Collocation methods for high-order well-balanced methods for systems of balance laws

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Outline

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- We consider 1d systems of balance laws of the form

$$U_t(x,t) + f(U(x,t))_x = S(U(x,t))H_x(x), \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

where $U(x,t)$ takes values in $\Omega \subset \mathbb{R}^N$, $f : \Omega \to \mathbb{R}^N$ is the flux function; $S : \Omega \to \mathbb{R}^N$; and H is a known function from $\mathbb{R} \to \mathbb{R}$ (possibly the identity function $H(x) = x$), which is supposed to be a continuous function.

- We suppose that system (1) is strictly hyperbolic, that is $D_f(U) = \dfrac{\partial f}{\partial U}(U)$ has N real different eigenvalues.

- **Stationary solutions:**

$$f(U)_x = S(U)H_x.$$

Main objective

To design high-order finite volume methods that solve exactly or with enhanced accuracy all the stationary solutions of the system (1) or, at least, a relevant family of them: Well-balanced high-order finite volume schemes.

Well-balanced high-order finite volume scheme

- We consider high-order finite volume numerical methods of the form:

$$\frac{dU_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(t) - F_{i-\frac{1}{2}}(t)\right) + \frac{1}{\Delta x} S_i, \quad (2)$$

where

- $U_i(t)$ is the approximation given by the numerical method of the average of the exact solution at the ith cell, $I_i = [x_{i-1/2}, x_{i+1/2}]$ at time t;
- $$S_i \approx \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(P_i^t(x)) H_x(x)\, dx.$$
- $P_i^t(x)$ is the approximation of the solution at the ith cell given by a reconstruction operator of order p from the sequence of cell averages $\{U_i(t)\}$;
- $F_{i+\frac{1}{2}} = \mathbb{F}(U^{t,-}_{i+\frac{1}{2}}, U^{t,+}_{i+\frac{1}{2}})$;
- $\mathbb{F}$ is a consistent first order numerical flux;
- $U^{t,-}_{i+1/2} = P_i^t(x_{i+1/2}), \quad U^{t,+}_{i+1/2} = P_{i+1}^t(x_{i+1/2})$;

Well-balanced high-order finite volume scheme

- We consider high-order finite volume numerical methods of the form:

$$\frac{dU_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(t) - F_{i-\frac{1}{2}}(t)\right) + \frac{1}{\Delta x}S_i, \tag{2}$$

- Given a continuous stationary solution U^* of (1), the following notation will be used:

$$\bar{U}_i^* = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U^*(x)\,dx, \quad U^*_{i+1/2} = U^*(x_{i+1/2}), \quad \forall i.$$

Definition

The numerical method (2) is said to be exactly well-balanced for a stationary solution U^* of (1) if the sequence of its cell-averages $\{\bar{U}_i^*\}$ is an equilibrium of the ODE system (2).

Definition

The numerical method (2) is said to be well-balanced for a stationary solution U^* of (1) if there exists an equilibrium $\{\widetilde{U}^*_{\Delta x,i}\}$ of (2) such that

$$\bar{U}_i^* = \widetilde{U}^*_{\Delta x,i} + O(\Delta x^q), \quad \forall i, \tag{3}$$

for some $q \geq p$.

- The equilibria $\{\widetilde{U}^*_{\Delta x,i}\}$ of (2) will be also called discrete stationary solutions.

Well-balanced high-order finite volume scheme

- We consider high-order finite volume numerical methods of the form:

$$\frac{dU_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(t) - F_{i-\frac{1}{2}}(t)\right) + \frac{1}{\Delta x}S_i, \qquad (2)$$

Definition

A numerical method is said to be fully exactly well-balanced (resp. fully well-balanced) if it is exactly well-balaced (resp. well-balanced) for every stationary solution U^*.

Remark

A numerical method that is exactly well-balanced for a stationary solution is well-balanced for that solution: it is enough to take

$$\widetilde{U}^*_{\Delta x,i} = \bar{U}^*_i, \quad \forall i.$$

Well-balanced high-order finite volume scheme

- We consider high-order finite volume numerical methods of the form:

$$\frac{dU_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}}(t) - F_{i-\frac{1}{2}}(t)\right) + \frac{1}{\Delta x} S_i, \qquad (2)$$

Definition

Given a stationary solution U^* of (1), the reconstruction operator is said to be well-balanced for U^* if

$$P_i(x) = U^*(x), \quad \forall x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \forall i,$$

where P_i is the approximation of U^* obtained by applying the reconstruction operator to the vector $\{\bar{U}_i^*\}$ of cell-averages of U^*.

Theorem

If the reconstruction operator is well-balanced for U^*, then the numerical method (2) with

$$S_i = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(P_i^t(x)) H_x(x)\, dx,$$

is exactly well-balanced for U^*.

Well-balanced high order reconstruction operator

But, a standard reconstruction operator is not expected in general to be well-balanced.

Reconstruction procedure: known stationary solutions and exact integration (see [CGLGP08])

Given a family of cell values $\{U_i\}$, at every cell $[x_{i-1/2}, x_{i+1/2}]$:

1. Look for the stationary solution $U_i^*(x)$ such that

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_i^*(x)\,dx = U_i. \tag{3}$$

2. Apply the reconstruction operator to the cell values $\{V_j\}_{j\in\mathcal{S}_i}$ given by

$$V_j = U_j - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U_i^*(x)\,dx,$$

   to obtain

$$Q_i(x) = Q_i(x;\{V_j\}_{j\in\mathcal{S}_i}).$$

3. Define

$$P_i(x) = U_i^*(x) + Q_i(x). \tag{4}$$

Well-balanced high order reconstruction operator

But, a standard reconstruction operator is not expected in general to be well-balanced.

- The reconstruction operator P_i in (4) is well-balanced for every stationary solution provided that the reconstruction operator Q_i is exact for the null function. Moreover, if Q_i is conservative, then P_i is conservative, that is,

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} P_i(x)\,dx = U_i,\ \forall i,$$

  and P_i is of the same order of accuracy p of Q_i provided that the stationary solutions are smooth.

Well-balanced high-order finite volume scheme: quadrature formula

- The well-balanced property of the method can be lost if a quadrature formula is used to compute the integral appearing at the right-hand side of (2).
- To avoid this problem the source term is written in the equivalent form:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(P_i^t(x))H_x(x)\,dx = f\left(U_i^{t,*}(x_{i+\frac{1}{2}})\right) - f\left(U_i^{t,*}(x_{i-\frac{1}{2}})\right) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left((S(P_i^t(x)) - S(U_i^{t,*}(x))\right) H_x(x)\,dx.$$

where $U_i^{t,*}$ is the stationary solution found at the first step of the reconstruction procedure at the ith cell and time t.

