In-cell Discontinuous Reconstruction path-conservative methods for non conservative hyperbolic systems - Second-order extension

Ernesto Pimentel-García[1], Manuel J. Castro[1], Christophe Chalons[2], Tomás Morales de Luna[3], Carlos Parés[1]

[1]University of Málaga

[2]Laboratoire de Mathématiques de Versailles

[3]University of Córdoba

Introduction

We consider first order quasi-linear PDE systems

$$\partial_t W + \mathcal{A}(W)\partial_x W = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{1}$$

The system is supposed to be strictly hyperbolic and the characteristic fields $R_i(W)$, $i = 1, \ldots, N$, are supposed to be either genuinely nonlinear:

$$\nabla \lambda_i(W) \cdot R_i(W) \neq 0, \quad \forall\, W \in \Omega,$$

or linearly degenerate:

$$\nabla \lambda_i(W) \cdot R_i(W) = 0, \quad \forall\, W \in \Omega.$$

Introduction

In the case of nonconservative systems important problems appear from an analytical and numerical point of view when $\mathcal{A} \neq 0$ and it is not the Jacobian of any function. Here we will follow the theory developed by Dal Maso, LeFloch and Murat (1995) that allows a notion of weak solution which satisfies (1) in the sense of Borel measures. This definition is based on the choice of a family of Lipschitz continuous paths $\Phi : [0,1] \times \Omega \times \Omega \to \Omega$ satisfying certain regularity and consistency properties. In particular

$$\Phi(0, W_l, W_r) = W_l, \quad \Phi(1, W_l, W_r) = W_r, \quad \forall (W_l, W_r) \in \Omega \times \Omega,$$

and

$$\Phi(\xi, W, W) = W, \quad \forall \xi \in [0,1], \quad \forall W \in \Omega.$$

We remember that the family of paths can be understood as a tool to give a sense to integrals of the form

$$\int_a^b \mathcal{A}(W(x))\, W_x(x)\, dx,$$

for functions W with jump discontinuities.

Introduction

More precisely, given a bounded variation function $W : [a,b] \mapsto \Omega$, it is defined:

$$\fint_a^b \mathcal{A}(W(x))\, W_x(x)\, dx = \int_a^b \mathcal{A}(W(x))\, W_x(x)\, dx + \sum_m \int_0^1 \mathcal{A}(\Phi(s; W_m^-, W_m^+)) \frac{\partial \Phi}{\partial s}(s; W_m^-, W_m^+)\, ds. \tag{2}$$

If such a mathematical definition of the nonconservative products is assumed to define the concept of weak solution, the generalized Rankine-Hugoniot condition:

$$\int_0^1 \mathcal{A}(\Phi(s; W^- W^+)) \frac{\partial \Phi}{\partial s}(s; W^-, W^+)\, ds = \sigma(W^+ - W^-), \tag{3}$$

has to be satisfied across an admissible discontinuity. Once the family of paths has been prescribed, a concept of entropy is required, as it happens for systems of conservation laws, that may be given by an entropy pair or by Lax's entropy criterion.

Problem statement

Question 1: Which is the 'good' election of the family of path?

When the hyperbolic system is the vanishing-viscosity limit of the parabolic problems

$$W_t^\epsilon + \mathcal{A}(W^\epsilon)\, W_x^\epsilon = \epsilon(\mathcal{R}(W^\epsilon) W_x^\epsilon)_x, \tag{4}$$

where $\mathcal{R}(W)$ is any positive-definite matrix, the adequate family of paths should be related to the viscous profiles: a function V is said to be a viscous profile for (4) linking the states W^- and W^+ if it satisfies

$$\lim_{\chi\to-\infty} V(\chi) = W^-, \quad \lim_{\chi\to+\infty} V(\chi) = W^+, \quad \lim_{\chi\to\pm\infty} V'(\chi) = 0 \tag{5}$$

and there exists $\sigma \in \mathbb{R}$ such that the travelling wave

$$W^\epsilon(x,t) = V\left(\frac{x-\sigma t}{\epsilon}\right), \tag{6}$$

is a solution of (4) for every ϵ. It can be easily verified that, in order to be a viscous profile, V has to solve the equation

$$-\xi V' + \mathcal{A}(V)\, V' = (\mathcal{R}(V)\, V')', \quad with \quad \text{(5)}. \tag{7}$$

Path-conservative methods

Let us consider a family of paths Φ. According to (Parés, 2006), a numerical method for solving (1) is said to be path-conservative if it can be written in the form

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\big(\mathcal{D}^+_{i-1/2} + \mathcal{D}^-_{i+1/2}\big), \tag{8}$$

where

$$W_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W(x,t_n)\,dr. \tag{9}$$

$$\mathcal{D}^{\pm}_{i+1/2} = \mathcal{D}^{\pm}\big(W_i^n, W_{i+1}^n\big),$$

where $\mathcal{D}^-$ and $\mathcal{D}^+$ satisfy

$$\mathcal{D}^{\pm}(W,W) = 0, \quad \forall W \in \Omega, \tag{10}$$

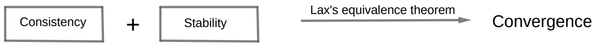
$$\mathcal{D}^-(W_l,W_r) + \mathcal{D}^+(W_l,W_r) = \int_0^1 \mathcal{A}\big(\Phi(s;W_l,W_r)\big)\frac{\partial \Phi}{\partial s}(s;W_l,W_r)\,ds, \tag{11}$$

for every set $W_l, W_r \in \Omega$.

Path-conservative methods: convergence issue

Question 2: Do the numerical schemes converge to the right solution?

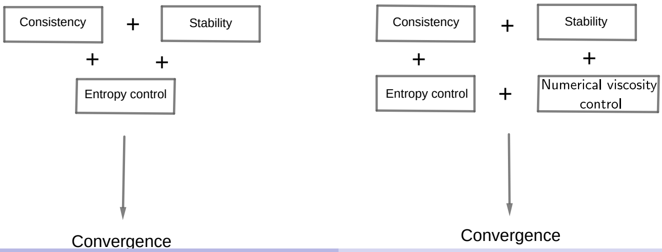
Lineal systems



Nonlinear systems

Systems of conservation laws

Nonconservative systems



Path-conservative methods: semi-discrete form

Following (Parés, 2006), first order path-conservative numerical schemes can be extended to high-order by using reconstruction operators:

$$W_i'(t) = -\frac{1}{\Delta x}\left(\mathcal{D}^-_{i+\frac{1}{2}}(t) + \mathcal{D}^+_{i-\frac{1}{2}}(t) + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^t(x))\frac{\partial}{\partial x}\mathbb{P}_i^t(x)\,dx\right), \tag{12}$$

where $\mathbb{P}_i^t(x)$ is the smooth approximation of the solution at the ith-cell provided by a high-order reconstruction operator from the sequence of cell values $\{W_i(t)\}$ and

$$\mathcal{D}^{\pm}_{i+1/2}(t) = \mathcal{D}^{\pm}_{i+1/2}(W^-_{i+1/2}(t), W^+_{i+1/2}(t)),$$

where $W^-_{i+1/2}(t) = \mathbb{P}_i^t(x_{i+\frac{1}{2}})$ and $W^-_{i+1/2}(t) = \mathbb{P}_{i+1}^t(x_{i+\frac{1}{2}})$ (see (Castro et. al., 2017) for details).

