q$, $N \geq N_n$, $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|+N}\varphi^*(\frac{|\beta|+N}{n})$.

Remark 3.4. *If $a(x, y, \xi) = 0$ for every $x, y \in \Omega$ and $|\xi| \geq M$ then $a \sim 0$. In particular, every (ω)-smoothing operator is associated to an amplitude equivalent to zero.*

Proposition 3.5. *Let A be the pseudodifferential operator defined by an amplitude $a \in AS^{m,\omega}_{\rho,\delta}(\Omega)$ which is equivalent to zero. Then A is an (ω)-smoothing operator.*

Proof: We show that $K(x,y) := \int e^{i(x-y)\xi}a(x,y,\xi)d\xi$ is a function in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$ and $(A\varphi)(x) = \int K(x,y)\varphi(y)dy$ for every $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$. We fix a compact set $Q \subset \Omega\times\Omega$, then

$$|D^\alpha_x D^\gamma_y a(x,y,\xi)| \leq C_n e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|+N}{n})}e^{m\omega(\xi)}|\xi|^{\delta|\alpha+\gamma|-(\rho-\delta)N}$$

for every $(x,y)\in Q$, $N \geq N_n$, and $\log(\frac{|\xi|}{R}) \geq \frac{n}{N}\varphi^*(\frac{N}{n})$. We now fix $n_0 \in \mathbb{N}$ and we take $0<\epsilon<1$ and $n\in\mathbb{N}$ with $\omega(\frac{t}{R}) \geq \epsilon\omega(t) - \frac{1}{\epsilon}$ and $\epsilon(\rho-\delta)n > 2n_0$. Then, for every $N \geq N_{8n}$ and $\frac{2n}{N}\varphi^*(\frac{N}{2n}) \leq \log(\frac{|\xi|}{R}) \leq \frac{2n}{N+1}\varphi^*(\frac{N+1}{2n})$ we have that $|D^\alpha_x D^\gamma_y\big(e^{i(x-y)\xi}a(x,y,\xi)\big)|$ is not greater than

$$C_{8n}\sum_{\substack{\beta\leq\alpha\\ \mu\leq\gamma}}\binom{\alpha}{\beta}\binom{\gamma}{\mu}e^{(\rho-\delta)n_0\varphi^*(\frac{|\alpha-\beta+\gamma-\mu|}{n_0})}|\xi|^{|\beta+\mu|+\delta|\alpha-\beta+\gamma-\mu|-(\rho-\delta)N}e^{m\omega(\xi)}e^{(\rho-\delta)4n\varphi^*(\frac{N}{4n})}.$$

Applying lemma 1.4, we have

$$|\xi|^{\delta|\alpha-\beta+\gamma-\mu|}e^{-\delta n_0\varphi^*(\frac{|\alpha-\beta+\gamma-\mu|}{n_0})} \leq e^{n_0\omega(\xi)}$$

and

$$|\xi|^{|\beta+\mu|} \leq e^{n_0\varphi^*(\frac{|\beta+\mu|}{n_0})}e^{n_0\omega(\xi)},$$

from where we conclude

$$|D^\alpha_x D^\gamma_y\big(e^{i(x-y)\xi}a(x,y,\xi)\big)| \leq$$
$$C_{8n}2^{|\alpha+\gamma|}e^{n_0\varphi^*(\frac{|\alpha+\gamma|}{n_0})}e^{(m+2n_0)\omega(\xi)}e^{(\rho-\delta)4n\varphi^*(\frac{N}{4n})}|\xi|^{-(\rho-\delta)N}.$$

An application of lemma 1.5(2) gives

$$|D^\alpha_x D^\gamma_y\big(e^{i(x-y)\xi}a(x,y,\xi)\big)| \leq D_{n_0}2^{|\alpha+\gamma|}e^{n_0\varphi^*(\frac{|\alpha+\gamma|}{n_0})}e^{(m+1-n_0)\omega(\xi)}.$$

Selecting n_0 large enough we conclude that $K \in \mathcal{E}_{(\omega)}(\Omega\times\Omega)$. To finish, it is easy to see that A coincides with the operator with kernel K. ∎

Lemma 3.6. ([24, p. 241]) *There is a sequence $(\Phi_\ell)_{\ell\geq1}$ and constants $C, D > 0$ such that $\Phi_\ell \in \mathcal{D}_{(\omega)}(\mathbb{R}^p)$, $|\Phi_\ell(\xi)| \leq 1$, $\Phi_\ell(\xi) = 1$ for $|\xi| \leq 2$, $\Phi_\ell(\xi) = 0$ for $|\xi| \geq 3$ and with the property that*

$$|\Phi^\alpha_\ell(\xi)| \leq C(\frac{D}{3})^{|\alpha|}\ell^{|\alpha|+1}$$

whenever $|\alpha| \leq \ell$.

We now fix a positive constant $R \geq 1$ and we put

$$\Psi_{j,n}(\xi) := 1 - \Phi_j\big(\frac{\xi}{Re^{\frac{n}{j}\varphi^*(\frac{j}{n})}}\big).$$

Then $\Psi_{j,n}(\xi) \neq 0$ implies $|\xi| \geq 2Re^{\frac{n}{j}\varphi^*(\frac{j}{n})}$, whereas if ξ is in the support of any derivative we also have $|\xi| \leq 3Re^{\frac{n}{j}\varphi^*(\frac{j}{n})}$. It follows that

$$|D_\xi^i \Psi_{j,n}(\xi)| \leq C(\frac{D}{|\xi|})^{|i|} j^{|i|+1}$$

for any multi-index i with $|i| \leq j$.

In order to construct an amplitude from a formal sum, the idea is the following: For a fixed n, as in [25] we could find a $C^\infty-$function a^n satisfying the estimates in 2.10 only for this fixed n. Since each a^n is obtained as a series involving the a_j's, we will take, for each n, a finite block of the series, in such a way that when we put together these blocks we obtain an amplitude which is equivalent to the formal sum. In some sense, this procedure reflects the fact that our amplitudes are 'tied up' to Fréchet spaces $\mathcal{E}_{(\omega)}(\Omega)$.