Well-balanced high-order finite volume scheme: quadrature formula

- Now, a high-order quadrature formula could be used to approximate the source term integral:

$$
\begin{aligned}
S_i = & f\left(U_i^{t,*}(x_{i+\frac{1}{2}})\right) - f\left(U_i^{t,*}(x_{i-\frac{1}{2}})\right) \\
& + \Delta x \sum_{m=1}^{M} b_m \left(S(P_i^t(x_i^m)) - S(U_i^{t,*}(x_i^m))\right) H_x(x_i^m),
\end{aligned} \tag{3}
$$

  where x_i^m, b_m, $m = 1, \ldots, M$ are respectively the points and the weight of the quadrature formula chosen in the cell I_i, whose order of accuracy s is bigger or equal than p.

- If the quadrature formula is used to compute cell averages, i.e.

$$
\bar{U}_i^* \approx \widetilde{U}_i^* = \sum_{m=1}^{M} b_m U^*(x_i^m),
$$

  the numerical method is still well-balanced provided that the reconstruction operator is computed as follows:

Well-balanced high-order finite volume scheme: quadrature formula

Reconstruction procedure: known stationary solutions, numerical integration (see [CP20])

Given a family of cell values $\{U_i\}$, at every cell I_i:

1. Look for the stationary solution $U_i^*(x)$ such that:

$$\sum_{m=1}^{M} b_m U_i^*(x_i^m) = U_i. \tag{3}$$

2. Apply the reconstruction operator to the cell values $\{V_j\}_{j\in\mathcal{S}_i}$ given by

$$V_j = U_j - \sum_{m=1}^{M} b_m U_i^*(x_j^m), \quad j \in \mathcal{S}_i,$$

   to obtain:

$$Q_i(x) = Q_i(x; \{V_j\}_{j\in\mathcal{S}_i}).$$

3. Define

$$P_i(x) = U_i^*(x) + Q_i(x).$$

Well-balanced high order reconstruction operator: first step

- The first step of the reconstruction procedure is equivalent to find the solution of the ODE system

$$f(U)_x = S(U)\, H_x, \tag{4}$$

with prescribed average in the integration domain.

- Sometimes the first step has to be computed numerically: we need a numerical solver with order of accuracy at least p for the following local problems:

Local problems

(LP) Given an index i and a state $W \in \Omega$ find, if it is possible, approximations

$$U^{*,m}_{i,j},\ m = 1, \ldots, M,\ j \in \mathcal{S}_i; \quad U^*_{i,i\pm 1/2};$$

of the values

$$U^*_i(x^m_j),\ m = 1, \ldots, M,\ j \in \mathcal{S}_i; \quad U^*_i(x_{i\pm 1/2});$$

where U^*_i is the stationary solution that satisfies

$$\sum_{m=1}^{M} b_m U^*_i(x^m_i) = W. \tag{5}$$

Well-balanced high order reconstruction operator: first step

Reconstruction procedure: unknown stationary solutions, numerical integration.

Given a family of cell values $\{U_i\}$, at every cell I_i:

1. Apply the local solver at the ith cell with $W = U_i$ to obtain
$$U^{*,m}_{i,j}, \quad m = 1, \ldots, M, \; j \in \mathcal{S}_i; \quad U^*_{i,i\pm 1/2}.$$
2. Apply the reconstruction operator to the cell values $\{V_j\}_{j\in\mathcal{S}_i}$ given by
$$V_j = U_j - \sum_{m=1}^{M} b_m U^{*,m}_{i,j}, \quad j \in \mathcal{S}_i,$$
to obtain:
$$Q_i(x) = Q_i(x; \{V_j\}_{j\in\mathcal{S}_i}).$$
3. Define
$$P^m_i = U^{*,m}_{i,i} + Q_i(x^m_i), \quad m = 1, \ldots, M,$$
$$U^+_{i-1/2} = U^*_{i,i-1/2} + Q_i(x_{i-1/2}),$$
$$U^-_{i+1/2} = U^*_{i,i+1/2} + Q_i(x_{i+1/2}).$$

Well-balanced high order reconstruction operator: first step

Definition

The last reconstruction operator is said to be well balanced for a sequence $\{\widetilde{U}_{\Delta x,i}\}$ if, for every i

$$\sum_{m=1}^{M} b_m U_{i,j}^{*,m} = \widetilde{U}_{\Delta x,j}, \quad j \in \mathcal{S}_i$$
$$U_{i,i+1/2}^{*} = U_{i+1,i+1/2}^{*},$$

where $U_{i,j}^{*,m}$, $U_{i,i\pm 1/2}^{*}$ is the output given by the solver of local problems for $W = \widetilde{U}_{\Delta x,i}$ at the ith cell.

Theorem

Let U^* be a stationary solution of (1). *If there exists a sequence $\{\widetilde{U}^*_{\Delta x,i}\}$ such that*

$$\bar{U}_i^* = \widetilde{U}^*_{\Delta x,i} + O(\Delta x^q), \quad \forall i,$$

for some $q \geq p$, and the reconstruction operator is well-balanced, then the numerical method is well-balanced for U^*.

Construct well-balanced approximations of a stationary solution

Definition

Let us consider a numerical solver of order $q \geq p$ that, given

$$\begin{cases} D_f(U)U_x = S(U)H_x, \\ U(x_{i_0-1/2}) = U_0. \end{cases} \tag{4}$$

provides approximations

$$U^{*,m}_{\Delta x,i},\ m = 1,\dots,M; \quad U^*_{\Delta x,i+1/2}, \quad \forall i,$$

of the values of the solution U^* at the quadrature points and the intercells

$$U^*(x_i^m),\ m = 1,\dots,M; \quad U^*(x_{i+1/2}), \quad \forall i.$$

The solver is said to be consistent with the local solver if, at every cell I_i, the solution of the local problem with $W = \widetilde{U}^*_{\Delta x,i}$, is given by

$$\begin{aligned} &U^{*,m}_{i,j} = U^{*,m}_{\Delta x,j}, \quad m = 1,\dots,M,\ j \in \mathcal{S}_i; \\ &U^*_{i,i\pm 1/2} = U^*_{\Delta x,i\pm 1/2}. \end{aligned}$$

Construct well-balanced approximations of a stationary solution

Theorem

If there exists a numerical solver for Cauchy problems (5) *consistent with the local solver, then the numerical method is fully well-balanced.*

- In the recent paper [GBCP20] a fully well-balanced method has been presented following these lines:
  - The Cauchy problems' solver was the standard RK4 method.
  - The local solver was based on the solution of control problems. Newton's method was used to solve these control problems and the RK4 method was used to solve both the state and the adjoint equations.

