EQUILIBRIUM WAGE/TENURE CONTRACTS WITH ON-THE-JOB LEARNING AND SEARCH.

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Abstract

This paper investigates equilibria in a labor market where firms post wage/tenure contracts and risk-averse workers, both employed and unemployed, search for better paid job opportunities. Different firms typically offer different contracts. Workers accumulate general human capital through learning-by-doing. With on-the-job search, a worker's wage evolves endogenously over time through experience effects, tenure effects and quits to better paid employment. This equilibrium approach suggests how to identify econometrically between experience and tenure effects on worker wages.

1 Introduction

This paper investigates individual wage dynamics in the context of an equilibrium labour market model where workers become more productive through learning-by-doing and firms post contracts where, inter alia, wages paid depend on tenure. The analysis leads to new insights into two important areas in labour economics - the nature of equilibrium in search markets, and the empirical decomposition of wages into experience and tenure effects.

Although there is no free lunch, it is accepted by many that individuals accumulate human capital freely by working. Typists become better typists while working as typists, economists become more productive by doing economics, etc. This seems both an important and intuitive idea. A related idea now common among labour economists is that human capital can be dichotomized into general human capital and firm specific human capital. A worker who enjoys an increase in general human capital becomes more productive at all jobs, whereas accumulating firm specific human capital implies a worker is only more productive at that firm. Workers who change job, or those who are laid off, lose their firm specific human capital but keep their general human capital. Putting the above two ideas together, plus assuming a worker's wage is an increasing function of both his/her general and specific human capital, leads to at least the rudiment of a theory of how the wages of workers evolve through their working lives.

There is a significant empirical literature which has attempted to decompose wages into experience effects (general human capital) and tenure effects (firm specific capital); see for example, Altonji and Shakotko, 1987, Topel, 1991, Altonji and Williams, 2005, and Dustmann and Meghir, 2005. The difficulty faced by this literature is that tenure and experience are perfectly correlated within any employment spell. As it is unreasonable to assume a quit, which resets tenure to zero, is an exogenous outcome which is orthogonal to the wage paid at the previous employer and at the new one, identifying between tenure and experience wage effects requires an equilibrium theory of wage formation and quit turnover.

Following seminal work by Burdett and Mortensen (1998), there have been three different approaches to explaining quit turnover and employee wage variation in frictional labour markets. Postel-Vinay and Robin (2002), Cahuc et al (2006) explain within firm wage dispersion by assuming firms respond to outside offers. As an employee with longer tenure is more likely to have received outside offers, it follows that wages will be positively correlated with tenure. Stevens (2004) and Burdett and Coles (2003),(2009) instead assume firms must treat employees equally, where employees with the same productivity and tenure are paid the same wage. In that case firms optimally set up a seniority wage scale where more senior employees are paid more. By raising wages paid with tenure, the firm increases the value of employment at the firm and can then pay relatively low wages to new hires.[1] Moscarini (200?) instead assumes worker productivity is uncertain and within firm wage variation arises as a firm learns over time about the productivity of each employee.

In this paper we extend the Burdett and Coles (2003) approach, B/C from now on, to the case that there is also learning-by-doing; i.e. firms offer contracts where wages paid depend on tenure and a worker's productivity also increases over time while employed. This structure not only generates non-trivial, idiosyncratic earnings profiles across each worker's lifetime, it also yields equilibrium wage dispersion as:
(i) workers are ex-ante heterogeneous - worker i has productivity y_i when first entering the labour market;
(ii) different employees have different current work experience x;
(iii) different employees have different current tenures/seniority τ;
(iv) there is dispersion in wage contracts offered, where firm j pays piece rate $\theta = \theta_j(\tau)$ to employees with tenure τ;
(v) there is sorting with age, where on-the-job search implies workers eventually find and quit to better paid employment.

Burdett et al (2009) consider a similar framework but restrict each firm to paying a single piece rate; i.e. there are no tenure effects.[2] That paper finds that learning-by-

[1] See Stevens (2004) for a complete discussion of optimal contracting structures when workers are risk neutral. Most recently Shi (2009) and Menzio and Shi (2009) extend this wage/tenure contracting approach to a directed search framework with aggregate productivity shocks.