Theorem 3.7. *Let* $\sum a_j \in FAS^{m,\omega}_{\rho,\delta}(U)$ *be given and let* Ω *be a relatively compact open subset of* U*. Then there is an amplitude* $a \in AS^{m,\omega}_{\rho,\delta}(\Omega)$ *such that* $a \sim \sum a_j$ *on* Ω*.*

Proof: We put

$$\Psi_{j,n}(\xi) := 1 - \Phi_j\Big(\frac{\xi}{Re^{\frac{n}{j}\varphi^*(\frac{j}{n})}}\Big)$$

where R will be determined later. Associated to R we put $R_1 := (2R)^{\rho-\delta}$ and we observe that $\Psi_{j,n}(\xi) \neq 0$ implies $e^{(\rho-\delta)n\varphi^*(\frac{j}{n})} \leq |\xi|^{j(\rho-\delta)}(\frac{1}{R_1})^j$, whereas if ξ is in the support of any derivative we also have $(\frac{|\xi|}{3})^{j(\rho-\delta)}(\frac{1}{R_1})^j \leq e^{(\rho-\delta)n\varphi^*(\frac{j}{n})}$. According to definition 3.1, we can select R large enough so that $\sum \frac{je^{Djp}}{R_1^j} < \infty$ and, for some sequence (C_n),

$$|D_x^\alpha D_y^\gamma D_\xi^\beta a_j(x,y,\xi)| \leq$$

$$C_n B^{|\beta|}\beta! e^{(\rho-\delta)2n\varphi^*(\frac{|\alpha+\gamma|}{2n})} e^{m\omega(\xi)} |\xi|^{|\alpha+\gamma|\delta - |\beta|\rho} e^{(\rho-\delta)2n\varphi^*(\frac{j}{2n})} |\xi|^{-(\rho-\delta)j}$$

whenever $(x,y) \in \Omega$, $\log(\frac{|\xi|}{R}) \geq \frac{n}{|\beta|+j}\varphi^*(\frac{|\beta|+j}{n})$.
We first assume $(x,y) \in \Omega$, $n \in \mathbb{N}$, $\log(\frac{|\xi|}{3R}) \geq \frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$ and $\Psi_{j,n}(\xi) \neq 0$. Then $\log(\frac{|\xi|}{R}) \geq \max\big(\frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n}), \frac{n}{j}\varphi^*(\frac{j}{n})\big) \geq \frac{2n}{|\beta|+j}\varphi^*(\frac{|\beta|+j}{2n})$. Moreover from $D_\xi^i \Psi_{j,n}(\xi) \neq 0$ we deduce that $\log(\frac{|\xi|}{3R}) \leq \frac{n}{j}\varphi^*(\frac{j}{n})$, and consequently, $|\beta| \leq j$ since $\varphi^*(t)/t$ is increasing. Since $|\xi| \geq 1$ and $0 \leq \rho \leq 1$ then $|\xi|^{-|i|} \leq |\xi|^{-|i|\rho}$ and we can estimate

$$|D_x^\alpha D_y^\gamma D_\xi^\beta\big(a_j(x,y,\xi)\Psi_{j,n}(\xi)\big)|e^{-m\omega(\xi)} \leq$$

$$\sum_{i\leq\beta}\binom{\beta}{i}|D_\xi^i\Psi_{j,n}(\xi) D_x^\alpha D_y^\gamma D_\xi^{\beta-i} a_j(x,y,\xi)|e^{-m\omega(\xi)}$$

which is less than

$CC_n e^{2n\varphi^*(\frac{|\alpha+\gamma|}{2n})}|\xi|^{|\alpha+\gamma|\delta-(\rho-\delta)j}e^{n\omega(\xi)}\sum_{i\le\beta}\frac{B^{|\beta-i|}D^{|i|}j^{|i|+1}}{i!}|\xi|^{-|i|-\rho|\beta-i|}\le$

$CC_nB^{|\beta|}\beta!e^{(\rho-\delta)2n\varphi^*(\frac{|\alpha+\gamma|}{2n})}|\xi|^{|\alpha+\gamma|\delta-|\beta|\rho}e^{(\rho-\delta)2n\varphi^*(\frac{j}{2n})}|\xi|^{-(\rho-\delta)j}\sum_{i\le\beta}\frac{D^{|i|}j^{|i|+1}}{i!}\le$

$CC_nB^{|\beta|}\beta!e^{(\rho-\delta)2n\varphi^*(\frac{|\alpha+\gamma|}{2n})}|\xi|^{|\alpha+\gamma|\delta-|\beta|\rho}(\frac{1}{R_1})^j je^{Djp}.$

We proceed by induction to select a sequence (j_n) of natural numbers in such a way that $j_1 := 0$, $j_n < j_{n+1}$, $\lim_{n\to\infty}\frac{j_n}{n} = +\infty$ and

$$C_{n+1}\sum_{j=j_{n+1}+1}^{\infty}\frac{je^{Djp}}{R_1^j}\le\frac{C_n}{2}\sum_{j=j_n+1}^{j_n+2}\frac{je^{Djp}}{R_1^j}.$$

Then

$$\overline{D}_n := C_n\sum_{j=j_n+1}^{j_{n+1}}\frac{je^{Djp}}{R_1^j}$$

satisfies that $\overline{D}_{n+1}\le\frac{\overline{D}_n}{2}$. We now prove that

$$a(x,y,\xi) := a_0(x,y,\xi)+\sum_{n=1}^{\infty}\sum_{j=j_n+1}^{j_{n+1}}\Psi_{j,n}(\xi)a_j(x,y,\xi)$$

is an amplitude. Since $j_n\le j$ and $\Psi_{j,n}(\xi)\ne 0$ implies

$$\frac{n}{j_n}\varphi^*(\frac{j_n}{n})\le\frac{n}{j}\varphi^*(\frac{j}{n})\le\log(\frac{|\xi|}{2R})$$

the condition $\lim_{n\to\infty}\frac{n}{j_n}\varphi^*(\frac{j_n}{n}) = +\infty$ permits to conclude that the sum defining a is locally finite. Hence a is a well defined C^∞ function. Let us assume $\log(\frac{|\xi|}{3R})\ge\frac{n}{|\beta|}\varphi^*(\frac{|\beta|}{n})$. Then, for every $n\in\mathbb{N}$,

$$|D_x^\alpha D_y^\gamma D_\xi^\beta(\textstyle\sum_{k=n}^{\infty}\sum_{j=j_k+1}^{j_{k+1}}\Psi_{j,k}(\xi)a_j(x,y,\xi))|\le$$

$$CB^{|\beta|}\beta!e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|}{n})}|\xi|^{|\alpha+\gamma|\delta-|\beta|\rho}e^{m\omega(\xi)}\textstyle\sum_{k=n}^{\infty}\overline{D}_k.$$

Since $a_0+\sum_{k=1}^{n-1}\sum_{j=j_k+1}^{j_{k+1}}\Psi_{j,k}a_j$ is a finite sum of amplitudes we conclude that $a(x,y,\xi)\in AS_{\rho,\delta}^{m,\omega}(\Omega)$.