Outline

Collocation methods: double interpretation

- Let us consider a Cauchy problem

$$\begin{cases} U_x = G(x, U), \\ U(x_0) = U_0, \end{cases} \tag{4}$$

First interpretation: standard RK methods based on a Butcher tableau

$$\begin{array}{c|ccc} c_1 & a_{1,1} & \dots & a_{1,s} \\ c_2 & a_{2,1} & \dots & a_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s,1} & \dots & a_{s,s} \\ \hline & b_1 & \dots & b_s. \end{array}$$

When these methods are applied to (4) in an uniform mesh of nodes $x_i = x_0 + i\Delta x$, $i = 0, 1, \dots$, the numerical solutions are updated as follows:

$$U_{i+1} = U_i + \Delta x \Phi_{\Delta x}(U_i), \quad i = 0, 1, \dots$$

Collocation methods: double interpretation

- Here

$$\Phi_{\Delta x}(U_i) = \sum_{j=1}^{s} b_j K_i^j.$$

- $K_i^1, \dots, K_i^s$ solve the nonlinear system

$$K_i^j = G\left(x_i^j, U_i + \Delta x \sum_{l=1}^{s} a_{j,l} K_i^l\right), \quad j = 1, \dots, s,$$

where

$$x_i^j = x_i + c_j \Delta x, \quad j = 1, \dots, s.$$

Collocation methods: double interpretation

Second interpretation

$$U_{i+1} = P_i(x_{i+1}),$$

where P_i is the only polynomial of degree s that satisfies:

$$\begin{cases} P_i(x_i) = U_i, \\ P_i'(x_i^j) = G(x_i^j, P(x_i^j)), \quad j = 1, \ldots, s. \end{cases}$$

- We will consider here Gauss-Legendre methods, in which $x_i^1, \ldots, x_i^s$ and $b_1, \ldots, b_s$ are respectively the quadrature points and the weights of the Gauss quadrature formula in the interval $[x_i, x_{i+1}]$.
- The order of accuracy of these methods is $2s$.
- That Gauss methods are symmetric or reversible in the following sense (see [HLW06]):

$$\Phi_{\Delta x} \circ \Phi_{-\Delta x} = Id, \quad \text{or equivalently} \quad \Phi_{\Delta x} = \Phi_{-\Delta x}^{-1}. \tag{4}$$

Collocation methods: local problems

- Let us suppose that the stencils are

$$\mathcal{S}_i = \{i - l, \ldots, i + r\}.$$

- For simplicity, let us describe how the local problem is solved for $i = 0$.

Local problem at I_0

Given a state W, we have to find approximations at the intercells of I_0 and at the quadrature points of I_j, $j = -l, \ldots, r$ of the stationary solution U^* that satisfies

$$\sum_{m=1}^{s} b_m U^*(x_0^m) = W.$$

- Let us use the following notation to design the extremes of the intervals:

$$I_j = [x_j^0, x_{j+1}^0], \quad j = -l, \ldots, r.$$

Collocation methods: local problems

Local problem

We look for $r+l+1$ polynomials P_j, $j=-l,\dots,r$, of degree s (with vector coefficients) such that:

$$\begin{cases} \displaystyle\sum_{m=1}^{s} b_m P_0(x_0^m) = W, \\ D_f(P_j(x_j^m))P_j'(x_j^m) = S(P_j(x_j^m))H_x(x_j^m), \quad j=-l,\dots,r, \quad m=1,\dots,s, \\ P_j(x_{j+1}^0) = P_{j+1}(x_{j+1}^0), \quad j=-l,\dots,r-1. \end{cases}$$

- This system has $(r+l+1)(s+1)$ vector unknowns (the coefficients of the polynomials) and $(r+l+1)(s+1)$ equations.
- If this nonlinear system can be solved, the solution to the local problem will be given by:

$$U_j^{*,m} = P_j(x_j^m),\ m=1,\dots,s,\ j=-l,\dots,r;\ U_{-1/2}^{*} = P_0(x_0^0),\ U_{1/2}^{*} = P_0(x_1^0).$$

Collocation methods: local problems

Local problem: equivalent form

Find U_0^0, K_j^m, $j = -l, \dots, r$, $m = 1, \dots, s$ such that

$$\begin{cases} \sum_{m=1}^{s} b_m U_0^m = W, \\ D_f(U_j^m) K_j^m = S(U_j^m) H_x(x_j^m), \quad j = -l, \dots, r, \quad m = 1, \dots, s, \end{cases}$$

where

$$\begin{aligned} U_{j+1}^0 &= U_j^0 + \Delta x \sum_{m=1}^{s} b_m K_j^m, \quad j = 0, \dots, r-1, \\ U_{-(j+1)}^0 &= U_{-j}^0 - \Delta x \sum_{m=1}^{s} b_m K_{-(j+1)}^m, \quad j = 0, \dots, l-1, \\ U_j^m &= U_j^0 + \Delta x \sum_{k=1}^{s} a_{m,k} K_j^k, \quad j = -l, \dots, r, \quad m = 1, \dots, s. \end{aligned}$$

Collocation methods: local problems

- The system has $s(l+r+1)+1$ vector unknowns and $s(l+r+1)+1$ equations.
- A fixed-point algorithm will be used to solve the nonlinear system.

Numerical solver for the local problems (LP) using collocation RK methods.

- Choose initial guesses: $U_0^{0,0}, K_j^{m,0}$, $j=-l,\dots,r$, $m=1,\dots,s$.
- For $n=0,1\dots$
  - Compute:

$$
\begin{aligned}
U_{j+1}^{0,n} &= U_j^{0,n} + \Delta x \sum_{m=1}^{s} b_m K_j^{m,n}, \quad j=0,\dots,r-1,\\
U_{-(j+1)}^{0,n} &= U_{-j}^{0,n} - \Delta x \sum_{m=1}^{s} b_m K_{-(j+1)}^{m,n}, \quad j=0,\dots,l-1,\\
U_j^{m,n} &= U_j^{0,n} + \Delta x \sum_{k=1}^{s} a_{m,k} K_j^{k,n}, \quad j=-l,\dots,r, \quad m=1,\dots,s.
\end{aligned}
$$

Collocation methods: local problems

Numerical solver for the local problems (LP) using collocation RK methods.

- For $n = 0, 1 \dots$
  - Compute $K_j^{m,n+1}$, $j = -l, \dots, r$, $m = 1, \dots, s$ by solving the linear systems:

$$D_f(U_j^{m,n})K_j^{m,n+1} = S(U_j^{m,n})H_x(x_j^m), \quad j = -l, \dots, r, \quad m = 1, \dots, s. \tag{5}$$

  - Compute $U_0^{0,n+1}$ such that

$$\sum_{m=1}^{s} b_m \left(U_0^{0,n+1} + \sum_{k=1}^{s} a_{m,k} K_0^{k,n+1} \right) = W. \tag{6}$$

- At every stage of the fixed-point algorithm, $s(l + r + 1)$ linear systems have to be solved.
- If the Jacobian is singular, these systems may have no solution or to have infinitely many.
- If the algorithm is stopped at the $\bar{n}$-th iteration the numerical solutions of the problem are:

$$U_j^{*,m} = U_j^{m,\bar{n}}, \ m = 1, \dots, s, \ j = -l, \dots r; \quad U_{-1/2}^{*} = U_0^{0,\bar{n}}, \quad U_{1/2}^{*} = U_1^{0,\bar{n}}.$$

Collocation methods: local problems

- Observe that a sensible initial guess is given by:

$$\begin{aligned} U_0^0 &= W, \\ K_j^{m,0} &= K, \quad i = -l, \dots, r, \quad m = 1, \dots, s, \end{aligned}$$

where K is the solution of the linear system:

$$D_f(W)K = S(W)H_x(x_0),$$

where x_0 represents the mid-point of the cell I_0.