[2] Also see Bunzel et al (2000) but who assume all learning is lost when the worker is laid-off. Barlevy (2008) considers a partial framework but estimates the underlying parameters using an

doing goes a long way to improving the empirical properties of the original Burdett and Mortensen (1998) framework. For example the density of wages paid becomes single peaked and the right tail is suitably fat (it has the Pareto distribution). Further as experience is valuable, firms can pay relatively low wage rates and the worker will still prefer employment to unemployment. Of course this reflects why many summer intern programs pay nothing yet still attract interns by offering "valuable work experience". This investment effect is important, however, as it helps resolve the issue raised in Hornstein et al (2008): that previous equilibrium wage dispersion models cannot adequately explain the difference between the lowest wage observed and the mean wage.[3]

Here we assume firms compete in piece rate tenure contracts and find observed individual wages can be decomposed as:

$$\log w_{ij}(x,\tau) = \log y_i + \log \theta_j(0) + \rho x + \log \frac{\theta_j(\tau)}{\theta_j(0)}, \tag{1}$$

where ρ is the rate of human capital accumulation while employed. The observed individual wage thus depends on the worker i fixed effect ($\log y_i$), the firm j fixed effect ($\log \theta_j(0)$ which describes firm j's piece rate paid to new hires), experience effect x and the tenure effect at firm j. Of course if tenure effects were constant within and across firms, so that $\theta_j(\tau) = e^{g\tau}\theta_j(0)$, we would obtain the standard wage regression equation of the form:

$$\log w_{ij}(x,\tau) = \log y_i + \log \theta_j(0) + \rho x + g\tau.$$

Our critical contribution is that $\theta_j(.)$ in (1) is chosen optimally by each firm j. Not surprisingly we show tenure effects are not uniform. Indeed, almost by definition, tenure effects are firm specific and a major econometric problem in (1) is separating the firm fixed effect from the firm specific tenure effect. We shall argue these fixed effects are negatively correlated: firms which pay low starting wages typically pay

interesting application of record statistics.

[3] B/C also 'explains' this puzzle. B/C generates a foot-in-the-door effect: as wages increase with seniority, an unemployed worker is willing to accept a low starting wage in order to step on the promotion ladder. Postel Vinay and Robin (2002) also has this effect.

wages which, at least at first, increase steeply with tenure. We shall further show with a numerical example that the above regression equation, which estimates the average return to tenure g across all employed workers, may find the average return to tenure is almost zero even though marginal tenure effects for new hires are large.

An important concept in the paper is the baseline piece rate scale. As in B/C, all optimal contracts correspond to a starting point on an underlying baseline piece rate scale, where different firms offer different starting points on that scale. Over time an employee simply rises up this scale through accumulated tenure, but quits whenever an outside offer is received which puts him/her on an even higher point. This not only implies quit rates fall with tenure but also firms that pay the highest wage rates enjoy the lowest quit rates, both of which are well established empirical facts. We show the least generous firms, those which offer the lowest value contracts in the market, offer contracts with the strongest positive tenure effect. By paying a low starting wage, such firms extract the search rents of each new hire. Paying a low wage, however, implies the new hire is at risk of being poached by a near competitor. By raising wages quickly with tenure, the optimal contract increases the new hire's expected value to remaining with this employer and so reduces the worker's quit rate. As the wage paid becomes more "competitive", i.e. closer to marginal product, the marginal tenure effect becomes small.

The paper is organised as follows. Section 2 describes the model and Section 3 characterises the set of optimal contracts offered by firms. This section formally establishes the baseline piece rate scale property of the model. Section 4 formally describes a Market Equilibrium and computes the steady state distributions of experience and payoffs across all workers in the economy. Section 5 characterises the Market Equilibrium and establishes existence. Section 6 discusses the insights yielded by the model and describes a numerical example. Section 7 concludes while section 8, the technical appendix, contains the more laborious proofs.

2 THE BASIC FRAMEWORK

Time is continuous with an infinite horizon and only steady-states are considered. There is a continuum of both firms and workers, each of measure one. All firms are equally productive and have a constant returns to scale technology. Each worker exits the market according to a Poisson process with parameter ϕ, while ϕ also describes the inflow of new labour market entrants. For the moment we simplify by assuming new entrants are ex-ante identical - each entrant i has the same initial productivity $y_i = y_0$. As shown below, the results generalize straightforwardly to the case that workers are ex-ante heterogenous with initial productivity y_i drawn from some population distribution A.

A worker's productivity y does not change while unemployed. Learning-by-doing implies productivity y increases at rate $\rho > 0$ while employed. Thus after x years of work experience, a type i worker's productivity is $y = y_i e^{\rho x}$. We restrict attention to $\rho < \phi$ so that lifetime payoffs are bounded. A worker with productivity y generates flow output y while employed. We normalize the price of the production good to one, so y also describes flow revenue.

Anti-discrimination legislation requires equal treatment of employees. In general a firm might pay wage $w = w(y, \tau)$ to equally productive workers y with the same tenure τ. We restrict attention to piece rate contracts: wages paid take the form $w = \theta(\tau)y$ where $\theta(\tau)$ is the piece rate paid at tenure τ.