2nd-order in-cell discontinuous reconstruction p-c methods

Idea: Combine a standard second-order reconstruction operator in smoothness regions and a discontinuous operator close to discontinuities so that no numerical viscosity is added in the non-smooth regions.

Semi-discrete method

Once the numerical approximations W_i^n of the averages of the solutions have been computed at time $t_n = n\Delta t$, the first step is to mark the cells I_i such that the solution of the Riemann problem consisting of (1) with initial conditions

$$W(x,0) = \begin{cases} W_{i-1}^n & \text{if } x < 0, \\ W_{i+1}^n & \text{if } x > 0, \end{cases} \qquad (13)$$

involves a shock wave. Let us denote by $\mathcal{M}_n$ the set of indices of the marked cells, i.e.

$$\mathcal{M}_n = \{i \text{ s.t. the solution of the Riemann problem (1), (13) involves a shock wave}\}. \qquad (14)$$

Semi-discrete method

We take $\mathbb{P}_i^n(x,t)$ in (12) as following:

- If $i-1, i, i+1 \notin \mathcal{M}_n$ then $\mathbb{P}_i^n$ is the approximation of the first degree Taylor polynomial of the solution given by:

$$\mathbb{P}_i^n(x,t) = W_i^n + \widetilde{\partial_x W}_i^n (x - x_i) - \mathcal{A}(W_i^n)\widetilde{\partial_x W}_i^n (t - t^n),$$

where

$$\left(\widetilde{\partial_x W}_i^n\right)_k = \mathrm{minmod}\left(\alpha \frac{\mathrm{w}_{\mathrm{i+1,k}}^{\mathrm{n}} - \mathrm{w}_{\mathrm{i,k}}^{\mathrm{n}}}{\Delta \mathrm{x}}, \frac{\mathrm{w}_{\mathrm{i+1,k}}^{\mathrm{n}} - \mathrm{w}_{\mathrm{i-1,k}}^{\mathrm{n}}}{2\Delta \mathrm{x}}, \alpha \frac{\mathrm{w}_{\mathrm{i,k}}^{\mathrm{n}} - \mathrm{w}_{\mathrm{i-1,k}}^{\mathrm{n}}}{\Delta \mathrm{x}}\right),$$

being α is a parameter with $1 \leq \alpha < 2$ and

$$\mathrm{minmod(a,b,c)} = \begin{cases} \min\{a,b,c\} & \text{if } a,b,c > 0, \\ \max\{a,b,c\} & \text{if } a,b,c < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, using the equation:

$$\partial_t W(x_i, t_n) = -\mathcal{A}(W(x_i,t_n))\partial_x W(x_i,t_n) \approx -\mathcal{A}(W_i^n)\widetilde{\partial_x W}_i^n.$$

Semi-discrete method

- If $i \in \mathcal{M}_n$ then

$$\mathbb{P}_i^n(x,t) = \begin{cases} W_{i,l}^n & \text{if } x \leq x_{i-1/2} + d_i^n \Delta x + \sigma_i^n (t - t_n), \\ W_{i,r}^n & \text{otherwise.} \end{cases}.$$

where d_i^n is chosen so that

$$d_i^n w_{i,l,k}^n + (1 - d_i^n) w_{i,r,k}^n = w_{i,k}^n, \tag{15}$$

for some index $k \in \{1, \dots, N\}$; and σ_i^n, $W_{i,l}^n$, and $W_{i,r}^n$ are chosen so that if W_{i-1}^n and W_{i+1}^n may be linked by an admissible discontinuity with speed σ, then

$$W_{i,l}^n = W_{i-1}^n, \quad W_{i,r}^n = W_{i+1}^n, \quad \sigma_i^n = \sigma. \tag{16}$$

This in-cell discontinuous reconstruction can only be done if $0 \leq d_i^n \leq 1$, i.e. if

$$0 \leq \frac{w_{i,r,k}^n - w_{i,k}^n}{w_{i,r,k}^n - w_{i,l,k}^n} \leq 1,$$

otherwise the MUSCL-Hancock reconstruction is applied in the cell. Moreover, if $d_i^n = 1$ and $\sigma_i^n > 0$ (resp. $d_i^n = 0$ and $\sigma_i^n < 0$) the cell is unmarked and the cell I_{i+1} (resp. I_{i-1}) is marked if necessary.

Semi-discrete method

- Otherwise (i.e. if $i \notin \mathcal{M}_n$ but $i-1 \in \mathcal{M}_n$ or $i+1 \in \mathcal{M}_n$) then
$$\mathbb{P}_i^n(x,t) = W_i^n.$$

Remark 1

In the case $i \in \mathcal{M}_n$, if one of the equations of system (1), say the kth one, is a conservation law, the index k is selected in (15), so that the corresponding variable is conserved. Moreover, if there is a linear combination of the unknowns $\sum_{k=1}^N \alpha_k w_k$ that is conserved, (15) may be replaced by:

$$d_i^n \sum_{k=1}^N \alpha_k w_{i,l,k}^n + (1-d_i^n) \sum_{k=1}^N \alpha_k w_{i,r,k}^n = \sum_{k=1}^N \alpha_k w_{i,k}^n. \tag{17}$$

If there are more than one conservation laws, the index k corresponding to one of them is selected in (15).

Choice of the discontinuous reconstruction

How to choose σ_i^n, $W_{i,l}^n$, $W_{i,r}^n$?
Two strategies:

- **First strategy:** If the solutions of the Riemann problems are explicitly known, in a marked cell σ_i^n, $W_{i,l}^n$, $W_{i,r}^n$ can be chosen as the speed, the left, and the right states of (one of the) discontinuous waves appearing in the solution of the Riemann problem with initial data W_{i-1}^n, W_{i+1}^n.
- **Second strategy:** If a Roe matrix is available, in a marked cell σ_i^n, $W_{i,l}^n$, $W_{i,r}^n$ can be chosen as the speed, the left, and the right states of one of the discontinuities appearing in the solution of the linearized Riemann problem with initial data W_{i-1}^n, W_{i+1}^n. More explicitly, an index k^* has to be selected and then

$$\sigma_i^n = \lambda_{k^*}(W_{i-1}^n, W_{i+1}^n), W_{i,l}^n = W_{i-1}^n + \sum_{k=1}^{k^*-1} \alpha_k R_k(W_{i-1}^n, W_{i+1}^n),$$

$$W_{i,r}^n = W_{i,l}^n + \alpha_{k^*} R_{k^*}(W_{i-1}^n, W_{i+1}^n),$$

where α_k, $k = 1, \ldots, N$ represent the coordinates of $W_{i+1}^n - W_{i-1}^n$ on the basis of eigenvectors of $\mathcal{A}_\Phi(W_{i-1}^n, W_{i+1}^n)$.