To finish we have to show that $a\sim\sum a_j$ on Ω. In order to do this, we assume that $(x,y)\in\Omega$ and $\log(\frac{|\xi|}{3R})\ge\frac{n}{|\beta|+N}\varphi^*(\frac{|\beta|+N}{n})$ and we estimate the derivatives of

$$a-\sum_{j<N}a_j = \sum_{k=1}^{\infty}\sum_{j=j_k+1}^{j_{k+1}}\Psi_{j,k}a_j-\sum_{j=1}^{N-1}a_j.$$

We will only consider the case $N > nj_n$. For every $j\in\mathbb{N}$ there is $k\in\mathbb{N}$ with $j_k < j\le j_{k+1}$. Then $k < n$ implies $j\le j_n(< N)$ and $\log(\frac{|\xi|}{3R})\ge\frac{n}{N}\varphi^*(\frac{N}{n})\ge\frac{1}{j_n}\varphi^*(j_n)\ge\frac{k}{j}\varphi^*(\frac{j}{k})$. On the other hand, $k\ge n$ and $N > j$ also imply $\log(\frac{|\xi|}{3R})\ge\frac{n}{N}\varphi^*(\frac{N}{n})\ge\frac{k}{j}\varphi^*(\frac{j}{k})$ and $\Psi_{j,k}(\xi) = 1$.

Consequently, $a - \sum_{j<N} a_j$ can be expressed as a sum of functions $\Psi_{j,k}a_j$ with $j \geq N$ and $k \geq n$.
It follows from the above estimates that

$$|D_x^\alpha D_y^\gamma D_\xi^\beta\big(a_j(x,y,\xi)\Psi_{j,k}(\xi)\big)|e^{-m\omega(\xi)}$$

is less than

$$C\overline{D}_k B^{|\beta|}\beta! e^{(\rho-\delta)2k\varphi^*(\frac{|\alpha+\gamma|+N}{2k})}|\xi|^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)N} e^{(\rho-\delta)k\varphi^*(\frac{j-N}{k})}|\xi|^{-(\rho-\delta)(j-N)} \leq$$

$$C\overline{D}_k B^{|\beta|}\beta! e^{(\rho-\delta)2k\varphi^*(\frac{|\alpha+\gamma|+N}{2k})}|\xi|^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)N}(\frac{1}{R_1})^{j-N}.$$

where the last inequality follows from the fact that $\Psi_{j,k}(\xi) \neq 0$ implies that $\log(\frac{|\xi|}{2R}) \geq \frac{k}{j}\varphi^*(\frac{j}{k}) \geq \frac{k}{j-N}\varphi^*(\frac{j-N}{k})$.
Finally we get

$$|D_x^\alpha D_y^\gamma D_\xi^\beta\big(a - \sum_{j<N} a_j\big)| \leq \sum_{k\geq n}\sum_{j\geq N}|D_x^\alpha D_y^\gamma D_\xi^\beta(\Psi_{j,k}a_j)|$$

which is dominated by

$$CB^{|\beta|}\beta! e^{(\rho-\delta)n\varphi^*(\frac{|\alpha+\gamma|+N}{n})}|\xi|^{|\alpha+\gamma|\delta-|\beta|\rho-(\rho-\delta)N} e^{m\omega(\xi)}\sum_{k\geq n}\overline{D}_k(\sum_{j\geq N}(\frac{1}{R_1})^{j-N}).$$

■

We observe that any other choice of the sequence (j_n) and the constant R satisfying the estimates in the proof of the previous result would give another amplitude defining the same pseudodifferential operator, modulus an (ω)-smoothing one. Hence, in what follows we can assume without loss of generality that (j_n) and R are as big as necessary.

Our next aim is to give the asymptotic expansion formula. From now on, we will always assume that $\mathcal{E}_{(\omega)}(\Omega)$ contains a Gevrey class $\Gamma^{\{s\}}(\Omega)$ for some $s > 1$. Then, for $\sigma(t) := t^{1/s}$, we have that $\mathcal{E}_{(\sigma)}(\Omega)$ is contained (and it is dense) in $\mathcal{E}_{(\omega)}(\Omega)$ (see [8]).
We assume that $\frac{n}{j}\varphi^*(\frac{j}{n}) \geq n$ for every $j \geq j_n$. We put $\varphi_j := \Psi_{j,n}$ if $j_n < j \leq j_{n+1}$, $\varphi_0(\xi) = 1$.

As in the proof of 2.2 we have the following

Lemma 3.8. *Let $a \in S^{m,\omega}_{\rho,\delta}(\Omega)$ and let A be the pseudodifferential operator defined by a. Then, for every $u \in \mathcal{D}_{(\omega)}(\Omega)$,*

$$A(u) = \sum_{j=0}^{\infty} A_j(u)$$

where A_j is the pseudodifferential operator with amplitude $a_j(x,y,\xi) := (\varphi_j - \varphi_{j+1})(\xi)a(x,y,\xi)$.

Lemma 3.9. *Let* $\sum_{j=0}^{\infty} p_j(x,\xi)$ *be a formal sum in* $FAS_{\rho,\delta}^{m,\omega}(U)$, Ω *a relatively compact open subset of* U *and* (j_n) *as in the proof of 3.7 and satisfying the additional assumption* $\frac{n}{j}\varphi^*(\frac{j}{n}) \geq \max(n, \log C_n)$ *for* $j \geq j_n$, (C_n) *being the constants of definition 3.1 relatives to the closure of* Ω. *We let*

$$p(x,\xi) := \sum_{j=0}^{\infty} \varphi_j(\xi) p_j(x,\xi)$$

which is a symbol in $AS_{\rho,\delta}^{m,\omega}(\Omega)$. *Then, the corresponding pseudodifferential operator* $P(x,D)$ *is the limit in* $L(\mathcal{D}_{(\omega)}(\Omega), \mathcal{D}'_{(\omega)}(\Omega))$ *of the sequence of operators* $P_N : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$, *where each* P_N *is the pseudodifferential operator with symbol* $\sum_{j=0}^{N}(\varphi_j - \varphi_{j+1})(\xi)(\sum_{l=0}^{j} p_l(x,\xi))$.

Proof: We first observe that

$$\sum_{j=0}^{N}(\varphi_j - \varphi_{j+1})(\xi)(\sum_{l=0}^{j} p_l(x,\xi)) = \sum_{j=0}^{N} \varphi_j(\xi)p_j(x,\xi) - \varphi_{N+1}(\xi)\sum_{j=0}^{N} p_j(x,\xi).$$

Let B be a bounded set in $\mathcal{D}_{(\omega)}(\Omega)$ and let K be a compact set in Ω. We will prove

(a) $\int (\sum_{j=0}^{N} \varphi_j(\xi)p_j(x,\xi)) e^{ix\xi}\hat{u}(\xi)d\xi \to \int (\sum_{j=0}^{\infty} \varphi_j(\xi)p_j(x,\xi)) e^{ix\xi}\hat{u}(\xi)d\xi$
and