Collocation methods: Cauchy problems' solver

- Let us consider the Cauchy problem

$$\begin{cases} D_f(U)U_x = S(U)H_x, \\ U(x_{i_0-1/2}) = U_0. \end{cases} \tag{5}$$

- The approximations of the solution of (5) at the intercells will be obtained by applying the adapted collocation method using the mesh whose points are $\{x_{i+1/2}\}$ and the approximations at the quadrature points will be given by the stages of the RK method.
- It can be easily proved that the solvers of both the Cauchy problems and the local problems are consistent. Then, the high-order numerical methods are fully well-balanced.

Third-order methods

- We will use the third-order CWENO reconstruction operator which uses 3-cell centered stencils, so that $l = r = 1$.
- In order to solve both the local and global problems, we apply the fourth-order 2-stage Gauss-Legendre method corresponding to the choices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{bmatrix},$$

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad b_1 = b_2 = \frac{1}{2}.$$

First and second order schemes

- Although the general algorithm can be used in particular to design well-balanced method with order of accuracy one or two, the local problems to solve are easier if, instead of a Gauss quadrature formula, the second order mid-point formula is used to compute averages.

- In both cases, the local problems and the Cauchy problems are solved by applying the described fixed-point algorithm in the mesh of step $\Delta x/2$ whose points are the extremes and the midpoints of the cells.

Resonant problems

- In the described algorithms, we have to solve linear systems of the form

$$D_f(U^*)K = S(U^*)H_x(x^*)$$

- If one of the eigenvalues of $D_f(U^*)$ vanishes the problem is resonant:
  - If $S(U^*)H_x(x^*)$ does not belong to the image of the linear application defined by the matrix $D_f(U^*)$, the system has no solution. This situation can arise:
    - Solving a local problem: the standard reconstruction will be used then.
    - Solving a Cauchy problem: it will be assumed that it has not a solution defined in the whole computational domain.
  - Otherwise, the system has infinitely many solutions

$$K^* + \alpha R, \quad \alpha \in \mathbb{R},$$

where K^* is a particular solution and R is an eigenvector associated to the null eigenvalue. Some extra information is needed to select the adequate solution. For instance, the limit

$$\lim_{x \to x^*} D_f(U)^{-1} S(U) H_x \qquad (6)$$

can be formally computed knowing that $\lim_{x \to x^*} U(x) = U^*$. The value of (6) will be then selected as the solution of the linear system.

Outline

Numerical experiments

The numerical methods are based on the Rusanov flux. The following symbols will be used to denote the different numerical methods of order *i* considered:

- SM*i*, $i = 1, 2, 3$: based on the standard reconstruction operators.
- WBM*i*, $i = 1, 2, 3$: based on the well-balanced reconstruction operator where the local problems are exactly solved and numerical integration is used.
- DWBM*i*, $i = 1, 2, 3$: based on the well-balanced reconstruction operator in which the Cauchy problems are solved using the standard RK4 method and the local problems are solved using control techniques: see [GBCP20].
- CDWBM*i*, $i = 1, 2, 3$: based on the well-balanced reconstruction operator in which Cauchy and local problems are solved by using the 2-stage Gauss-Legendre collocation method.

Problem 1: Burgers equation with a nonlinear source term

- Let us consider the Burgers equation with a non-linear source term

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = S(u)H_x, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x). \end{cases}$$

$S(u) = u^2$

- Let us consider $S(u) = u^2$.
- Stationary solutions are given by

$$u^*(x) = C_0 e^{H(x)}.$$

- We consider $x \in [-1, 1]$, $t \in (0, 5]$, $H(x) = x$ and $U_0(x) = e^x$ as the initial condition.

Problem 1: Burgers equation with a nonlinear source term

- **Errors:**

Cells	SM1: Error	Order	SM2: Error	Order	SM3: Error	Order
100	7.53E-2	-	2.44E-3	-	7.66E-6	-
200	3.78E-2	0.995	8.09E-4	1.591	9.62E-7	2.993
400	1.89E-2	1.002	2.16E-4	1.905	1.21E-7	2.995
800	9.43E-3	1.000	5.54E-5	1.963	1.51E-8	2.998

Table: Problem 1.1. Errors in L^1 norm and convergence rates for SMi, $i = 1, 2, 3$.

Cells	Order 1: Error			Order 2: Error			Order 3: Error		
	WBM	DWBM	CDWBM	WBM	DWBM	CDWBM	WBM	DWBM	CDWBM
100	4.21E-15	3.55E-15	1.53E-15	8.87E-16	3.63E-16	2.66E-16	3.20E-16	1.43E-14	4.54E-14
200	2.90E-15	5.54E-13	8.09E-16	4.42E-16	1.23E-15	2.46E-16	2.54E-16	2.43E-14	1.21E-14
400	1.84E-14	2.05E-14	3.44E-13	1.82E-15	3.64E-16	4.15E-13	7.40E-14	4.47E-14	5.02E-14
800	4.45E-16	2.67E-15	2.33E-15	1.83E-16	2.03E-16	6.13E-15	2.61E-15	9.48E-14	5.45E-13

Table: Problem 1.1. Errors in L^1 norm for WBMi, DWBMi, CDWBMi, $i = 1, 2, 3$.

Problem 1: Burgers equation with a nonlinear source term

- **Computational time:**

Cells	Order (i)	SMi	WBMi	DWBMi	CDWBMi
100	1	20	30	70	60
	2	30	60	140	130
	3	40	190	240	230
200	1	20	60	230	160
	2	40	190	330	320
	3	110	480	530	540
400	1	50	180	520	410
	2	100	530	1150	1250
	3	350	1680	1980	1990
800	1	140	570	2020	1280
	2	270	2040	3580	4170
	3	1080	5540	6600	5960

Table: Problem 1.1. Computational times (milliseconds).

Problem 1: Burgers equation with a nonlinear source term

$S(u) = \sin(u(x))$

- Let us consider $S(u) = \sin(u(x))$.
- Stationary solutions are given by the ODE

$$\frac{du}{dx}(x) = \frac{\sin(u(x))}{u(x)}.$$

- We consider $x \in [-1, 1]$, $t \in (0, 5]$, $H(x) = x$.
- Initial condition: solution of the ODE

$$\begin{cases} \dfrac{du}{dx}(x) = \dfrac{\sin(u(x))}{u(x)}, \\ u(-1) = 2. \end{cases}$$

Problem 1: Burgers equation with a nonlinear source term

- **Errors:**

Cells	Error ($i = 1$)	Order	Error ($i = 2$)	Order	Error ($i = 3$)	Order
100	2.72E-3	-	1.43E-4	-	2.53E-5	-
200	1.34E-3	1.021	2.43E-6	5.879	1.74E-8	10.503
400	6.58E-4	1.026	8.19E-7	1.569	1.14E-10	7.250
800	3.24E-4	1.022	2.34E-7	1.806	1.41E-11	3.016

Table: Problem 1.2. Errors in L^1 norm and convergence rates for SMi, $i = 1, 2, 3$.