Time step

The time step Δt_n is chosen as follows:

$$\Delta t_n = \min(\Delta t_n^c, \Delta t_n^r), \tag{18}$$

where

$$\Delta t_n^c = CFL \min\left(\frac{\Delta x}{\max_{i,l} |\lambda_{i,l}|}\right), \tag{19}$$

where $CFL \in (0,1)$ is the stability parameter and $\lambda_{i,l}, \ldots, \lambda_{i,N}$ represent the eigenvalues of $\mathcal{A}(W_i^n)$; and

$$\Delta t_n^r = \min_{i \in \mathcal{M}_n} \begin{cases} \dfrac{1 - d_i^n}{|\sigma_i^n|}\Delta x, & if \quad \sigma_i^n > 0, \\ \\ \dfrac{d_i^n}{|\sigma_i^n|}\Delta x, & if \quad \sigma_i^n < 0. \end{cases} \tag{20}$$

Observe that, besides the stability requirement, this choice of time step ensures that no discontinuous reconstruction leaves a marked cell.

Fully discrete method

Once the time step is chosen, (12) is integrated in the interval $[t^n, t^{n+1}]$, with $t^{n+1} = t^n + \Delta t_n$, to obtain:

$$W_i^{n+1} = W_i^n - \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \left(\mathcal{D}^-_{i+\frac{1}{2}}(t) + \mathcal{D}^+_{i-\frac{1}{2}}(t) + \fint_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^n(x,t))\partial_x \mathbb{P}_i^n(x,t)dx \right) dt,$$

and the mid-point rule is used to approximate the integrals in time:

$$W_i^{n+1} = W_i^n - \frac{\Delta t_n}{\Delta x} \left(\mathcal{D}^-_{i+\frac{1}{2}}(t^{n+\frac{1}{2}}) + \mathcal{D}^+_{i-\frac{1}{2}}(t^{n+\frac{1}{2}}) + \fint_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^n(x, t^{n+1/2}))\partial_x \mathbb{P}_i^n(x, t^{n+1/2})\, dx \right), \tag{21}$$

Fully discrete method

The computation of the dashed integral in this expression depends on the cell:

1. If $i-1, i, i+1 \notin \mathcal{M}_n$ the mid-point rule is used again to approximate the integral:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^n(x, t^{n+1/2}))\partial_x \mathbb{P}_i^n(x, t^{n+1/2})\, dx \approx \Delta x \mathcal{A}(W_i^{n+\frac{1}{2}})\widetilde{\partial_x W}_i^n, \tag{22}$$

where

$$W_i^{n+\frac{1}{2}} = \mathbb{P}_i^n(x_i, t^{n+\frac{1}{2}}) = W_i^n - \frac{\Delta t}{2}\mathcal{A}(W_i^n)\widetilde{\partial_x W}_i^n.$$

Fully discrete method

2 If $i \in \mathcal{M}_n$, taking into account the definition of the dashed integrals (2), one has:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^n(x, t^{n+1/2}))\partial_x \mathbb{P}_i^n(x, t^{n+1/2})\, dx = \int_0^1 \mathcal{A}(\Phi(s; W_{i,l}^n, W_{i,r}^n))\partial_s \Phi(s; W_{i,l}^n, W_{i,r}^n)\, ds. \tag{23}$$

Observe that, if $W_{i,1}^n$ and $W_{i,r}^n$ can be linked by a shock whose speed is σ_i^n, then the generalized Rankine-Hugoniot condition (3) leads to

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^n(x, t^{n+1/2}))\partial_x \mathbb{P}_i^n(x, t^{n+1/2})\, dx = \sigma_j^n \left(W_{i,r}^n - W_{i,l}^n \right). \tag{24}$$

Fully discrete method

3 If $i \notin \mathcal{M}_n$ but $i-1 \in \mathcal{M}_n$ or $i+1 \in \mathcal{M}_n$ then

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{A}(\mathbb{P}_i^n(x, t^{n+1/2}))\partial_x \mathbb{P}_i^n(x, t^{n+1/2})\, dx = 0. \quad (25)$$

The final expression of the fully discrete numerical method is as follows:

$$W_i^{n+1} = W_i^n - \frac{\Delta t_n}{\Delta x}\left(\mathcal{D}^-_{i+\frac{1}{2}}(t^{n+\frac{1}{2}}) + \mathcal{D}^+_{i-\frac{1}{2}}(t^{n+\frac{1}{2}}) + \mathcal{D}_i\right), \quad (26)$$

where

$$\mathcal{D}_i = \begin{cases} \Delta x \mathcal{A}(W_i^{n+\frac{1}{2}})\widetilde{\partial_x W}_i^n & \text{if } i-1, i, i+1 \notin \mathcal{M}_n; \\ \int_0^1 \mathcal{A}(\Phi(s; W_{i,l}^n, W_{i,r}^n))\partial_s \Phi(s; W_{i,l}^n, W_{i,r}^n)\, ds & \text{if } i \in \mathcal{M}_n; \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

Shock-capturing property

Theorem

Assume that W_l and W_r can be joined by an entropy shock of speed σ. Then, the numerical method provides an exact numerical solution of the Riemann problem with initial conditions

$$W(x,0) = \begin{cases} W_l & \text{if } x < 0, \\ W_r & \text{otherwise,} \end{cases}$$

in the sense that

$$W_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} W(x,t^n)\, dx, \quad \forall i, n \tag{28}$$

where $W(x,t)$ is the exact solution.

Numerical methods

We consider the following nonconservative systems:

- Coupled Burgers system.
- Gas dynamics in Lagrangian coordinates.
- Simplified shallow water equations.

Notation:

- O1_noDisRec: standard first-order path-conservative Roe or Godunov (if it is indicated between parentheses) methods.
- O1_DisRec: first-order path-conservative method with discontinuous reconstruction;
- O2_noDisRec: second-order extension standard of the first order path-conservative method based on the MUSCL-Hancock reconstruction;
- O2_DisRec: second-order path-conservative method that combines MUSCL-Hancock and discontinuous reconstruction.

Coupled Burgers equations

$$\left\{ \begin{array}{rcl} \partial_t u + \partial_x \left(\dfrac{u^2}{2} \right) + u \partial_x v & = & 0, \\ \partial_t v + \partial_x \left(\dfrac{v^2}{2} \right) + v \partial_x u & = & 0, \end{array} \right. \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \qquad (29)$$

introduced in (Castro, Macías & Parés, 2001), where $W = (u,v)^T$ belongs to the state space $\Omega = \{W \in \mathbb{R}^2, u + v > 0\}$. Nonconservative form (1) with

$$\mathcal{A}(W) = \left[\begin{array}{cc} u & u \\ v & v \end{array} \right].$$

The system is strictly hyperbolic in Ω with eigenvalues and eigenvectors

$$\lambda_1(W) = 0, \quad \lambda_2(W) = u + v.$$

$$R_1(W) = [1, -1]^T, \quad R_2(W) = [u, v]^T,$$

that are respectively linearly degenerate and genuinely nonlinear.
The sum $u + v$ is conserved because satisfies the standard Burgers equation

$$\partial_t (u+v) + \partial_x \left(\frac{1}{2} (u+v)^2 \right) = 0.$$

Coupled Burgers equations

Once we choose the family of paths, the simple waves of this system are:

- Stationary contact discontinuities linking states W_l, W_r such that

$$u_l + v_l = u_r + v_r.$$

- Rarefactions waves joining states W_l, W_r such that

$$u_l + v_l < u_r + v_r, \quad \frac{u_l}{v_l} = \frac{u_r}{v_r}.$$

- Shock waves joining states W_l and W_r such that

$$u_l + v_l > u_r + v_r$$

  that satisfy the jump condition:

$$\begin{aligned} \sigma[u] &= \left[\frac{u^2}{2}\right] + \int_0^1 \phi_u(s; W_l, W_r)\partial_s \phi_v(s; W_l, W_r)\, ds, \\ \sigma[v] &= \left[\frac{v^2}{2}\right] + \int_0^1 \phi_v(s; W_l, W_r)\partial_s \phi_u(s; W_l, W_r)\, ds. \end{aligned}$$

  This leads, independently of ϕ, to $\sigma = \frac{u_l + v_l + u_r + v_r}{2}$.

Coupled Burgers equations

If, for instance, the family of straight segments is chosen

$$\phi_u(s;W_l,W_r) = u_l + s(u_r - u_l); \quad \phi_v(s;W_l,W_r) = v_l + s(v_r - v_l), \quad (30)$$

the jump conditions reduce to:

$$\begin{aligned} \sigma[u] &= \left(\frac{u_l + u_r}{2}\right)(u_r - u_l + v_r - v_l), \\ \sigma[v] &= \left(\frac{v_l + v_r}{2}\right)(u_r - u_l + v_r - v_l), \end{aligned}$$

and two states can be joined by an admissible shock if

$$u_l + v_l > u_r + v_r, \quad \frac{u_l}{v_l} = \frac{u_r}{v_r}.$$

A Roe matrix is given in this case by:

$$\mathcal{A}(W_l,W_r) = \begin{bmatrix} 0.5(u_l + u_r) & 0.5(u_l + u_r) \\ 0.5(v_l + v_r) & 0.5(v_l + v_r) \end{bmatrix}. \quad (31)$$

Coupled Burgers equations: Test 1: straight segments

In this test case we consider the definition of weak solution related to the family of straight segments (30) and the corresponding Roe matrix (31). Let us consider the following initial condition

$$W_0(x) = (u,v)_0(x) = \begin{cases} (2.0, 2.0) & \text{if } x < 0.5, \\ (1.0, 1.0) & \text{otherwise.} \end{cases}$$

The solution of the Riemann problem in this case consists of a shock wave joining the left and right states. A 1000-cell mesh and CFL=0.5 have been used.

Coupled Burgers equations: Test 1: straight segments

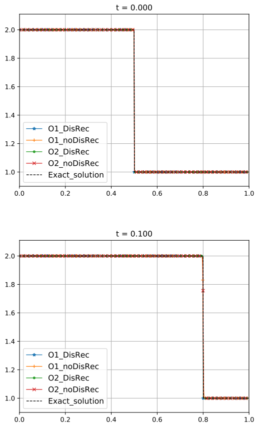


Figure: Variable u

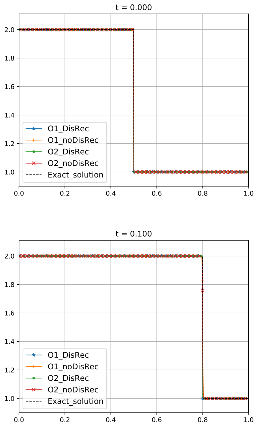


Figure: Variable v

Coupled Burgers equations

Let us consider, for instance, the family of paths given by the viscous profiles of the regularized system:

$$\left\{ \begin{array}{rcl} \partial_t u_+ \partial_x \left(\dfrac{u^2}{2} \right) + u \partial_x v & = & \epsilon u_{xx}, \\ \partial_t v + \partial_x \left(\dfrac{v^2}{2} \right) + v \partial_x u & = & \epsilon v_{xx}, \end{array} \right. \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{32}$$

introduced in (Berthon, 2002).
To apply this technique, a cell is marked if

$$u_{i-1}^n + v_{i-1}^n > u_{i+1}^n + v_{i+1}^n.$$

Strategy 1 (based on the exact solutions of the Riemann problems) is followed here. More precisely:

$$\sigma_i^n = \frac{1}{2}(u_{i-1}^n + v_{i-1}^n + u_{i+1}^n + v_{i+1}^n), \quad W_{i,l}^n = W^*(W_{i-1}^n, W_{i+1}^n), \quad W_{i,r}^n = W_{i+1}^n,$$

where $W^*(W_{i-1}^n, W_{i+1}^n)$ represents the state at the left of the shock wave appearing in the solution of the Riemann problem. Finally, the conserved variable $u + v$ is chosen to determine d_i^n, i.e.

$$d_i^n(u_{i,l}^n + v_{i,l}^n) + (1 - d_i^n)(u_{i,r}^n + v_{i,r}^n) = (u_i^n + v_i^n).$$

Coupled Burgers equations: Test 2: isolated shock

Let us consider the following initial condition taken from (Castro, Fjordholm, Mishra, Parés, 2013)

$$W_0(x) = (u,v)_0(x) = \begin{cases} (7.99, 11.01) & if x < 0.5, \\ (0.25, 0.75) & \text{othewise.} \end{cases}$$

The solution of the Riemann problem consists of a shock wave joining the left and right states. We use first a 100-cell and then a 1000-cell mesh. CFL=0.5 has been used.

Coupled Burgers equations: Test 2: isolated shock

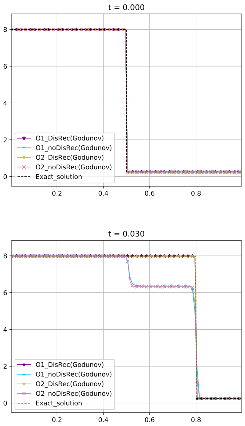


Figure: Variable u. 100-cell mesh

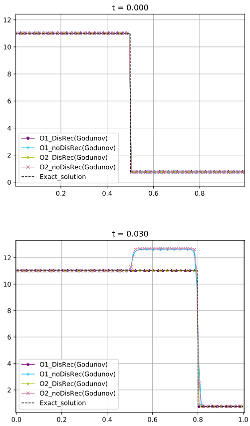


Figure: Variable v. 100-cell mesh

Coupled Burgers equations: Test 2: isolated shock

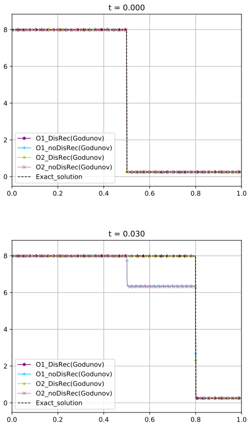


Figure: Variable u. 1000-cell mesh

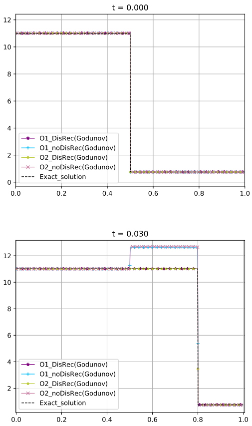


Figure: Variable v. 1000-cell mesh

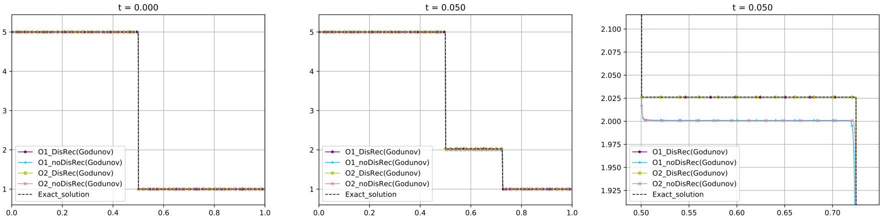
Coupled Burgers equations: Test 3: Contact discontinuity + shock wave