Workers are either unemployed or employed and obtain new job offers at Poisson rate λ, independent of their employment status. Any job offer is fully described by the piece rate contract $\theta(\cdot)$ offered by the firm; i.e., an offer is a function $\theta(\tau) \geq 0$ defined for all tenures $\tau \geq 0$. There is no recall should a worker quit or reject a job offer. Thus given an outside contract offer, say $\widetilde{\theta}(\cdot)$, the worker compares the value of remaining at his/her current firm on contract $\theta(.)$ with current tenure τ, or switching to the new firm on contract $\widetilde{\theta}(\cdot)$ with zero tenure.

There are job destruction shocks in that each employed worker is displaced into unemployment according to a Poisson process with parameter $\delta > 0$. As typically done in this literature (e.g. Postel-Vinay and Robin (2001)), a worker with productivity y

enjoys income flow by while unemployed, where $0 < b < 1$.

In the absence of job destruction shocks, workers would find their earnings always increase over time. An optimal consumption strategy with liquidity constraints would then imply workers consume current earnings $\theta(\tau)y$. Job destruction shocks, however, generate a precautionary savings motive. For tractability we simplify by assuming workers can neither borrow nor save; i.e. consumption equals earnings at all points in time. We further assume a flow utility function with constant relative risk aversion; i.e. $u(w) = w^{1-\sigma}/(1-\sigma)$ with $\sigma \geq 0$. As typically done we assume the worker's continuation payoff is zero in the event of death.[4]

For simplicity assume firms and workers have a zero rate of time preference. Firms are risk neutral and, with no discounting, the objective of each firm is to maximize steady state flow profit. Each worker chooses a search and quit strategy to maximize expected total lifetime utility where the exit process implies each discounts the future at rate ϕ.

Workers

Let $V = V(y, \tau|\theta)$ denote the expected lifetime value of a worker with current productivity y and tenure τ on piece rate contract $\theta(\cdot)$. As piece rate contracts imply wages paid are always proportional to productivity y and given a CRRA utility function with parameter σ, the following identifies equilibria where V is multiplicatively separable with form

$$V = y^{1-\sigma} U(\tau|\theta).$$

$U(\tau|\theta)$ is referred to as the piece rate value of the contract with tenure τ, and $U(0|\widetilde{\theta})$ is the piece rate value of the outside offer $\widetilde{\theta}$.[5]

Let $V^U(y)$ denote the expected lifetime value of an unemployed worker with productivity y. Given a CRRA utility function, we characterize equilibria where V^U takes the form

$$V^U = y^{1-\sigma} U^U,$$

[4] As $\sigma > 1$ implies flow utility is negative while alive, we further assume no free exit (no suicide).

[5] Note, equilibrium will find $U(\tau|\theta) > 0 (< 0)$ when $\sigma < 1 (> 1)$ so that payoffs are always increasing in productivity y.

where U^U is a constant to be determined.

Although firms offer piece rate contracts $\widetilde{\theta}$, workers only care about the piece rate value $U_0 = U(0|\widetilde{\theta})$ of accepting each contract offer. As firms may offer different contracts, let $F(U_0)$ denote the proportion of firms in the market whose job offer, if accepted, yields piece rate value no greater than U_0. Search is random in that given a job offer, $F(U_0)$ is the probability the offer has piece rate value no greater than U_0. Let $\underline{U}$ and $\overline{U}$ denote the infimum and supremum of this distribution function.

Standard arguments imply the value of being an unemployed worker using an optimal search strategy satisfies the Bellman equation

$$\phi V^U = u(by) + \lambda \int_{\underline{U}}^{\overline{U}} \max[y^{1-\sigma}U_0 - V^U, 0]dF(U_0),$$

where $V^U = y^{1-\sigma}U^U$. Substituting in the functional form for $u(.)$, CRRA ensures all the y terms cancel out and this Bellman equation reduces to the following equation for U^U :

$$\phi U^U = \frac{b^{1-\sigma}}{1-\sigma} + \lambda \int_{U^U}^{\overline{U}} [U_0 - U^U]dF(U_0). \tag{2}$$

Thus each unemployed worker accepts any job offer with piece rate value $U_0 \geq U^U$; i.e. CRRA and competition in piece rate contracts ensures each worker's optimal search strategy is independent of productivity y.