Cells	Order 1: Error		Order 2: Error		Order 3: Error	
	DWBM	CDWBM	DWBM	CDWBM	DWBM	CDWBM
100	9.71E-14	2.78E-15	1.76E-13	4.00E-16	1.99E-13	1.22E-14
200	7.56E-15	3.49E-15	3.46E-15	8.55E-14	2.97E-14	2.52E-14
400	4.00E-15	3.26E-15	7.53E-16	1.23E-15	3.31E-14	1.39E-13
800	5.97E-15	4.30E-15	8.54E-16	2.04E-15	6.63E-14	1.01E-13

Table: Problem 1.2. Errors in L^1 norm for DWBMi and CDWBMi, $i = 1, 2, 3$

Problem 1: Burgers equation with a nonlinear source term

- **Computational time:**

Cells	Order(i)	SMi	DWBMi	CDWBMi
100	1	10	340	150
	2	20	690	390
	3	40	1390	570
200	1	30	1280	330
	2	60	2350	950
	3	180	5190	1680

Table: Problem 1.2. Computational times (milliseconds). $t = 5s$.

Problem 2: shallow water equations.

- We consider the shallow water model, which is a particular case of

$$U_t + f(U)_x = S(U)H_x, \tag{7}$$

corresponding to the choices $N = 2$,

$$U = \begin{pmatrix} h \\ q \end{pmatrix}, \quad f(U) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{g}{2}h^2 \end{pmatrix}, \quad S(U) = \begin{pmatrix} 0 \\ gh \end{pmatrix}.$$

The variable x makes reference to the axis of the channel and t is the time; $q(x,t)$ and $h(x,t)$ are the discharge and the thickness, respectively; g is the gravity and $H(x)$ is the depth function measured from a fixed reference level.

- The system of ODE satisfied by the stationary solutions is:

$$\begin{cases} q_x = 0, \\ \left(\dfrac{q^2}{h} + \dfrac{1}{2}gh^2\right)_x = ghH_x. \end{cases}$$

Perturbation of a stationary solution

Setting of the experiment

- $x \in [0,3]$, $t \in (0,5]$ and $CFL = 0.9$.
- Depth function:

$$H(x) = \begin{cases} -0.25(1+\cos(5\pi(x+0.5))) & \text{if} \quad 1.3 \leq x \leq 1.7, \\ 0 & \text{otherwise.} \end{cases}$$

- Initial condition: the initial condition $U_0(x) = (h_0(x), q_0(x))^T$ is given by:

$$h_0(x) = \begin{cases} h^*(x) + 0.1, & \text{if } 0.7 \leq x \leq 1.0, \\ h^*(x), & \text{otherwise,} \end{cases}$$

$$q_0(x) = q^*(x),$$

where $U^*(x) = (h^*(x), q^*(x))^T$ is the subcritical stationary solution of

$$\begin{cases} q_x = 0, \\ h_x = \dfrac{ghH_x}{-u^2 + gh}, \\ h(0) = 2, \; q(0) = 3.5. \end{cases} \tag{8}$$

Perturbation of a stationary solution

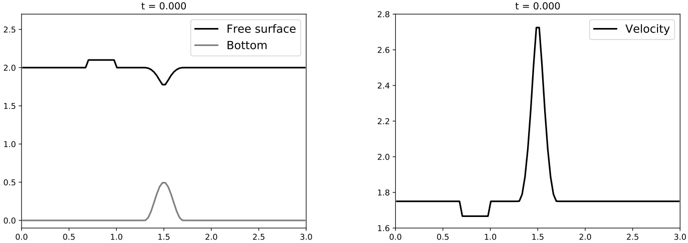


Figure: Problem 2. Initial condition.

Perturbation of a stationary solution

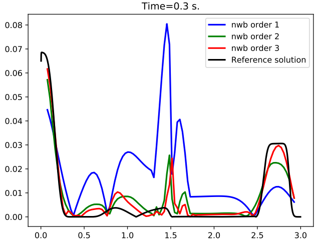


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 0.3s$ for h.(100 cells) .

Perturbation of a stationary solution

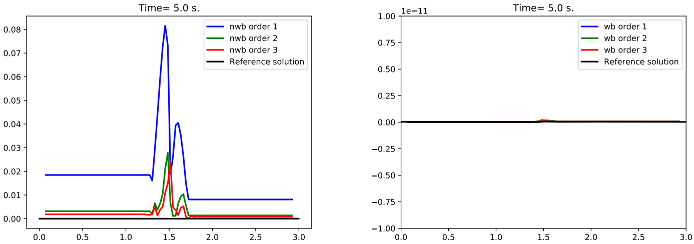


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 5s$ for h.(100 cells) .

- **Errors:**

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	5.15E-2	1.94E-1	9.39E-3	3.51E-2	6.01E-3	2.12E-2
CDWBMi	1.12E-14	2.45E-14	1.41E-13	1.61E-14	1.38E-13	1.39E-13

Table: Problem 2. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationay solution for the 100-cell mesh at time $t = 5s$.

Problem 3: Shallow water equations: transcritical stationary solutions.

- The system to be solved when U^* is a critical state reduces to:

$$\begin{bmatrix} 0 & 1 \\ 0 & 2u \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ ghH_x(x) \end{bmatrix}.$$

- The system has solutions only if $H_x(x) = 0$:

$$K = \alpha[1,0]^T, \quad \alpha \in \mathbb{R}.$$

- A smooth stationary solution can only reach a critical state at a minimum point x^c of the depth function H.
- The value of the critical state has to be $U^* = [h^c(q), q]^T$ where

$$h^c(q) = \frac{q^{2/3}}{g^{1/3}}. \qquad (8)$$

Problem 3: Shallow water equations: transcritical stationary solutions.

- Following the numerical treatment of resonant situations previously discussed, let us compute the limit

$$\lim_{x \to x^c} \frac{ghH_x}{-u^2 + gh},$$

to determine the value of h_x at x^c: the L'Hôpital's rule and some easy computations lead to

$$h_x(x^c) = \pm \sqrt{\frac{q^{2/3} H_{xx}(x^c)}{3g^{1/3}}}.$$

Problem 3: Shallow water equations: transcritical stationary solutions.

- If a system has to be solved in the adapted Gauss-Legendre collocation method algorithm in which

$$|F_r(U^*) - 1| < \epsilon,$$

  at a point $\bar{x}$ then
  - If $\bar{x}$ is not close to a minimum point of H, it is assumed that is not possible to find a smooth stationary solution that solves the problem and the algorithm is stopped.
  - Otherwise we choose:
    - $K = \left[\sqrt{\frac{q^{2/3} H_{xx}(\bar{x})}{3g^{1/3}}}, 0\right]^T$ if h is increasing close to $\bar{x}$.
    - $K = \left[-\sqrt{\frac{q^{2/3} H_{xx}(\bar{x})}{3g^{1/3}}}, 0\right]^T$ if h is decreasing close to $\bar{x}$.

    and the algorithm goes on.