We consider now the initial condition

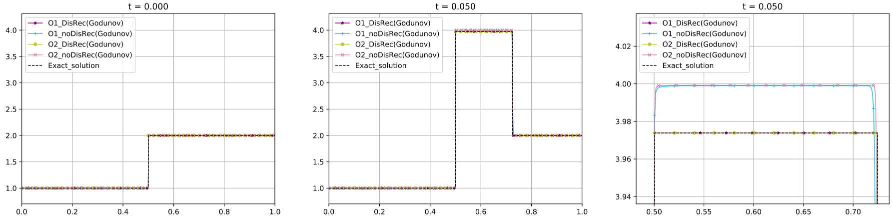
$$W_0(x) = (u, v)_0(x) = \begin{cases} (5, 1) & \text{if } x < 0.5, \\ (1, 2) & \text{otherwise.} \end{cases}$$

The solution of the corresponding Riemann problems consists of a stationary contact discontinuity followed by a shock. A 1000-cell mesh and CFL=0.5 have been used.

Coupled Burgers equations: Test 3: Contact discontinuity + shock wave. Variable u



Coupled Burgers equations: Test 3: Contact discontinuity + shock wave. Variable v



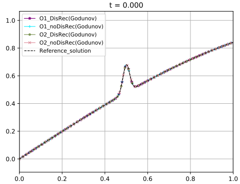
Coupled Burgers equations: Test 4: Perturbed stationary solution

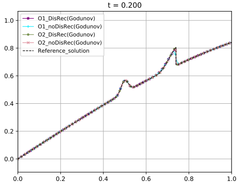
We consider finally the initial condition

$$W_0(x) = (u, v)_0(x) = (\sin(x) + 0.2e^{-2000(r-0.5)^2}, 1 - \sin(x)), \quad (33)$$

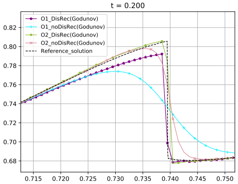
that is a stationary solution with a perturbation in the variable u. A 1000-cell mesh and CFL=0.5 have been used. It has been used a reference solution with a 10000-cell mesh.

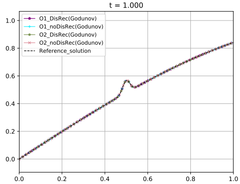
Coupled Burgers equations: Test 4: Perturbed stationary solution. Variable u



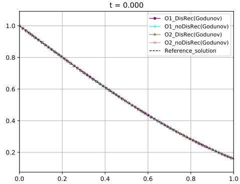


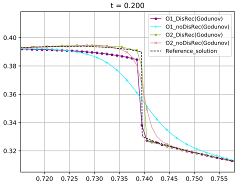
(a) Zoom



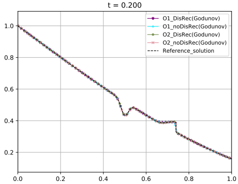


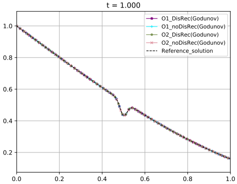
Coupled Burgers equations: Test 4: Perturbed stationary solution. Variable v





(a) Zoom





Gas dynamics equations in Lagrangian coordinates

The gas dynamics equations in Lagrangian coordinates can be written in nonconservative form (1) with

$$W = \begin{pmatrix} \tau \\ u \\ e \end{pmatrix}, \quad \mathcal{A}(W) = \begin{pmatrix} 0 & -1 & 0 \\ -\dfrac{(\gamma-1)e}{\tau^2} & 0 & \dfrac{\gamma-1}{\tau} \\ 0 & \dfrac{(\gamma-1)e}{\tau} & 0 \end{pmatrix},$$

where $\tau > 0$ represents the inverse of the density, u is the velocity, $p = p(\tau, e) > 0$ is the pressure, e is the internal energy, and $E = e + u^2/2$ the total energy. For the sake of simplicity, we consider a perfect gas equation of state $p(\tau, e) = (\gamma - 1)e/\tau$ where $\gamma > 1$.

Gas dynamics equations in Lagrangian coordinates

The system is strictly hyperbolic with eigenvalues

$$\lambda_1(W) = -\sqrt{\gamma p/\tau}, \quad \lambda_2(W) = 0, \quad \lambda_3(W) = \sqrt{\gamma p/\tau},$$

whose characteristic fields are given by the eigenvectors

$$R_1(W) = [1, \sqrt{\gamma p/\tau}, -p]^T, \quad R_2(W) = [1, 0, p/(\gamma-1)], \quad R_3(W) = [1, -\sqrt{\gamma p/\tau}, -p]$$

$R_2(W)$ is linearly degenerate and $R_i(W)$, $i = 1, 3$ genuinely nonlinear: see (Godlewski & Raviart, 1995). The admissible solutions are selected by Lax entropy inequalities, which here are equivalent to:

$$\sigma(\tau_+ - \tau_-) \geq 0. \tag{34}$$

Gas dynamics equations in Lagrangian coordinates

The simple waves of this system are:

- Stationary contact discontinuities linking states W_l, W_r such that $u_l = u_r$.
- Rarefactions waves joining states W_l, W_r such that $u_l < u_r$, and the relations given by the Riemann invariants:
  - 1-rarefactions:

$$2\sqrt{\frac{\gamma e_l}{\gamma - 1}} + u_l = 2\sqrt{\frac{\gamma e_r}{\gamma - 1}} + u_r, \quad \frac{e_l}{\tau_l^{\gamma-1}} = \frac{e_r}{\tau_r^{\gamma-1}}.$$

  - 2-rarefactions:

$$2\sqrt{\frac{\gamma e_l}{\gamma - 1}} - u_l = 2\sqrt{\frac{\gamma e_r}{\gamma - 1}} - u_r, \quad \frac{e_l}{\tau_l^{\gamma-1}} = \frac{e_r}{\tau_r^{\gamma-1}}.$$

- Shock waves joining W_l, W_r such that $u_l > u_r$ that satisfy the jump conditions:

$$\begin{aligned} \sigma[\tau] &= -[u], \\ \sigma[u] &= [p], \\ \sigma[e] &= \int_0^1 \phi_p(s; W_l, W_r)\partial_s \phi_u(s; W_l, W_r)\, ds. \end{aligned}$$