Now consider the value of being employed with piece rate contract $\theta(.)$. As formally established in B/C, an optimal contract implies it is never optimal for a worker to quit into unemployment.[6] Thus given an optimal contract, standard arguments imply $V = V(y, \tau|\theta(.))$ evolves according to

$$\phi V(y,\tau|\theta(.)) = u(\theta(\tau)y) + \frac{\partial V}{\partial y}\rho y + \frac{\partial V}{\partial \tau} + \lambda \int_{\underline{U}}^{\overline{U}} \max[y^{1-\sigma}U_0 - V, 0]dF(U_0) + \delta[V^U - V], \tag{3}$$

[6]Suppose an optimal contract implies the worker quits into unemployment at tenure $T \geq 0$. Thus at tenure T, the firm's continuation profit is zero and the worker obtains V^U. The same contract but which instead offers piece rate $\theta(t) = b$ for all tenures $t \geq T$ is strictly profit increasing - on-the-job learning implies the worker obtains an improved payoff no lower than V^U at T and, by not quitting, the firm's continuation payoff is strictly positive (as $b < 1$). This latter contract then makes greater expected profit which contradicts optimality of the original contract.

where $V(.) = y^{1-\sigma}U(\tau|\theta)$. An employed worker enjoys flow payoff $u(\theta(\tau)y)$ while employed at this firm, enjoys increasing value through on-the-job learning, enjoys changing value as tenure at the firm increases over time, at rate λ receives an outside offer with piece-rate value U_0 and quits whenever such an offer yields value exceeding V, and at rate δ becomes unemployed through a job destruction shock. CRRA and an equilibrium with the above functional forms imply the y terms all cancel out and this Bellman equation reduces to the following differential equation for $U = U(\tau|\theta)$:

$$[\delta + \phi - \rho(1-\sigma)]U - \frac{dU}{d\tau} = \frac{[\theta(\tau)]^{1-\sigma}}{1-\sigma} + \delta U^U + \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0 \quad (4)$$

Again this preference structure ensures the worker's optimal quit strategy is independent of productivity y : the worker quits to any outside offer which has piece rate value greater than current value; i.e. when $U_0 \geq U = U(\tau|\theta)$. Thus each employee with tenure s at a firm with contract $\theta(\cdot)$ leaves at rate $\phi + \delta + \lambda[1 - F(U(s|\theta(.))]$. The probability a new hire survives to be an employee with tenure τ is then

$$\psi(\tau|\theta) = e^{-\int_0^\tau [\phi+\delta+\lambda(1-F(U(s|\theta)))]ds}. \quad (5)$$

Firms

Let $\overline{ue}$ denote the steady state unemployment rate and let $N(x)$ denote the fraction of unemployed workers who have experience no greater than x. Measure $1 - \overline{ue}$ of workers are thus employed and let $H(x, U)$ denote the proportion of employed workers who have experience no greater than x and piece rate value no greater than U. Each of these objects are determined endogenously.

Consider now a firm which posts contract $\theta(.)$ with starting piece rate value $U_0 = U(0|\theta)$. If $U_0 < U^U$ all potential employees prefer being unemployed to accepting this job offer and so such an offer yields zero profit. Suppose instead $U_0 \geq U^U$. As there is no discounting, the firm's steady state flow profit can be written as[7]

$$\Omega(\theta) = \lambda \left[\begin{array}{c} \overline{ue} \int_{x=0}^{\infty} \left[\int_0^\infty \psi(\tau|\theta))[1-\theta(\tau)][y_0 e^{\rho x}]e^{\rho\tau} d\tau \right] dN(x) \\ +(1-\overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} \left[\int_0^\infty \psi(\tau|\theta)[1-\theta(\tau)][y_0 e^{\rho x}]e^{\rho\tau} d\tau \right] dH(x, U') \end{array} \right].$$

[7] See Burdett and Coles (2003) for the relevant argument.

The firm's steady state flow profit is composed of two terms. The first term describes the profit obtained by attracting unemployed workers, where the bracketed inside integral is the expected total profit per new hire, and each new hire from the unemployment pool has starting productivity $y_0 e^{\rho x}$ with experience x considered as a random draw from $N(.)$. The second term describes the profit obtained by attracting employed workers who have piece rate values $U' < U_0$ and so accept the job offer. This condition can be re-expressed as

$$\begin{aligned}\Omega(\theta) &= \lambda y_0 \left[\int_0^\infty \psi(t|\theta)[1-\theta(t)]e^{\rho t}dt\right] \\ &\quad \times \left[\overline{ue}\int_{x=0}^\infty e^{\rho x}dN(x) + (1-\overline{ue})\int_{U'=\underline{U}}^{U_0}\int_{x=0}^\infty e^{\rho x}dH(x,U')\right].\end{aligned}$$