Transcritical smooth stationary solution

Setting of the experiment

- $x \in [0, 3]$, $t \in (0, 1]$ and $CFL = 0.9$.
- Depth function:

$$H(x) = \begin{cases} -0.25(1 + \cos(5\pi(x + 0.5))) & \text{if} \quad 1.3 \leq x \leq 1.7, \\ 0 & \text{otherwise.} \end{cases}$$

- Initial condition: transcritical stationary solution which solves the ODE

$$\begin{cases} q_x = 0, \\ (-u^2 + gh)h_x = ghH_x, \\ h(0) = 1.67750727, \\ q(0) = 2.5, \end{cases}$$

  that reaches a critical state at $x^c = 1.5$.
- 4 uniform meshes with 100, 200, 400 and 800 cells are considered.

Transcritical smooth stationary solution

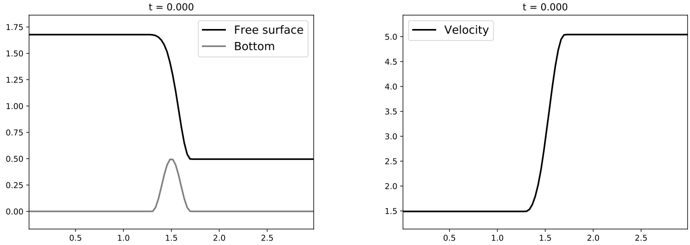


Figure: Problem 3.1. Initial condition: a transcritical stationary solution.

Transcritical smooth stationary solution

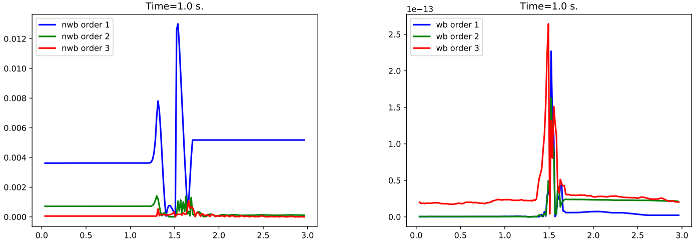


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 1s$ for h.(200 cells) .

Transcritical smooth stationary solution

- **Errors:**

Cells	Error ($i=1$) h	Order	Error ($i=2$) h	Order	Error ($i=3$) h	Order
100	4.99E-2	-	7.63E-3	-	5.99E-3	-
200	1.31E-2	1.923	1.27E-3	2.583	8.86E-4	2.757
400	3.87E-3	1.766	1.84E-4	2.790	6.20E-5	3.838
800	1.56E-3	1.314	5.31E-5	1.794	6.88E-6	3.172
Cells	Error ($i=1$) q	Order	Error ($i=2$) q	Order	Error ($i=3$) q	Order
100	1.28E-1	-	1.89E-2	-	1.72E-2	-
200	2.81E-2	2.188	3.18E-3	2.575	2.35E-3	2.866
400	9.29E-3	1.161	4.81E-4	2.724	2.09E-4	3.489
800	4.09E-3	1.168	1.37E-4	1.812	2.29E-5	3.190

Table: Problem 3.1. Errors in L^1 norm and convergence rates for SMi, $i=1,2,3$.

Transcritical smooth stationary solution

Cells	Error ($i=1$)		Error ($i=2$)		Error ($i=3$)	
	h	q	h	q	h	q
100	2.00E-14	9.15E-15	3.71E-14	7.33E-15	2.41E-14	6.73E-14
200	1.78E-14	6.92E-15	4.03E-14	3.27E-15	6.55E-14	6.11E-14
400	8.66E-14	1.69E-15	6.13E-14	1.34E-14	8.89E-14	1.45E-13
800	2.18E-15	6.22E-15	8.24E-14	2.16E-11	1.68E-13	2.61E-13

Table: Problem 3.1. Errors in L^1 norm for CDWBMi, $i = 1, 2, 3$.

Perturbation of a smooth transcritical stationary solution

Setting of the experiment

The only difference with Problem 3.1. is that now, a perturbation of size $\Delta h = 0.02$ is imposed to the thickness h in the interval $[1.1, 1.2]$.

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	1.01E-2	2.13E-2	4.83E-4	2.00E-3	2.70E-4	1.30E-3
CDWBMi	5.11E-14	7.06E-14	6.04E-14	3.50E-15	1.24E-13	1.25E-13

Table: Problem 3.2. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 200-cell mesh at time $t = 5s$.

Problem 4: shallow water equations with Manning friction

- Let us consider the shallow water equations with Manning friction:

$$\begin{cases} h_t + q_x = 0, \\ q_t + \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right)_x = g h H_x - \dfrac{k q |q|}{h^\eta}. \end{cases}$$

The variable x makes reference to the axis of the channel and t is the time; $q(x,t)$ and $h(x,t)$ are the discharge and the thickness, respectively; g is the gravity and $H(x)$ is the depth function measured from a fixed reference level; k is the Manning friction coefficient and η is a parameter equal to $\frac{7}{3}$. Remember that $u = q/h$ is the depth-averaged velocity.

- The system of ODE satisfied by the stationary solutions is:

$$\begin{cases} q_x = 0, \\ \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right)_x = g h H_x - \dfrac{k q |q|}{h^\eta}, \end{cases}$$

which can be written as follows:

$$\begin{cases} q_x = 0, \\ \left(-u^2 + g h \right) h_x = g h H_x - \dfrac{k q |q|}{h^\eta}. \end{cases}$$

Preservation of the friction-only steady states.

Setting of the experiment

- $x \in [0, 1]$, $t \in (0, 1]$ and $CFL = 0.9$.
- We assume a flat topography, i.e., $H_x = 0$ and the Manning coefficient is $k = 1$.
- Initial condition: the subcritical branch of the steady state obtained by assuming $q(0) = -1$ and $h(0)$ obtained by considering a zero of the nonlinear function $\xi(h)$ defined by

$$\xi(h) = -\frac{q_0^2}{\eta - 1}\left(h^{\eta-1} - h_0^{\eta-1}\right) + \frac{g}{\eta+2}\left(h^{\eta+2} - h_0^{\eta+2}\right) + kq_0|q_0|(x - x_0),$$

  where $x_0 = -\Delta x$, $x = 0$, $q_0 = q(0)$ and $h_0 = h^c$, with h^c defined by

$$h^c = \left(\frac{q_0^2}{g}\right)^{\frac{1}{3}},$$

  (see [MDBCF15]).
- The space domain discretization is made of 200 cells.
- A reference solution has been computed with a first order well-balanced scheme on a fine mesh (1600 cells).

Preservation of the friction-only steady states.

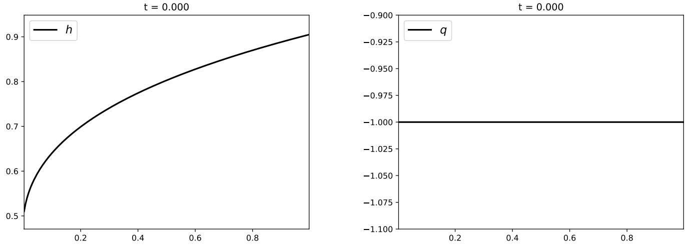


Figure: Problem 4.1. Initial condition: a subcritical stationary solution.