Gas dynamics equations in Lagrangian coordinates

If, for instance, the family of straight segments is chosen for the variables τ, u, p, the jump conditions reduce to:

$$\begin{aligned} \sigma[\tau] &= (u_l - u_r), \\ \sigma[u] &= p_r - p_l, \\ \sigma[e] &= \frac{1}{2}(p_r + p_l)(u_r - u_l). \end{aligned}$$

It can be easily checked that these jump conditions are equivalent to the standard Rankine-Hugoniot conditions corresponding to the conservative formulation and thus, the weak solutions are the same. A Roe matrix is given in this case by:

$$\mathcal{A}(W_l, W_r) = \mathcal{A}(\bar{W}), \quad \bar{W}(W_l, W_r) = (\bar{\tau}, \bar{u}, \bar{p}),$$

with $\bar{\tau} = \frac{\tau_l + \tau_r}{2}, \quad \bar{u} = \frac{u_l + u_r}{2}, \quad \bar{e} = \frac{\bar{p}\bar{\tau}}{\gamma - 1}, \quad \bar{p} = \frac{p_l + p_r}{2}$, see (Munz, 1994).

Gas dynamics equations in Lagrangian coordinates

To apply this technique, a cell is marked if

$$u^n_{i-1} \geq u^n_{i+1}.$$

The second strategy to select the speed, and the left and right states of the discontinuous reconstruction based on the Roe matrix is used here. More precisely:

- If $u^n_{i-1} = u^n_{i+1}$ then

$$\sigma^n_i = 0, \quad W^n_{i,l} = W^n_{i-1}, \quad W^n_{i,r} = W^n_{i+1}.$$

- If $u^n_{i-1} > u^n_{i+1}$ and $\tau^n_{i+1} - \tau^n_{i-1} < 0$ then

$$\sigma^n_i = -\sqrt{\gamma\bar{p}/\bar{\tau}}, \quad W^n_{i,l} = W^n_{i-1}, \quad W^n_{i,r} = W^n_{i-1} + \alpha_1 R_1(W^n_{i-1}, W^n_{i+1}).$$

- If $u^n_{i-1} > u^n_{i+1}$ and $\tau^n_{i+1} - \tau^n_{i-1} > 0$ then

$$\sigma^n_i = \sqrt{\gamma\bar{p}/\bar{\tau}}, \quad W^n_{i,l} = W^n_{i+1} - \alpha_3 R_3(W^n_{i-1}, W^n_{i+1}), \quad W^n_{i,r} = W_{i+1}.$$

This method is extended here to second order.

Test 1: isolated 1-shock

Let us consider the following initial condition taken from (Chalons, 2019)

$$(\tau, u, p)_0(x) = \begin{cases} (2.09836065573770281, 2.3046638387921279, 1) & \text{if } x < 0.5, \\ (8, 0, 0.1) & \text{otherwise.} \end{cases}$$

The solution of the Riemann problem consists of a 1-shock wave joining the left and right states. A 300-cell mesh and CFL=0.5 have been used.

Test 1: isolated shock

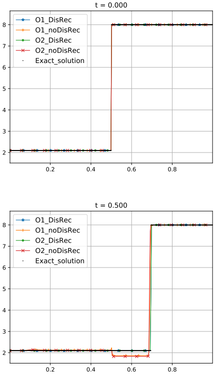


Figure: Variable τ

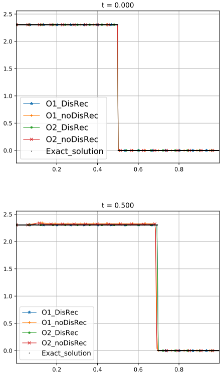


Figure: Variable u

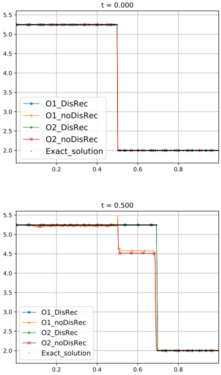


Figure: Variable e

Test 2: 1-shock + contact discontinuity + 3-shock

Let us consider the following initial condition taken from (Chalons, 2019)

$$(\tau, u, p)_0(x) = \begin{cases} (5, 3.323013993227, 0.481481481481) & \text{if } x < 0.5, \\ (8, 0, 0.1) & \text{otherwise.} \end{cases}$$

The solution of the Riemann problem consists of a 1-shock wave with negative speed, a stationary contact discontinuity, and a 3-shock that coincides with the one in the first test problem. A 300-cell mesh and CFL=0.5 have been used.

Test 2: 1-shock + contact discontinuity + 3-shock

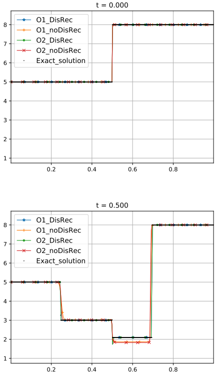


Figure: Variable τ

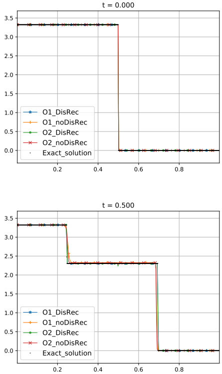


Figure: Variable u

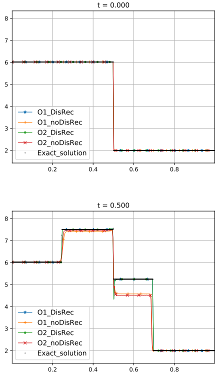


Figure: Variable e

Modified shallow water equations

Let us consider the modified Shallow Water system introduced in (Castro, LeFloch, Muñoz-Ruiz, Parés, 2008):

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} \right) + qh\partial_x h = 0, \end{cases} \tag{35}$$

where $W = (h, q)^t$ belongs to $\Omega = \{W \in \mathbb{R}^2 | \quad 0 < q, \ 0 < h < (16q)^{1/3}\}$. This system can be written in the form (1) with

$$\mathcal{A}(W) = \begin{bmatrix} 0 & 1 \\ -u^2 + uh^2 & 2u \end{bmatrix},$$

being $u = q/h$. The system is strictly hyperbolic Ω with eigenvalues

$$\lambda_1(W) = u - h\sqrt{u}, \quad \lambda_2(W) = u + h\sqrt{u},$$

whose characteristic fields, given by the eigenvectors

$$R_1(W) = [1, u - h\sqrt{u}]^T, \quad R_2(W) = [1, u + h\sqrt{u}]^T,$$

are genuinely nonlinear.