To determine the contract that maximizes Ω we follow B/C and use the following two step procedure. First we identify a firm's piece rate contract which maximizes

$$\left[\int_0^\infty \psi(t|\theta(.))[1-\theta(t)]e^{\rho t}dt\right],$$

conditional on the contract yielding piece rate value U_0. Such a contract is termed an optimal contract. Assuming an optimal contract exists, let $\theta^*(.|U_0)$ denote it, where $\theta^*(\tau|U_0)$ is the optimal piece rate paid at tenure τ. If we define maximized profit per hire

$$\Pi^*(0|U_0) = \int_0^\infty \psi(t|\theta^*)[1-\theta^*(t|U_0)]e^{\rho t}dt,$$

then an optimal contract yields steady-state flow profits

$$\Omega^*(U_0) = \lambda y_0 \Pi^*(0|U_0)\left[\overline{ue}\int_{x=0}^\infty e^{\rho x}dN(x) + (1-\overline{ue})\int_{U'=\underline{U}}^{U_0}\int_{x=0}^\infty e^{\rho x}dH(x,U')\right].$$

Given an optimal contract for each U_0, the firm's optimization problem then reduces to choosing a starting payoff U_0 to maximize $\Omega^*(U_0)$. Before formally defining an equilibrium, it is convenient first to characterise the optimal contract θ^* for each U_0.

3 Optimal Piece Rate Tenure Contracts.

A useful preliminary insight is that because the arrival rate of offers is independent of a worker's employment status, an unemployed worker will always accept a contract which offers $\theta(\tau) = b$ for all t. Further, as $b < 1$ by assumption, a firm can always obtain strictly positive profit by offering this contract. Thus, without loss of generality, we assume (a) all firms make strictly positive profit; $\Omega^* > 0$, (b) $\underline{U} \geq U^U$ (as an offer $U_0 < U^U$ attracts no workers and so makes zero profit). We further simplify the exposition by assuming F has a connected support.

For any starting value $U_0 \geq U^U$, an optimal contract $\theta^*(.|U_0)$ solves the program

$$\max_{\theta(.)} \int_0^\infty \psi(t|\theta(.))e^{\rho t}[1-\theta(t)]dt \tag{6}$$

subject to (a) $\theta(.) \geq 0$, (b) $U(0|\theta(.)) = U_0$ and (c) the optimal quit strategies of workers which determine the survival probability $\psi(.|\theta)$. In what follows we assume the constraint $\theta \geq 0$ is never a binding constraint. Theorem 3 below shows a Market Equilibrium of this type always exists whenever $\sigma \geq 1$. In contrast for $\sigma < 1$, equilibrium exist where some optimal contracts have an initial phase where $\theta = 0$ binds (e.g. Stevens (2004) when workers are risk neutral). For ease of exposition, however, we do not consider such situations.[8]

Given an optimal contract θ^* which yields starting value U_0, let $U \equiv U^*(\tau|U_0)$ denote the worker's corresponding piece rate value of employment at duration τ.Of course $U^*(0|U_0) = U_0$. Similarly given an optimal contract θ^* which yields starting value U_0, let $\Pi^*(\tau|U_0)$ denote the firm's continuation profit given an employee with current tenure τ; i.e.

$$\Pi^*(\tau|U_0) = \int_\tau^\infty \frac{\psi(t|\theta^*)}{\psi(\tau|\theta^*)}[1-\theta^*(t|U_0)]e^{\rho t}dt.$$

Theorem 1

For any $U_0 \geq \underline{U}$, an optimal contract $\theta^*(.|U_0)$ and corresponding worker and firm payoffs U^* and Π^* are solutions to the dynamical system $\{\theta, U, \Pi\}$ where

[8]The analysis generalises straightforwardly but is not particularly interesting and doesnot seem empirically relevant. All that happens is the baseline piece rate scales (described below) may have an initial part which is zero.

(a) θ is determined by

$$\frac{\theta^{1-\sigma}}{1-\sigma} + \theta^{-\sigma}\left[1-\theta+[\rho-\phi-\delta-\lambda(1-F(U))]\Pi\right] \tag{7}$$
$$= [\delta+\phi-\rho(1-\sigma)]U - \delta U^U - \lambda\int_U^{\overline{U}}[1-F(U_0)]dU_0.$$

(b) Π is given by

$$\Pi(t) = \int_t^\infty e^{-\int_t^s[\delta+\phi-\rho+\lambda(1-F(U(\tau))]d\tau}(1-\theta(s))ds, \text{ and} \tag{8}$$

(c) U evolves according to the differential equation

$$\frac{dU}{dt} = -\theta^{-\sigma}\frac{d\Pi}{dt} \tag{9}$$

with initial value $U(0) = U_0$.

Proof is in the Appendix.