(b) $\int \varphi_{N+1}(\xi)(\sum_{j=0}^{N} p_j(x,\xi)) e^{ix\xi}\hat{u}(\xi)d\xi \to 0$
as N goes to infinity, uniformly on $x \in K$ and $u \in B$. By the Paley Wiener theorem ([8, 3.4]) there is $D > 0$ with $|\hat{u}(\xi)| \leq De^{-(m+3)\omega(\xi)}$ for all $u \in B$. It follows from definition 3.1 that there is a sequence (C_n) with

$$|p_j(x,\xi)| \leq C_n \frac{e^{(\rho-\delta)n\varphi^*(\frac{j}{n})}}{(1+|\xi|)^{(\rho-\delta)j}} e^{m\omega(\xi)}$$

whenever $x \in K$ and $\log(\frac{|\xi|}{R}) \geq \frac{n}{j}\varphi^*(\frac{j}{n})$. Since $\varphi_j(\xi) \neq 0$ and $j_n < j \leq j_{n+1}$ imply $\log(\frac{|\xi|}{2R}) \geq \frac{n}{j}\varphi^*(\frac{j}{n})$ we get

$$|\varphi_j(\xi)p_j(x,\xi)\hat{u}(\xi)| \leq \frac{C_n}{(2R)^{j(\rho-\delta)}} De^{-3\omega(\xi)}.$$

We can assume $e^{-\omega(\xi)} \leq \frac{1}{|\xi|} \leq \frac{1}{e^{\frac{n}{j}\varphi^*(\frac{j}{n})}}$ for $\xi \in \mathrm{supp}\varphi_j$. For a fixed N one can find l such that $j_l < N \leq j_{l+1}$ and we have, using that $\frac{n}{j}\varphi^*(\frac{j}{n}) \geq \log C_n$ for every $n \in \mathbb{N}$ and $j_n < j \leq j_{n+1}$,

$$\sum_{j=N+1}^{\infty} \int |\varphi_j(\xi)p_j(x,\xi)\hat{u}(\xi)|d\xi \quad \leq D\sum_{n=l}^{\infty}\sum_{j=j_n+1}^{j_{n+1}} \frac{1}{(2R)^{j(\rho-\delta)}} \int e^{-2\omega(\xi)}d\xi,$$

which proves (a). To prove (b), given N we take n with $j_n+1 \leq N+1 \leq j_{n+1}$ and we note that $\varphi_{N+1}(\xi) \neq 0$ implies $\log(\frac{|\xi|}{2R}) \geq \frac{n}{N+1}\varphi^*(\frac{N+1}{n})$. Then

$$\begin{aligned} |\varphi_{N+1}(\xi)(\textstyle\sum_{j=0}^{N} p_j(x,\xi))| &\leq C_n \textstyle\sum_{j=0}^{N} \frac{e^{n(\rho-\delta)\varphi^*(\frac{j}{n})}}{|\xi|^{j(\rho-\delta)}} e^{m\omega(\xi)} \\ &\leq C_n \textstyle\sum_{j=0}^{\infty} \left(\frac{1}{2R}\right)^{(\rho-\delta)j} e^{m\omega(\xi)}. \end{aligned}$$

Hence

$$|\varphi_{N+1}(\xi)(\sum_{j=0}^{N} p_j(x,\xi))\hat{u}(\xi)| \le Ce^{-\omega(\xi)}e^{-\frac{n}{N+1}\varphi^*(\frac{N+1}{n})}$$

from where it follows (b), since $j_n+1 \le N+1 \le j_{n+1}$ implies $\frac{n}{N+1}\varphi^*(\frac{N+1}{n}) \ge n$. ∎

Lemma 3.10. *For every $n \in \mathbb{N}$ we have*

$$\lim_{N\to\infty} \frac{N}{e^{\frac{n}{N}\varphi^*(\frac{N}{n})}} = 0.$$

Proof: If this is not the case, we find $0 < \epsilon < 1$ and an unbounded sequence (N_k) of natural numbers such that

$$\epsilon^{-N_k}(N_k)^{N_k} \ge e^{n\varphi^*(\frac{N_k}{n})}.$$

An application of Stirling formula and [12, 2.1.2] gives a contradiction. ∎

Lemma 3.11. *Let $m \ge n$ and $\frac{1}{e}e^{\frac{m}{j}\varphi^*(\frac{j}{m})} \le t \le e^{\frac{n}{j}\varphi^*(\frac{j}{n})}$. Then*

$$|t|^{j+1} \ge e^{n\omega(t)}e^{2m\varphi^*(\frac{j}{2m})}e^{-j}.$$

In particular

$$e^{n\varphi^*(\frac{j}{n})} \ge e^{(n-1)\omega(t)}e^{2n\varphi^*(\frac{j}{2n})}$$

for j large enough.

Proof: By lemma 1.4 (2) we know that

$$n\omega(t) \le \log(t) + \sup_{k\in\mathbb{N}_0}\{k\log(t) - n\varphi^*(\frac{k}{n})\}$$

holds for $t \ge 1$. Since $0 < t \le e^{\frac{n}{j}\varphi^*(\frac{j}{n})}$ and $\frac{\varphi^*(t)}{t}$ is an increasing function we deduce that $n\omega(t) \le \log(t) + l\log(t) - n\varphi^*(\frac{l}{n})$ for some $0 \le l \le j$ and $\frac{m}{j}\varphi^*(\frac{j}{m}) \ge \frac{m}{j-l}\varphi^*(\frac{j-l}{m})$. Hence

$$\begin{aligned} t^j &= t^l e^{-n\varphi^*(\frac{l}{n})}t^{j-l}e^{n\varphi^*(\frac{l}{n})} \\ &\ge e^{n\omega(t)-\log(t)}\Big(\frac{e^{\frac{m}{j}\varphi^*(\frac{j}{m})}}{e}\Big)^{j-l}e^{m\varphi^*(\frac{l}{m})} \\ &\ge e^{n\omega(t)-\log(t)}e^{m\varphi^*(\frac{j-l}{m})+m\varphi^*(\frac{l}{m})}e^{-j} \\ &\ge e^{n\omega(t)-\log(t)}e^{-j}e^{2m\varphi^*(\frac{j}{2m})} \end{aligned}$$

the last inequality being a consequence of 1.4(1). The second statement in the lemma follows as the above inequality with $t = e^{\frac{n}{j}\varphi^*(\frac{j}{n})}$. ∎

Lemma 3.12. *Let $\sigma(t) = t^d$, $0 < d < 1$, and let ω be a weight function such that $\omega(t) = o(\sigma(t))$. Then there are $\lambda > 0$ and a sequence (j_n) of natural numbers such that*

$$\lambda\sigma(e^{\frac{n}{j}\varphi^*(\frac{j}{n})}) \ge j$$

for every $j \ge j_n$.