Modified shallow water equations

Once the family of paths has been chosen, the simple waves of this system are:

- 1-rarefaction waves joining states W_l, W_r such that

$$h_r < h_l, \quad \sqrt{u_l} + h_l/2 = \sqrt{u_r} + h_r/2,$$

  and 2-rarefaction waves joining states W_l, W_r such that

$$h_l < h_r, \quad \sqrt{u_l} - h_l/2 = \sqrt{u_r} - h_r/2.$$

- 1-shock and 2-shock waves joining states W_l and W_r such that $h_l < h_r$ or $h_r < h_l$ respectively, that satisfy the jump conditions:

$$\begin{aligned} \sigma[h] &= [q]\,, \\ \sigma[q] &= \left[\frac{q^2}{h}\right] + \int_0^1 \phi_q(s; W_l, W_r)\phi_h(s; W_l, W_r)\partial_s\phi_h(s; W_l, W_r)\, ds. \end{aligned}$$

Modified shallow water equations

If, for instance, the following family of path is chosen:

$$\phi(s; W_l, W_r) = \begin{bmatrix} \phi_h(s; W_l, W_r) \\ \phi_q(s; W_l, W_r) \end{bmatrix} = \begin{cases} \begin{bmatrix} h_l + 2s(h_r - h_l) \\ q_l \end{bmatrix} & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \begin{bmatrix} h_r \\ q_l + (2s-1)(q_r - q_l) \end{bmatrix} & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

the jump conditions reduce to:

$$\begin{aligned} \sigma[h] &= [q]\,, \\ \sigma[q] &= \left[\frac{q^2}{h}\right] + q_l \left[\frac{h^2}{2}\right]. \end{aligned}$$

If this family of paths and the Lax's entropy criterion is used we can obtain the expression of the simple waves curves.

Modified shallow water equations

Although we are not going to show these expression, the criterion to mark the cells is divided in the following cases:

Case 1: If the solution of the Riemann problem consists of a 1-shock and a 2-rarefaction waves: the cell is marked.

Case 2: If the solution of the Riemann problem consists of a 1-rarefaction and a 2-shock waves: the cell is marked.

Case 3 : If the solution of the Riemann problem consists of a 1-shock and a 2-shock waves: the cell is marked.

Case 4: Otherwise the solution of the Riemann problem consists of two rarefactions and the cell is not marked.

Modified shallow water equations: Roe strategy

A Roe matrix is given in this case by

$$\mathcal{A}(W_l, W_r) = \begin{bmatrix} 0 & 1 \\ -\bar{u}^2 + q_l \bar{h} & 2\bar{u} \end{bmatrix},$$

where

$$\bar{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}, \quad \bar{h} = \frac{h_l + h_r}{2}.$$

The variable h is selected for obtaining d_i^n. The following strategy based on the Roe matrix is used to select the speed, and the left and right states of the discontinuous reconstruction:

- Case 1:

$$\sigma_i^n = \bar{u} - h_{i-1}^n \sqrt{\bar{u}}, \quad W_{i,l}^n = W_{i-1}^n, \quad W_{i,r}^n = W_{i-1}^n + \alpha_1 R_1(W_{i-1}^n, W_{i+1}^n).$$

- Case 2:

$$\sigma_i^n = \bar{u} + h_{i-1}^n \sqrt{\bar{u}}, \quad W_{i,l}^n = W_{i+1}^n - \alpha_2 R_2(W_{i-1}^n, W_{i+1}^n), \quad W_{i,r}^n = W_{i+1}^n.$$

- Case 3: we select the 'dominant' one:
  - If $|\alpha_1| \leq |\alpha_2|$ then:

$$\sigma_i^n = \bar{u} + h_{i-1}^n \sqrt{\bar{u}}, \quad W_{i,l}^n = W_{i+1}^n - \alpha_2 R_2(W_{i-1}^n, W_{i+1}^n), \quad W_{i,r}^n = W_{i+1}.$$

  - If $|\alpha_1| > |\alpha_2|$ then:

$$\sigma_i^n = \bar{u} - h_{i-1}^n \sqrt{\bar{u}}, \quad W_{i,l}^n = W_{i-1}^n, \quad W_{i,r}^n = W_{i-1}^n + \alpha_1 R_1(W_{i-1}, W_{i+1}).$$

Test 1: Isolated 1-shock

Let us consider the following initial condition taken from (Castro, LeFloch, Muñoz-Ruiz, Parés, 2008)

$$(h,q)_0(x) = \begin{cases} (1,1) & \text{if } x < 0, \\ (1.8, 0.530039370688997) & \text{otherwise.} \end{cases}$$

The solution of the Riemann problem consists of a 1-shock wave joining the left and right states. A 1000-cell mesh and CFL=0.5 have been used.

Test 1: Isolated 1-shock

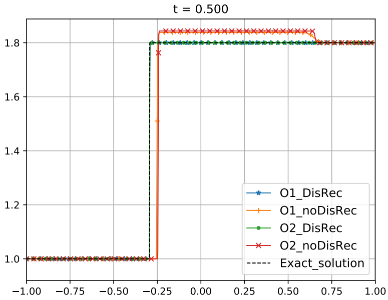


Figure: Variable h

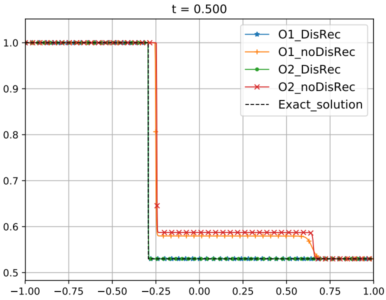


Figure: Variable q

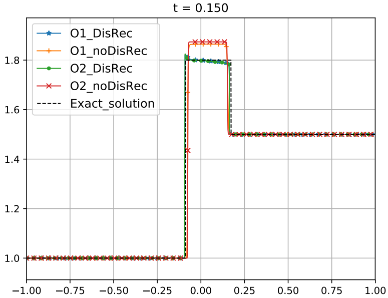
Test 2: left-moving 1-shock + right-moving 2-shock

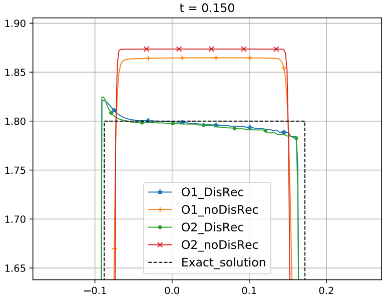
Let us consider the following initial condition

$$(h,q)_0(x) = \begin{cases} (1,1) & \text{if } x < 0, \\ (1.5, 0.1855893974385) & \text{otherwise.} \end{cases} \tag{36}$$

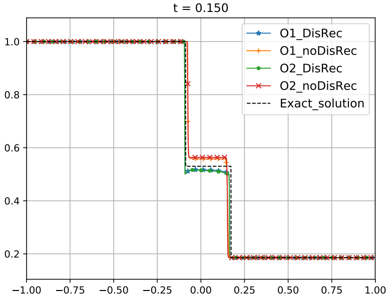
The solution of the Riemann problem consists of a 1-shock wave with negative speed and a 2-shock with positive speed with intermediate state $W_* = [1.8, 0.530039370688997]^T$. A 1000-cell mesh and CFL=0.5 have been used.