The above characterization of an optimal contract is very general - it allows mass points in F and the density of F need not exist. In the equilibrium described in Theorem 3 below, however, the density of offers F' exists. In that case, a more intuitive structure arises if we totally differentiate (7) and (8) with respect to t and so obtain the following autonomous differential equation system for (θ, Π, U):

$$\dot{\theta} = \frac{\lambda\left[\theta^{1-\sigma}\right]}{\sigma}F'(U)\Pi - \rho\theta \tag{10}$$

$$\dot{\Pi} = [\delta+\phi-\rho+\lambda(1-F(U))]\Pi - (1-\theta) \tag{11}$$

$$\dot{U} = -\theta^{-\sigma}\dot{\Pi} \tag{12}$$

As described in detail in B/C, the optimal contract involves a trade-off between lower wage variation (smoother consumption) and reducing marginal quit incentives. (10) describes the optimal speed at which piece rates increase with tenure. This depends on the continuation profit Π by retaining the employee, the degree of risk aversion σ and on the marginal number of competing firms $F'(U)$ who might attract this employee through an outside offer. As a new hire with initial experience x_0 enjoys wage $w = \theta(\tau)y_0e^{\rho(x_0+\tau)}$ at future tenure $\tau \geq 0$, then (10) implies the wage paid along the optimal contract evolves over time according to

$$\frac{d}{dt}\log w = \frac{\lambda F'(U)\Pi}{\sigma\theta^{\sigma}}. \tag{13}$$

As it is never efficient to pay a wage above marginal product, an optimal contract always implies strictly positive continuation profit Π. Thus wages are always increasing within the employment spell, and increase strictly while the density of competing outside offers $F'(U) > 0$. With no learning by doing, B/C find the most generous contract offered in the market, $U_0 = \overline{U}$, implies a constant wage (perfect consumption smoothing) and the worker never quits to a competing firm. Here instead a constant wage (perfect consumption smoothing) would require a piece rate $\theta(\tau)$ which declines at rate ρ. Thus even though an optimal contract implies wages must always increase within an employment spell, learning-by-doing implies the direct tenure effect may now be negative; i.e. $\theta^*(.)$ might be a decreasing function, though never declining faster than rate ρ.

As in B/C, when F is differentiable (which is true in equilibrium) the optimal contract is a saddle path solution to the differential equation system (10)-(12). Let $(\theta^\infty, \Pi^\infty, U^\infty)$ denote the stationary point of the dynamical system (10)-(12); i.e. $(\theta^\infty, \Pi^\infty, U^\infty)$ solves:

$$[\theta^\infty]^\sigma = \frac{\lambda}{\rho\sigma}F'(U^\infty)\Pi^\infty \tag{14}$$

$$\Pi^\infty = \frac{1-\theta^\infty}{\delta+\phi-\rho+\lambda(1-F(U^\infty))}. \tag{15}$$

There are two types of optimal contracts, initially generous ones whose value converges to U^∞ from above, and initially ungenerous ones whose value converge to U^∞ from below. Figure 1 depicts the corresponding contracts $\theta^*(.)$.

Figure 1 here.

Figure 1 depicts two optimal contracts. The bottom curve, denoted $\theta^{sl}(.)$, depicts the optimal contract for the least generous firm, the one which offers starting payoff $U_0 = \underline{U} < U^\infty$. As in B/C the least generous contract corresponds to a path $\theta(.)$ which increases with tenure and converges to the limit value θ^∞. As piece rates increase with tenure, it follows U increases with tenure (and converges to U^{∞}), while continuation profit Π decreases with tenure (and converges to Π^{∞}). The top curve, denoted $\theta^{sh}(.)$ instead depicts the most generous contract which offers $U_0 = \overline{U} > U^{\infty}$. Although the wage paid always increases over the employment spell, tenure effects are negative and the piece rate paid converges to θ^{∞} from above. Along that path U also decreases with tenure (and converges to U^{∞} from above).

It is useful to define the above contracts as the *baseline piece rate scales.* Corresponding to each tenure point on those salary scales is a unique piece rate value, which we denote $U^s(t) \in [\underline{U}, \overline{U}]$ and continuation profit $\Pi^s(t)$. Optimality of the baseline piece rate scale now yields a major simplification. Suppose a firm wishes to offer starting payoff $U_0 \in [\underline{U}, \overline{U}]$. If in addition $U_0 < U^{\infty}$, then optimality of the baseline piece rate scale implies the optimal contract yielding U_0 corresponds to starting point t_0 on the lower baseline piece rate scale, where $U^s(t_0) = U_0$, and piece rates paid at tenure t correspond to point $(t_0 + t)$ on the lower baseline piece rate scale. Conversely if $U_0 > U^{\infty}$, optimality of the baseline piece rate scale implies the optimal contract yielding U_0, is the starting point t_0 on the higher baseline piece rate scale where $U^s(t_0) = U_0$, and corresponding piece rate payments at points $(t_0 + t)$ along the higher baseline piece rate scale for all tenures $t \geq 0$.

Given this characterization of the baseline piece rate scales, we can now define and characterize a Market Equilibrium.