Proof: For every $n \in \mathbb{N}$ there is $A_n > 0$ with $\omega(t) \leq A_n + \frac{1}{n}\sigma(t)$ for all $t \geq 0$. Hence $\frac{n}{j}\varphi^*_\omega(\frac{j}{n}) \geq -\frac{nA_n}{j} + \frac{1}{j}\varphi^*_\sigma(j)$. We take j_n satisfying $\frac{nA_n}{j_n} \leq 1$. Now the conclusion follows from the fact that $\frac{1}{j}\varphi^*_\sigma(j) = \frac{1}{d}\log(\frac{j}{ed})$. ■

Theorem 3.13. *Let ω be a weight such that $\omega(t) = o(t^d)$, $d \leq \rho - \delta$, $d < 1$. Let $a \in AS^{m,\omega}_{\rho,\delta}(U)$ with associated pseudodifferential operator A and let Ω be a relatively compact open subset of U. Then there are a pseudodifferential operator $P(x,D) : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ and a (ω)-smoothing operator $R : \mathcal{E}'_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ such that $A\varphi = P(x,D)\varphi + R\varphi$ for every $\varphi \in \mathcal{D}_{(\omega)}(\Omega)$. Moreover*

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_j(x,\xi)$$

where $p_j(x,\xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha_\xi \partial^\alpha_y a(x,y,\xi)_{|y=x}$.

Proof: For $p_j(x,\xi)$ as above, we take $p(x,\xi)$ as in lema 3.9 and $P := P(x,D)$. According to the previous lemmata, the operator $A-P : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ can be represented as $A - P = \sum_{N=0}^{\infty} P_N$, where $P_N(u)(x) = \int K_N(x,y)u(y)dy$ and the series is convergent in $L(\mathcal{D}_{(\omega)}(\Omega), \mathcal{D}'_{(\omega)}(\Omega))$. Here

$$K_N(x,y) := \int (\varphi_N - \varphi_{N+1})(\xi)(a(x,y,\xi) - \sum_{j=0}^{N} p_j(x,\xi))e^{i(x-y)\xi}d\xi$$

is a function in $\mathcal{E}_{(\omega)}(\Omega \times \Omega)$, as easily follows from the fact that $\varphi_N - \varphi_{N+1}$ has compact support. Hence, each operator P_N is (ω)-smoothing and our aim is to show that also $\sum_{N=0}^{\infty} P_N$ is (ω)-smoothing. To do this we need to obtain a different representation for this series. There is no loss of generality to assume that Ω is convex (in fact, in view of theorem 2.17 we only have to show that every point $x \in \Omega$ admits a neighborhood W such that the kernel distribution of $A-P$ is an ultradifferentiable function of (ω)-class in $W \times W$.)
Proceeding as in [25, 2.25] we get, for $N \geq 1$, $K_N = \sum_{|\alpha|=1}^{N} A^N_\alpha + R_N$, where

$$A^N_\alpha(x,y) := \sum_{0 \neq \beta \leq \alpha} \frac{1}{\beta!(\alpha-\beta)!} \int e^{i(x-y)\xi} \sigma_{N,\alpha,\beta}(x,\xi)d\xi$$

for

$$\sigma_{N,\alpha,\beta}(x,\xi) := D^\beta_\xi(\varphi_N(\xi) - \varphi_{N+1}(\xi))D^{\alpha-\beta}_\xi \partial^\alpha_y a(x,x,\xi)$$

and

$$R_N(x,y) := \sum_{|\alpha|=N+1} \sum_{\beta \leq \alpha} \frac{1}{\beta!(\alpha-\beta)!} \int e^{i(x-y)\xi} \tau_{N,\alpha,\beta}(x,y,\xi)d\xi$$

for

$$\tau_{N,\alpha,\beta}(x,y,\xi) := D^\beta_\xi(\varphi_N(\xi) - \varphi_{N+1}(\xi))D^{\alpha-\beta}_\xi \omega_\alpha(x,y,\xi).$$

Here

$$\omega_\alpha(x,y,\xi) := (N+1)\int_0^1 \partial_y^\alpha a(x,x+t(y-x),\xi)(1-t)^N dt.$$

On the other hand $\sum_{r=1}^N \sum_{|\alpha|=1}^r A_\alpha^r = \sum_{j=1}^N I_j - W_N$, where

$$I_j(x,y) := \sum_{|\alpha|=j}\sum_{0\neq\beta\leq\alpha}\frac{1}{\beta!(\alpha-\beta)!}\int e^{i(x-y)\xi}D_\xi^\beta\varphi_j(\xi)D_\xi^{\alpha-\beta}\partial_y^\alpha a(x,x,\xi)d\xi$$

and

$$W_N(x,y) := \sum_{|\alpha|=1}^N\sum_{0\neq\beta\leq\alpha}\frac{1}{\beta!(\alpha-\beta)!}\int e^{i(x-y)\xi}D_\xi^\beta\varphi_{N+1}(\xi)D_\xi^{\alpha-\beta}\partial_y^\alpha a(x,x,\xi)d\xi.$$

Hence $\sum_{j=1}^N K_j = \sum_{j=1}^N I_j + \sum_{j=1}^N R_j - W_N$. To finish the proof of the Theorem we will show that $\sum_{j=1}^\infty R_j(x,y)$ and $\sum_{j=1}^\infty I_j(x,y)$ converge in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$ and that the sequence of operators defined by the kernels (W_N) converges to the null operator as N goes to infinity.

(a) Let $j_n < j \leq j_{n+1}$. Then $|D_x^\mu D_y^\nu I_j(x,y)|$ is not greater than

$$\sum_{|\alpha|=j}\sum_{0\neq\beta\leq\alpha}\sum_{\gamma\leq\mu}\binom{\mu}{\gamma}\frac{1}{\beta!(\alpha-\beta)!}\int|\xi|^{|\gamma+\nu|}|D_\xi^\beta\varphi_j(\xi)||D_x^{\mu-\gamma}(D_\xi^{\alpha-\beta}\partial_y^\alpha a(x,x,\xi))|d\xi.$$