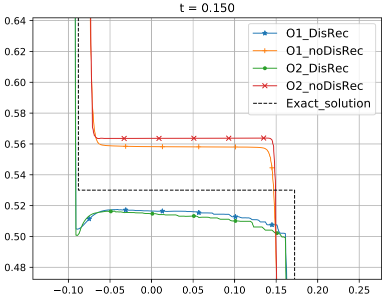
Test 2: left-moving 1-shock + right-moving 2-shock. Variable h



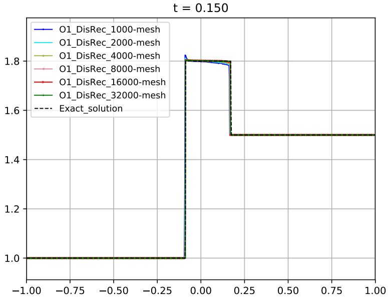


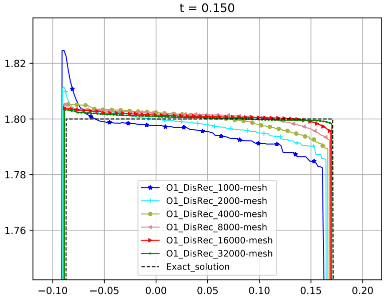
Test 2: left-moving 1-shock + right-moving 2-shock. Variable q



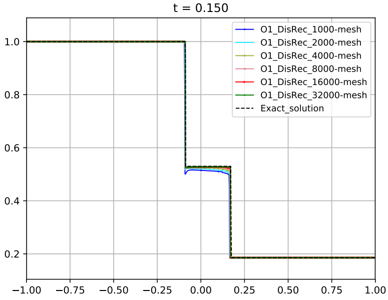


Test 2: left-moving 1-shock + right-moving 2-shock. Variable h: comparison





Test 2: left-moving 1-shock + right-moving 2-shock. Variable q: comparison



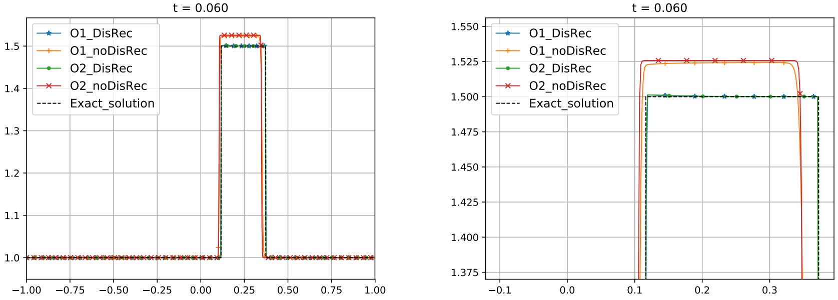
Test 3: right-moving 1-shock + right-moving 2-shock

Let us consider the following initial condition

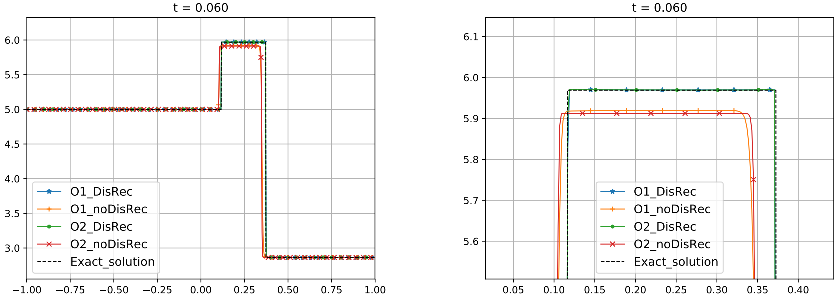
$$(h,q)_0(x) = \begin{cases} (1,1) & \text{if } x < 0, \\ (5, 2.86423084288) & \text{otherwise.} \end{cases} \quad (37)$$

The solution of the Riemann problem consists of a 1-shock and a 2-shock waves with positive speed and intermediate state $W_* = [1.5, 5.96906891076]^T$. A 1000-cell mesh and CFL=0.5 have been used.

Test 3: right-moving 1-shock + right-moving 2-shock. Variable h



Test 3: right-moving 1-shock + right-moving 2-shock. Variable q



Modified shallow water equations: Exact strategy

A more sophisticated strategy based on the exact solution of the Riemann problems allows one to handle correctly with these situations. The key ingredients are:

- The solution of the Riemann problem with initial data $W_{i-1,r}^{n-1}$ and $W_{i+1,l}^{n-1}$ is used to mark the cells instead of the one corresponding to the initial data W_{i-1}^{n} and W_{i+1}^{n}, where $W_{i-1,r}^{n-1}$ and $W_{i+1,l}^{n-1}$ are the states selected in the discontinuous reconstruction in the previous time step.
- The exact intermediate state is used when the solution of the Riemann problem involves two shock waves.
- If the solution of this Riemann problem involves two shock waves traveling in the same direction, a reconstruction with two discontinuities (one for each of the shock waves) is considered, so that the complete structure of the Riemann solution is imposed.

Test 2: left-moving 1-shock + right-moving 2-shock. Exact strategy

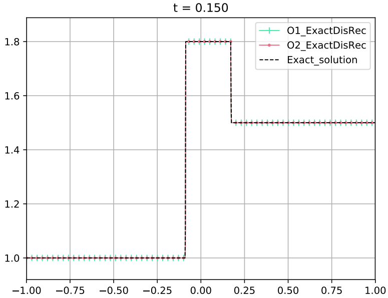


Figure: Variable h.

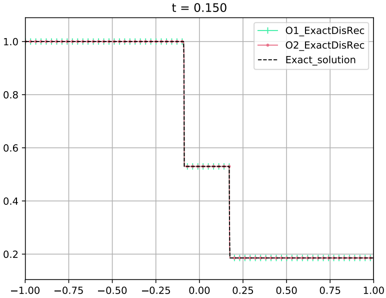


Figure: Variable q

Test 3: left-moving 1-shock + right-moving 2-shock. Exact strategy

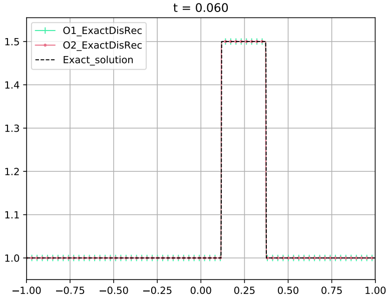


Figure: Variable h.

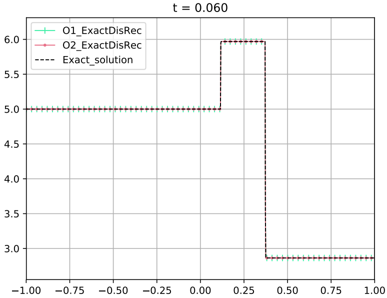


Figure: Variable q

Conclusions and future work

Conclusions:

- We extend the strategy developed in (Chalons, 2019) to second order of accuracy.
- This extension is based on the combination of the first-order in-cell reconstruction and the MUSCL-Hancock reconstruction.
- The isolated shock-capturing property is enunciated, proved and tested.

Future work:

- Extend this technique to arbitrary order of accuracy.
- Capture correctly non isolated shocks.
- Apply the methods to more complex models.
- Develop new Discontinuous Galerkin (DG) solvers based on discontinuous reconstructions.
- Explore the extension to multidimensional problems.

End

Thank you for your attention