4 MARKET EQUILIBRIUM

A moment's reflection establishes that new hires do not care about the particular tenure contract that is offered, only the value U_0 obtained by accepting it. To proceed we transform the equations obtained above into value space (U).

Recall that for any starting value $U_0 \in [\underline{U}, \overline{U}]$, we can identify a unique starting point on the baseline piece rate scales where the optimal contract yields starting payoff U_0. We can thus define $\theta = \widehat{\theta}(U_0)$ as the corresponding piece rate paid when the worker enjoys U_0 on the baseline piece rate scales, and $\Pi = \widehat{\Pi}(U_0)$ as the firm's correspnding continuation profit. Using the conditions of Theorem 1, Claim 1 identifies $\widehat{\Pi}(U)$ and $\widehat{\theta}(U)$.

Claim 1

For $U \in [\underline{U}, \overline{U}]$, $\widehat{\Pi}$ evolves according to the differential equation

$$\frac{d\widehat{\Pi}}{dU} = -\widehat{\theta}^{\sigma} \tag{16}$$

while $\widehat{\theta}$ satisfies

$$\begin{aligned} & \frac{[\widehat{\theta}]^{1-\sigma}}{1-\sigma} + \widehat{\theta}^{-\sigma}\left[1 - \widehat{\theta} + [\rho - \phi - \delta - \lambda(1 - F(U))]\widehat{\Pi}\right] \\ = \quad & [\delta + \phi - \rho(1-\sigma)]U - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)] dU_0. \end{aligned} \tag{17}$$

Proof Claim 1 follows directly from Theorem 1 and the definitions of $\widehat{\theta}, \widehat{\Pi}$.

By construction, each firm's optimized steady state flow profit by offering $U_0 \in [\underline{U}, \overline{U}]$ is

$$\Omega^*(U_0) = \lambda \widehat{\Pi}(U_0)[\overline{ue} \int_{x=0}^{\infty} y_0 e^{\rho x} dN(x) + (1 - \overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} y_0 e^{\rho x} dH(x, U')]. \tag{18}$$

We now formally define a Market Equilibrium.

A **Market Equilibrium** is a distribution of optimal contract offers, with corresponding value distribution $F(U)$, such that optimal job search by workers and steady state turnover implies the constant profit condition:

$$\begin{aligned} & \Omega^*(U_0) = \overline{\Omega} > 0 \text{ if } dF(U_0) > 0, \\ & \Omega^*(U_0) \leq \overline{\Omega}, \text{otherwise}. \end{aligned} \tag{19}$$

In an equilibrium, the constant profit condition requires that all optimal contracts offered by firms must make the same profit $\overline{\Omega} > 0$, while all other contracts must make no greater profit. We next use steady state turnover arguments to determine the equilibrium unemployment rate $\overline{ue}$ and distribution functions N, H. Identifying a Market Equilibrium then requires finding $F(.)$ so that the above constant profit condition is satisfied. We perform this task using a series of lemmas. Lemma 1 first specifies some (well known) technical results which much simplify the exposition.

Lemma 1. A Market Equilibrium implies:

(a) $\underline{U} = U^u$;

(b) $\overline{ue} = (\phi + \delta)/(\lambda + \phi + \delta)$

(c) $F(\overline{U}) = H(\overline{U}, \infty) = 1$; i.e. there are no mass points at $U = \overline{U}$.

Proof. Lemma 1(a) implies the lowest value offer in the market equals the value of unemployment. Its proof uses simple contradiction arguments : $\underline{U} < U^u$ is inconsistent with strictly positive profit (firms offering starting value $U_0 < U^u$ make zero profit), while $\underline{U} > U^u$ is inconsistent with the constant profit condition (offering $U_0 = \underline{U}$ is dominated by offering $U_0 = U^U$ as both offers only attract the unemployed and offering $U^u < \underline{U}$ generates greater profit per hire). Lemma 1(b) follows as all unemployed workers accept their first job offer and steady state turnover implies the stated condition. Lemma 1(c) is also established with contradiction arguments. If there is a mass point in F at $\overline{U}$, then offering contract with starting value $\overline{U}$ is dominated by instead offering starting value $U_0 = \overline{U}^+$ with the following deviating contract: $\theta = \theta^{sh}(\tau) + \varepsilon$ for $\tau \leq \delta$, and $\theta = \theta^{sh}(\tau)$ for all $\tau > \delta$, where $\varepsilon, \delta > 0$ but small. As a mass of firms offer starting contract with value $\overline{U}$, this more generous deviating contract reduces the new hire's initial quit rate by a discrete amount and so raises steady state flow profit by an amount which is of order δ. As the increase in wage paid over this interval implies a flow profit loss of order $\varepsilon\delta$ then, for ε small enough, this deviating contract strictly increases profit which contradicts the constant profit condition. The same contradiction argument implies H cannot contain a mass point at $\overline{U}$, otherwise the above deviating contract yields a discrete increase in the firm's hiring rate while the loss in profit per new hire is arbitrarily small. This completes the proof of Lemma 1.