We fix $k\in\mathbb{N}$ and we take $n\geq k$ and $\ell := 2n$. If $D_\xi^\beta\varphi_j(\xi)\neq 0$ then

$$2Re^{\frac{n}{j}\varphi^*(\frac{j}{n})} \leq |\xi| \leq 3Re^{\frac{n}{j}\varphi^*(\frac{j}{n})}$$

and we have that $|D_x^{\mu-\gamma}D_\xi^{\alpha-\beta}\partial_y^\alpha a(x,x,\xi)|$ is less than or equal to

$$\begin{aligned}&C_{2\ell}(\alpha-\beta)!B^{|\alpha-\beta|}e^{(\rho-\delta)2l\varphi^*(\frac{|\mu-\gamma+\alpha|}{2l})}e^{m\omega(\xi)}|\xi|^{\delta(|\mu-\gamma|+|\alpha|)-\rho|\alpha-\beta|}\leq\\ &C_{2\ell}(\alpha-\beta)!B^{|\alpha-\beta|}e^{(\rho-\delta)(k\varphi^*(\frac{|\mu-\gamma|}{k})+\ell\varphi^*(\frac{j}{\ell}))}e^{m\omega(\xi)}|\xi|^{\delta|\mu-\gamma|+\rho|\beta|-(\rho-\delta)j}\end{aligned}$$

by using 1.3(1) and $k\leq\ell$. We now use that $|D_\xi^\beta\varphi_j(\xi)|\leq C(\frac{D}{|\xi|})^{|\beta|}j^{|\beta|+1}$and $|\xi|^{\delta|\mu-\gamma|}e^{-\delta k\varphi^*(\frac{|\mu-\gamma|}{k})}\leq e^{\delta k\omega(\xi)}$. Moreover $e^{-(n-1)\omega(\frac{\xi}{3R})}e^{n\varphi^*(\frac{j}{n})}\geq e^{2n\varphi^*(\frac{j}{2n})}$ (see lemma 3.11) and $|\xi|^{|\nu+\gamma|}\leq e^{k\varphi^*(\frac{|\nu+\gamma|}{k})+k\omega(\xi)}$, thus $|D_x^\mu D_y^\nu I_j(x,y)|$ is less than or equal to

$$\sum_{|\alpha|=j}\frac{C_{4n}B^je^{k\varphi^*(\frac{|\mu+\nu|}{k})}}{(2R)^{(\rho-\delta)j}}\sum_{\substack{0\neq\beta\leq\alpha\\ \gamma\leq\mu}}\binom{\mu}{\gamma}\frac{1}{\beta!}CD^{|\beta|}j^{|\beta|+1}\int e^{(2k+m)\omega(\xi)-(n-1)(\rho-\delta)\omega(\frac{\xi}{3R})}d\xi.$$

Given k we can select n large enough in order to ensure that the integral above is less than 1. Then, for $j_n < j \leq j_{n+1}$, we obtain

$$|D_x^\mu D_y^\nu I_j(x,y)|\leq CC_{4n}j^{p+1}B^j\frac{e^{Djp^2}}{(2R)^{(\rho-\delta)j}}e^{|\mu|+k\varphi^*(\frac{|\mu+\nu|}{k})}.$$

Proceeding as in the proof of 3.7 we can select the sequence (j_n) and the constant $R>0$ in order to guarantee the convergence of $\sum I_j(x,y)$ in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$.

(b) With a similar argument it is possible to prove that $\sum_{j=1}^{\infty} R_j(x,y)$ converges in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$ for a suitable choice of (j_n) and $R>0$. In fact, we recall that

$$R_j(x,y)=\sum_{|\alpha|=j+1}\sum_{\beta\leq\alpha}\frac{1}{\beta!(\alpha-\beta)!}\int e^{i(x-y)\xi}\tau_{j,\alpha,\beta}(x,y,\xi)d\xi.$$

Hence
$|D_x^\mu D_y^\nu R_j(x,y)|\leq$

$$\sum_{|\alpha|=j+1}\sum_{\beta\leq\alpha}\sum_{\gamma\leq\mu}\sum_{r\leq\nu}\binom{\mu}{\gamma}\binom{\nu}{r}\frac{1}{\beta!(\alpha-\beta)!}\int|\xi|^{|\gamma+r|}|D_x^{\mu-\gamma}D_y^{\nu-r}\tau_{j,\alpha,\beta}(x,y,\xi)|d\xi.$$

Now, for a fixed $k\in\mathbb{N}$ we take $n\geq k$ and $\ell=2n+2$. Then we have that $|D_x^{\mu-\gamma}D_y^{\nu-r}\tau_{j,\alpha,\beta}(x,y,\xi)|$ is less than or equal to the product of

$$|D_\xi^\beta(\varphi_j-\varphi_{j+1})(\xi)|(j+1)$$

by

$$\sum_{s\leq\mu-\gamma}\binom{\mu-\gamma}{s}\int_0^1|D_y^{\alpha+\nu-r}D_x^sD_y^{\mu-\gamma-s}D_\xi^{\alpha-\beta}a(x,x+t(y-x),\xi)|dt.$$

The above integral is dominated by

$$C_{2\ell}B^{|\alpha-\beta|}(\alpha-\beta)!\frac{e^{(\rho-\delta)\left(k\varphi^*(\frac{|\mu+\nu-r-\gamma|}{k})+\ell\varphi^*(\frac{j+1}{\ell})\right)}}{|\xi|^{\rho|\alpha-\beta|-\delta|\mu+\nu+\alpha-r-\gamma|}}e^{m\omega(\xi)}.$$

Having in mind

$$|\xi|^{\delta|\mu+\nu-r-\gamma|}e^{-\delta k\varphi^*(\frac{|\mu+\nu-r-\gamma|}{k})}\leq e^{\delta k\omega(\xi)}$$

(lemma 1.4) and

$$|\xi|^{|\gamma+r|}e^{\rho k\varphi^*(\frac{|\mu+\nu-r-\gamma|}{k})}\leq e^{k\omega(\xi)}e^{k\varphi^*(\frac{|\mu+\nu|}{k})}$$

(1.3 and 1.4) we conclude that

$$|\xi|^{|\gamma+r|}|D_x^{\mu-\gamma}D_y^{\nu-r}\tau_{j,\alpha,\beta}(x,y,\xi)|$$

is less than or equal to the product of $|D_\xi^\beta(\varphi_j-\varphi_{j+1})(\xi)|$ by

$$\frac{e^{(k+m+\delta k)\omega(\xi)}}{|\xi|^{\rho(j+1-|\beta|)-\delta(j+1)}}2^{|\mu-\gamma|}B^{j+1-|\beta|}C_{2\ell}(\alpha-\beta)!(j+1)e^{k\varphi^*(\frac{|\mu+\nu|}{k})}e^{(\rho-\delta)\ell\varphi^*(\frac{j+1}{\ell})}.$$

An application of the previous lemmata permits to conclude that, we can select n in such a way that the above estimate is less than

$$CD^{|\beta|}(j+1)^{|\beta|}C_{2\ell}\frac{e^{-\omega(\xi)}}{(R)^{(\rho-\delta)(j+1)}}2^{|\mu-\gamma|}B^{j+1-|\beta|}(\alpha-\beta)!(j+1)^2e^{k\varphi^*(\frac{|\mu+\nu|}{k})}.$$

Hence

$$|D_x^\mu D_y^\nu R_j(x,y)|\leq(\frac{e^{p^2D}B}{R^{\rho-\delta}})^{j+1}(j+1)^{p+2}C_{2\ell}2^{2|\mu+\nu|}e^{k\varphi^*(\frac{|\mu+\nu|}{k})}.$$

From where it follows that, after choosing (j_n) and R in the proper way, the series $\sum R_j$ converges in $\mathcal{E}_{(\omega)}(\Omega\times\Omega)$.