Preservation of the friction-only steady states.

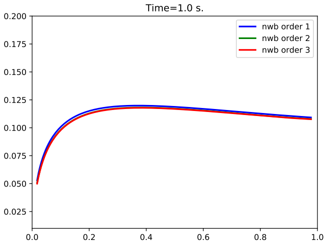


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 1s$ for h.(200 cells) .

- **Errors:**

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	1.11E-1	1.14E-2	1.09E-1	7.39E-3	1.08E-1	7.38E-3
CDWBMi	2.18E-14	2.50E-15	2.92E-14	2.95E-15	3.06E-14	4.93E-14

Table: Problem 4.1. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 200-cell mesh at time $t = 5s$.

Perturbation of a friction-only stationary solution

Setting of the experiment

- The initial condition $U_0(x) = (h_0(x), q_0(x))^T$ is the following perturbation of the stationary solution $U^*(x) = [h^*(x), q^*(x)]^T$ of the previous Test:

$$h_0(x) = \begin{cases} h^*(x) + 0.2, & \text{if } \frac{3}{7} \leq x \leq \frac{4}{7}, \\ h^*(x), & \text{otherwise,} \end{cases}$$

$$q_0(x) = q^*(x).$$

- We use a 100-cell mesh for the numerical simulation on the domain $[0, 1]$.
- The computations are carried out until $t = 9s$.

Perturbation of a friction-only stationary solution

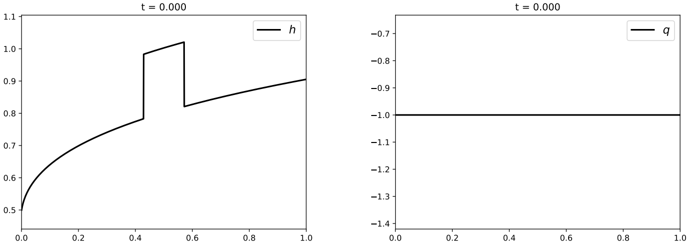


Figure: Initial condition.

Perturbation of a friction-only stationary solution

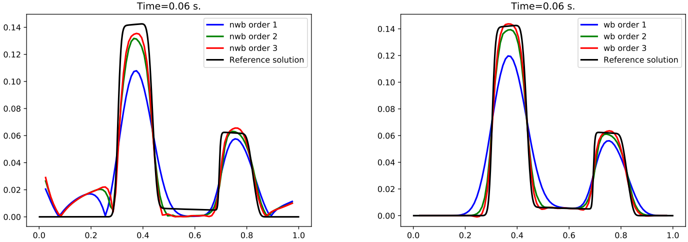


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 0.06s$ for h. (100 cells) .

Perturbation of a friction-only stationary solution

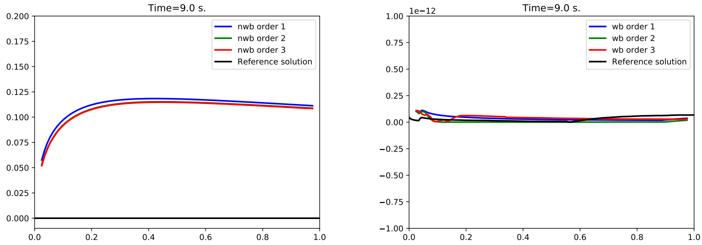


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 9s$ for h. (100 cells) .

- **Errors:**

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	1.10E-1	1.03E-2	1.07E-1	3.04E-3	1.06E-1	3.04E-3
CDWBMi	3.76E-14	6.32E-16	1.11E-14	3.20E-15	4.29E-14	1.79E-14

Table: Problem 4.1. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 100-cell mesh at time $t = 9s$.

Preservation of steady states involving both a varying topography and the bottom friction.

Setting of the experiment

- $x \in [0,1]$, $t \in (0,1]$, $CFL = 0.9$ and $k = 0.01$.
- The topography is given by

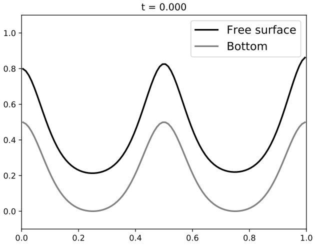
$$H(x) = 1 - \frac{1}{2}\frac{e^{\cos(4\pi x)} - e^{-1}}{e - e^{-1}}. \tag{8}$$

- Initial condition: supercritical stationary solution corresponding

$$\begin{cases} \left(-u^2 + gh\right) h_x = ghH_x - \dfrac{kq|q|}{h^\eta}, \\ q_x = 0, \\ q(0) = 1, h(0) = 0.3. \end{cases} \tag{9}$$

- The space domain discretization is made of 100 cells.

Preservation of steady states involving both a varying topography and the bottom friction.



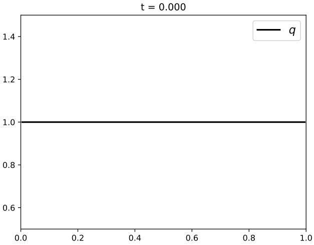


Figure: Initial condition: a supercritical stationary solution.

Preservation of steady states involving both a varying topography and the bottom friction.

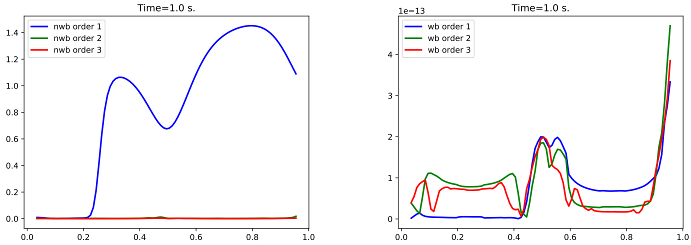


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 1s$ for h. (100 cells) .

- **Errors:**

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	8.30E-1	1.54	3.57E-3	4.89E-3	1.39E-3	4.30E-4
CDWBMi	8.48E-14	5.54E-16	1.13E-13	2.61E-15	9.66E-14	1.25E-14

Table: Problem 4.2. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 100-cell mesh at time $t = 1s$.

Perturbation of a stationary solution involving both a varying topography and the bottom friction.

Setting of the experiment

- The initial condition $U_0(x) = [h_0(x), q_0(x)]^T$ is given by:

$$h_0(x) = \begin{cases} h^*(x) + 0.05, & \text{if } x \in \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right], \\ h^*(x), & \text{otherwise,} \end{cases}$$

$$q_0(x) = \begin{cases} q^*(x) + 0.5, & \text{if } x \in \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right], \\ q^*(x), & \text{otherwise,} \end{cases}$$

  where $U^*(x) = [h^*(x), q^*(x)]^T$ is the stationary solution considered in the previous test.
- We use a 100-cell mesh for the numerical simulation on the domain $[0, 1]$.
- The computations are carried out until $t = 2s$.

Perturbation of a stationary solution involving both a varying topography and the bottom friction.