(*c*) Let $T_N : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ be the operator with kernel W_N. Since $\sum_{N=0}^{\infty} P_N$ converges in $L(\mathcal{D}_{(\omega)}(\Omega), \mathcal{D}'_{(\omega)}(\Omega))$, we deduce from (*a*) and (*b*) that (T_N) converges to an operator $T : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{D}'_{(\omega)}(\Omega)$ in $L(\mathcal{D}_{(\omega)}(\Omega), \mathcal{D}'_{(\omega)}(\Omega))$. In order to show that $T = 0$ it is sufficient to prove that T vanishes on the dense subset $\mathcal{D}_{(\sigma)}(\Omega)$, $\sigma(t) := t^d$. To do this, we fix $N \in \mathbb{N}$, $j_n < N+1 \leq j_{n+1}$, and we put $a_N := Re^{\frac{n}{N+1}\varphi^*(\frac{N+1}{n})}$. Then $D_\xi^\beta \varphi_{N+1}(\xi) \neq 0$ implies that $2a_N \leq |\xi| \leq 3a_N$. For every $u \in \mathcal{D}_{(\sigma)}(\Omega)$ we have

$$\begin{aligned} |T_N(u)(x)| &\leq \sum_{|\alpha|=1}^N \sum_{0\neq\beta\leq\alpha} \frac{1}{\beta!(\alpha-\beta)!} \Big| \int D_\xi^\beta \varphi_{N+1}(\xi) D_\xi^{\alpha-\beta} \partial_y^\alpha a(x,x,\xi)\hat{u}(\xi) d\xi \Big| \\ &\leq \sum_{|\alpha|=1}^N \sum_{0\neq\beta\leq\alpha} \frac{D^{|\beta|}(N+1)^{|\beta|+1}}{\beta!} \frac{B^{|\alpha-\beta|}C_n}{(2R)^{(\rho-\delta)|\alpha|}} \int_{|\xi|\geq 2a_N} e^{m\omega(\xi)} |\hat{u}(\xi)| d\xi. \end{aligned}$$

Let λ be given as in lema 3.12. Then $\lambda\sigma(a_N) \geq N+1$. For every $u \in \mathcal{D}_{(\sigma)}(\Omega)$ we have $|\hat{u}(\xi)| \leq e^{-(m+\lambda p^2+1)\sigma(\xi)}$ for $|\xi|$ large enough. Since $\log(t) = o(\sigma(t))$ we obtain

$$\int_{|\xi|\geq 2a_N} e^{m\omega(\xi)} |\hat{u}(\xi)| d\xi \leq \frac{e^{-\lambda p^2 \sigma(2a_N)}}{(2a_N)^2} < \frac{1}{2a_N e^{p^2(N+1)}}$$

for N large enough.
On account of lemma 3.10, we can assume that $\frac{j}{e^{\frac{n}{j}\varphi^*(\frac{j}{n})}} \leq \frac{1}{2^n}$ for $j \geq j_n$. Consequently, since $\sum_{\beta\leq\alpha} \frac{(N+1)^{|\beta|}}{\beta!} \leq e^{p^2(N+1)}$, we get

$$|T_N(u)(x)| \leq \frac{1}{2^n} \sum_{|\alpha|=1}^N \Big(\frac{BD}{2R^{\rho-\delta}}\Big)^{|\alpha|} \frac{N+1}{a_N} \frac{C_n}{2a_N} \leq \frac{1}{2^n} \sum_{j=1}^N \Big(\frac{pDB}{R^{\rho-\delta}}\Big)^j$$

from where we deduce that $T_N(u)(x)$ converges to 0 uniformly on $x \in \Omega$ as N goes to infinity. ■

In order to compose pseudodifferential operators it is useful to consider operations with formal sums.

Proposition 3.14. *Let $P(x,D)$ be the operator associated to $p(x,\xi) \in AS_{\rho,\delta}^{m,\omega}(U)$ and let Ω be a relatively compact open subset of U. Then the transposed operator (restricted to $\mathcal{D}_{(\omega)}(\Omega)$) can be decomposed as $P(x,D)^t = Q(x,D) + R$, where R is (ω)-smoothing and $Q(x,D)$ is defined by a symbol $q(x,\xi) \sim \sum q_j$, and we have $q_j(x,\xi) := \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha p(x,-\xi)$.*

Proof: We already know that $P(x,D)^t$ is the operator associated to $p(y,-\xi)$. Then, it suffices to apply 3.13. ■

Given $\sum p_j \in FAS_{\rho,\delta}^{m,\omega}(\Omega)$, standard calculations ([25]) and the properties of φ^* permit to prove that $\sum q_j$, where $q_j(x,\xi) := \sum_{|\alpha|+h=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha p_h$, is a formal sum.

Analogously, if $\sum p_j \in FAS_{\rho,\delta}^{m_1,\omega}(\Omega)$, $\sum q_j \in FAS_{\rho,\delta}^{m_2}$, one can prove that $\sum r_j \in FAS_{\rho,\delta}^{m_1+m_2,\omega}(\Omega)$, where $r_j(x,\xi) = \sum_{|\alpha|+k+h=j} \frac{1}{\alpha!} \partial_\xi^\alpha p_h D_x^\alpha q_j$, is a formal sum.

Definition 3.15. (1) *For* $\sum p_j \in FAS^{m,\omega}_{\rho,\delta}(\Omega)$ *we define* $(\sum p_j)^t$ *as the formal sum* $\sum_j q_j$, *where* q_j *is as before.*
(2) *For* $\sum p_j \in FAS^{m_1,\omega}_{\rho,\delta}(\Omega)$, $\sum q_j \in FAS^{m_2}_{\rho,\delta}$ *we define* $(\sum p_j) \circ (\sum q_j) = \sum r_j$, *where* r_j *is as above.*

The two following results are straightforward, therefore we omit their proof ([25]).

Proposition 3.16. (1) $((\sum p_j)^t)^t \sim \sum p_j$.
(2) *If* $\sum p_j \sim \sum p'_j$ *and* $\sum q_j \sim \sum q'_j$, *then* $(\sum p_j)\circ(\sum q_j) \sim (\sum p'_j)\circ(\sum q'_j)$.

Lemma 3.17. *Let* $\Omega \subset \mathbb{R}^p$ *be an open bounded set, and let* $p(x,\xi), q(x,\xi) \in AS^{m,\omega}_{\rho,\delta}(\Omega)$. *Assume that* $b(x,\xi) \in AS^{m,\omega}_{\rho,\delta}(\Omega)$ *satisfies* $b(x,\xi) \sim q^t(x,-\xi)$ *and that* $r(x,\xi) \in AS^{2m,\omega}_{\rho,\delta}(\Omega)$ *is equivalent to* $\sum_j \sum_{|\alpha|=j} \frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_y(p(x,\xi)b(y,\xi))|_{y=x}$. *Then,* $r(x,\xi) \sim p(x,\xi) \circ q(x,\xi)$.