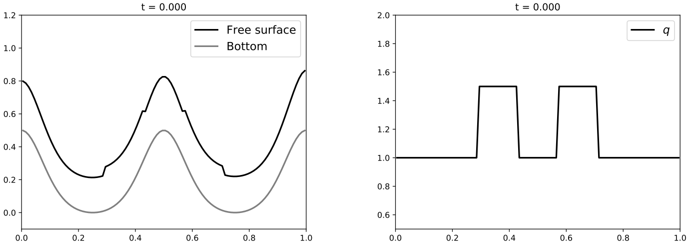


Figure: Initial condition.

Perturbation of a stationary solution involving both a varying topography and the bottom friction.

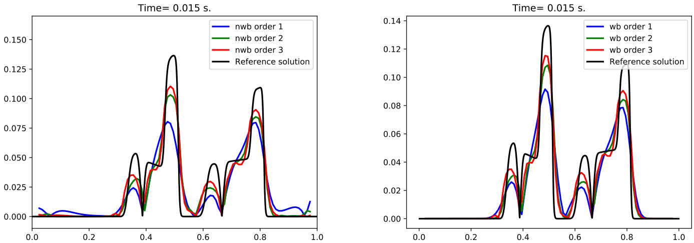
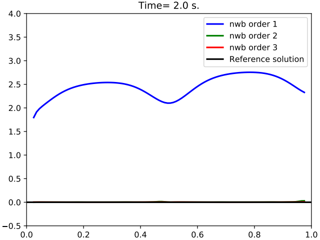


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 0.015s$ for h. (100 cells)
.

Perturbation of a stationary solution involving both a varying topography and the bottom friction.



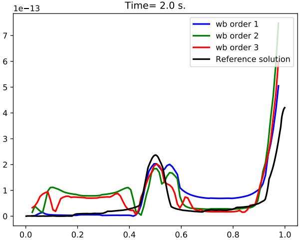


Figure: Differences between the stationary solution and the numerical solutions for SM$i = 1, 2, 3$ (left) and CDWBMi, $i = 1, 2, 3$ (right) at time $t = 2s$ for h. (100 cells) .

- **Errors:**

Method	Error ($i = 1$)		Error ($i = 2$)		Error ($i = 3$)	
	h	q	h	q	h	q
SMi	2.42	6.12	3.57E-3	4.87E-3	1.39E-3	4.30E-4
CDWBMi	8.60E-14	1.03E-15	1.12E-13	1.26E-15	9.69E-14	1.22E-14

Table: Problem 4.2. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 100-cell mesh at time $t = 2s$.

Problem 5: Compressible Euler equations with gravitational force

- These equations are a particular case of
$$U_t + f(U)_x = S(U)H_x, \tag{8}$$
corresponding to the choices $N = 3$,
$$U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad f(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix}, \quad S(U) = \begin{pmatrix} 0 \\ -\rho \\ -\rho u \end{pmatrix}.$$

- The system of ODE satisfied by the stationary solutions can be written in the following form, as we suppose that the system is strictly hyperbolic:
$$\begin{cases} q_x = 0, \\ \dfrac{d\hat{U}}{dx} = G(x, \hat{U}), \end{cases} \tag{9}$$
where
$$\hat{U} = \begin{pmatrix} \rho \\ E \end{pmatrix}, \quad G(x, \hat{U}) = -\begin{pmatrix} \dfrac{\rho}{c^2 - u^2} \\ \dfrac{\rho}{\gamma - 1}\left(1 + \dfrac{3-\gamma}{2}\dfrac{u^2}{c^2 - u^2}\right) \end{pmatrix} H_x,$$
where
$$c = \sqrt{\gamma \frac{p}{\rho}}$$
is the wave speed.

Stationary solution

Setting of the experiment

- $x \in [-1, 1]$, $t \in (0, 5]$ and $CFL = 0.9$.
- The gravity potential is the identity function $H(x) = x$.
- Initial condition: the supersonic stationary solution which solves the Cauchy problem:

$$\begin{cases} q_x = 0, \\ \dfrac{d\hat{U}}{dx} = G(x, \hat{U}), \\ \rho(-1) = 1,\ q(-1) = 10,\ E(-1) = 52, \end{cases} \tag{10}$$

  computed with the 2-stage collocation RK method.
- Inflow boundary condition is set at $x = -1$ and free boundary conditions are set at $x = 1$.
- 4 uniform meshes with 100, 200, 400 and 800 cells are considered.

Stationary solution

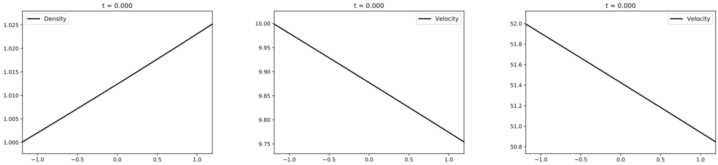


Figure: Initial condition. Density (left), velocity (middle) and energy (right).

Stationary solution

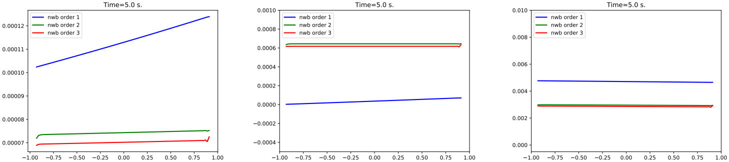


Figure: Differences between the stationary solution and the numerical solutions for SMi at time $t = 5s$. Number of cells: 100.

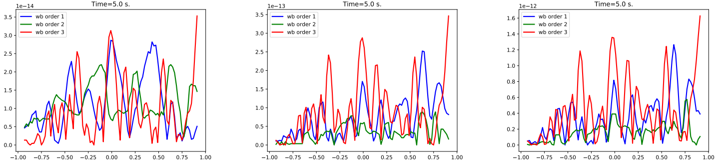


Figure: Differences between the stationary solution and the numerical solutions for CDWBMi at time $t = 5s$. Number of cells: 100.

Stationary solution

Method	Error ($i = 1$)	Error ($i = 2$)	Error ($i = 3$)
		h	
SMi	2.23E-4	9.36E-5	9.50E-3
CDWBMi	2.21E-14	2.28E-14	2.02E-14
		q	
SMi	1.45E-4	1.28E-3	5.95E-3
CDWBMi	1.38E-13	6.26E-14	1.97E-13
		E	
SMi	1.38E-4	1.22E-3	5.76E-3
CDWBMi	6.58E-13	3.08E-13	9.67E-13

Table: Problem 5. Errors in L^1 norm for SMi and CDWBMi ($i = 1, 2, 3$) with respect to the stationary solution for the 100-cell mesh at time $t = 5s$.

Outline

Conclusions

- A general strategy to obtain a well-balanced reconstruction operator has been proposed.
- This technique have been successfully applied to problems for which the explicit expression of the stationary solutions are not known: what we propose here is to build both the stationary solutions and the local solvers to be solved in the first step on the basis of the collocation RK methods.
- We have also introduce a general technique that allows us to deal with resonant problems for any 1d systems of balance laws.
- The tests put on evidence that the strategy based on the collocation RK methods introduced in this work is more efficient than the one presented in [GBCP20].

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