Theorem 3.18. *Let* $p(x,\xi), q(x,\xi) \in AS^{m,\omega}_{\rho,\delta}(U)$ *be given and let* Ω *be an open set which is relatively compact set in* U. *Let us denote by* P *and* Q *the corresponding pseudodifferential operators and assume that either* P *or* Q *is properly supported. Then,* $P \circ Q : \mathcal{D}_{(\omega)}(\Omega) \to \mathcal{E}_{(\omega)}(\Omega)$ *coincides, modulo an* (ω)*-smoothing operator, with the pseudodiferential operator associated to* $(2\pi)^p(p(x,\xi) \circ q(x,\xi))$.

Proof: Let us assume that P is properly supported. We take Ω_1, relatively compact and open in U containing $\overline{\Omega}$.

We know that $Q = (Q^t)^t$ and that Q^t is given by $q(y,-\xi)$. Therefore $Q^t = Q' + T'$ on Ω_1, where T' is (ω)-smoothing and Q' is given by a symbol q' on Ω_1, which is equivalent to q^t. Since the class of (ω)-smoothing operators is closed by taking transposes on Ω_1, Q coincides, modulo some (ω)-smoothing operator, with the operator Q_1 associated to $b(y,\xi) := q'(y,-\xi) \sim q^t(y,-\xi)$. As composing P with any (ω)-smoothing operator is again (ω)-smoothing, $P \circ Q - P \circ Q_1$ is (ω)-smoothing.

Given $f \in \mathcal{D}_{(\omega)}(\Omega)$ we have that $Q_1 f \in \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^p)$ (prop. 2.16), therefore $P(Q_1 f)(x) = \int p(x,\xi)\widehat{Q_1 f}(\xi)d\xi$ (prop. 2.4). But, from 2.16, $Q_1 f(x) = \hat{I}(-x)$, therefore $\widehat{Q_1 f}(\xi) = (2\pi)^p I(\xi)$. That is, $P \circ Q_1$ is the pseudodifferential operator associated to $a(x,y,\xi) = (2\pi)^p p(x,\xi)b(y,\xi)$. We apply 3.13 and 3.17 to conclude. ∎

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References

[1] **J.J. Betancor, C. Fernández, A. Galbis**, *Beurling ultradistributions of* L_p*-growth*, J. Math. Anal. Appl. **279** (2003), 246-265.

[2] **G. Björk**, *Linear partial differential operators and generalized distributions*, Ark. Mat., **6** (1965), 351-407.

[3] **J. Bonet, C. Fernández, R. Meise**, *Characterization of the ω-hypoelliptic convolution operators on ultradistributions,* Ann. Acad. Sci. Fen. Math. **25** (2000), 261-284.

[4] **J. Bonet, C. Fernández, R. Meise**, *Operators of solution for convolution equations*, (Proceedings of the Second International Workshop held at Trier University 1997), Note di Mat. **17** (1997), 1-12.

[5] **L. Boutet de Monvel**, *Opérateurs pseudo-différentiels analytiques et opérateurs d'ordre infini*, Ann. Inst. Fourier, **22** (1972), 229-268.

[6] **L. Boutet De Monvel, P. Kree**, *Pseudodifferential operators and Gevrey classes*, Ann. Inst. Fourier, **17** (1967), 295-323.

[7] **R.W. Braun**, *An extension of Komatsu's second structure theorem for ultradistributions*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **40** (1993), 411-417.

[8] **R.W. Braun, R. Meise and B.A. Taylor**, *Ultradifferentiable functions and Fourier analysis*, Result. Math., **17** (1990), 206-237.

[9] **A.P. Calderón**, *Lecture notes on pseudo-differential operators and elliptic boundary value problems*, Consejo Nacional de Investigaciones Científicas y Técnicas. Instituto Argentino de Matemática. Buenos Aires (1976).

[10] **C.C. Chou**, *La transformation de Fourier complexe et l'equation de convolution,* Lecture Notes in Math., **325**, Springer, Berlin, 1973.

[11] **C. Fernández, A. Galbis, M.C. Gómez-Collado**, *Elliptic convolution operators on non-quasianalytic classes,* Arch. Math., **76** (2001), 133-140.

[12] **M.C. Gómez-Collado**, Ultradistribuciones casi-periódicas y operadores de convolucion elípticos, PhD. Universidad de Valencia (2000).

[13] **A. Grigis, J.A. Sjöstrand**, *Microlocal analysis for differential operators. An introduction*, London Mathematical Society Lecture Note Series, **196**. Cambridge University Press, Cambridge (1994).

[14] **L. Hörmander**, *Pseudo-differential operators*, Comm. Pure Appl. Math **18** (1965), 129-209.

[15] **S. Hashimoto, T. Matsuzawa, Y. Morimoto**, *Operateurs pseudo-differentiels et classes de Gevrey*, Comm. in Part. Diff. Eq., **8** (1983), 1277-1289.

[16] **H. Komatsu**, *Ultradistributions I, Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo, Sec. I.A, **20** (1973), 25-105.

[17] **H. Komatsu**, *An introduction to the theory of generalized functions*, Department of Mathematics. Science University of Tokyo (2000).

[18] **J.J. Kohn, L. Nirenberg**, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math **18** (1965), 269-305.

[19] **M. Langenbruch**, *Continuation of Gevrey regularity for solutions of partial differential operators,* In:"Functional Analysis" (Proceedings of the First International Workshop held at Trier University 1994), Walter de Gruyter (1996), 249-280.

[20] **O. Liess, L. Rodino**, *Inhomogeneous Gevrey classes and related pseudodifferential operators*, Boll. Un. Mat. Ital. Analisi Funz. e Appl., Serie VI, vol. III-C, n. 1 (1984), 233-323.

[21] **W. Matsumoto**, *Theory of pseudo-differential operators of ultradifferentiable class*, J. Math. Kyoto Univ. **27** (1987), 453-500.

[22] **R. Meise, B.A. Taylor**, *Whitney's extension theorem for ultradifferentiable functions of Beurling type*, Ark. Mat. **26** (1988), 265-287.

[23] **L. Rodino**, *Linear partial differential operators in Gevrey spaces*, World Scientific Pub. (1993).

[24] **F. Treves**, *Introduction to pseudodifferential and Fourier integral operators I*, Plenum Press, New York (1980).

[25] **L. Zanghirati**, *Pseudodifferential operators of infinite order and Gevrey classes*, Ann. Univ. Ferrara, Sez. VII, Sc. Mat. **31** (1985), 197-219.

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