The next step is to characterize steady state $N(x)$ and $H(x, U)$. The turnover arguments in Burdett et al (2009) imply the distribution of experience across unemployed workers is:

$$N(x) = 1 - \frac{\lambda\delta}{(\phi + \lambda)(\phi + \delta)} e^{-\frac{\phi(\phi+\delta+\lambda)x}{(\phi+\lambda)}}. \tag{20}$$

Let $N_0 = N(0)$ and note it is strictly positive: N_0 describes the proportion of unemployed workers who have never had a job and so have zero experience. For $x > 0$, the distribution of experience across unemployed worker is described by the exponential distribution. Burdett et al (2009) also determines the distribution of experience across all employed workers, which here is written as

$$H(x,\overline{U}) = 1 - e^{-\frac{\phi(\phi+\delta+\lambda)x}{(\phi+\lambda)}}. \tag{21}$$

This distribution is also exponential but, in contrast to N, note that $H(0,\overline{U}) = 0$: in a steady state the measure of employed workers with zero experience must be zero. Lemma 2 now characterizes $H(.)$ for all $x > 0$, $U \in [\underline{U},\overline{U}]$.

Lemma 2. For $x > 0$ and $U \in [\underline{U},\overline{U}]$, $H = H(x,U)$ satisfies the partial differential equation:

$$[\phi + \delta + \lambda(1 - F(U)]H + \frac{\partial H}{\partial x} + \dot{U}\frac{\partial H}{\partial U} = (\phi + \delta)F(U)N(x),$$

where along the baseline piece rate scale $\dot{U} = \dot{U}(U)$ is given by:

$$\dot{U} = \widehat{\theta}^{-\sigma}[(1 - \widehat{\theta}) - [\delta + \phi - \rho + \lambda(1 - F(U))]\widehat{\Pi}] \tag{22}$$

and H satisfies the boundary conditions

$$\begin{aligned} H(0,U) &= 0 \text{ for all } U \in [\underline{U},\overline{U}]; \\ H(x,\underline{U}) &= 0 \text{ for all } x \geq 0. \end{aligned}$$

The Proof of Lemma 2 is in the Appendix.

Although H is described by a relatively straightforward first order partial differential equation, a closed form solution does not exist. Nevertheless it still possible to characterize fully a Market Equilibrium. Given $\overline{ue}$ obtained in Lemma 1 and $N(x)$ given by (20), then (18) describing Ω^* implies the constant profit condition requires finding F such that

$$\widehat{\Pi}(U_0)\left[\begin{array}{c} \frac{\phi(\phi+\delta-\rho)}{\phi(\phi+\delta+\lambda)-\rho(\phi+\lambda)} \\ +\frac{\lambda}{\lambda+\phi+\delta}\int_{U'=\underline{U}}^{U_0}\int_{x'=0}^{\infty} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU' \end{array}\right] = \frac{\overline{\Omega}}{\lambda y_0} \text{ for all } U_0 \in [\underline{U},\overline{U}] \tag{23}$$

with H given by lemma 2.

Fortunately the problem dichotomises. In what follows, Theorem 2 and Lemma 3 below solve (23) for equilibrium $\widehat{\theta}(.), \widehat{\Pi}(.)$. Equation (17) in Claim 1, which describes the optimal contract, will then determine equilibrium F.

Theorem 2. In any Market Equilibrium, the constant profit condition is satisfied if and only if

$$\widehat{\Pi} = \frac{1}{\phi + \delta - \rho} \sqrt{(1 - \overline{\theta})(1 - \widehat{\theta})} \text{ for all } U_0 \in [\underline{U}, \overline{U}],$$

where $\overline{\theta} = \widehat{\theta}(\overline{U})$ is the highest piece rate offered in the market.

Proof of Theorem 2 is in the Appendix.

Putting $\rho = 0$ finds this solution is the same as that found in B/C. Our next step is to use Theorem 2 to characterize a Market Equilibrium and so establish existence.

5 Existence and Characterization of a Market Equilibrium.

Although an analytic solution does not exist, solving for a Market Equilibrium is relatively straightforward. The approach is first to hypothesize an equilibrium value for $\overline{\theta}$, the highest piece rate paid in the market, and then use backward induction to map out the equilibrium outcomes. The free choice of $\overline{\theta}$ is tied down by Lemma 1(a), which requires $U^U = \underline{U}$.

Given an equilibrium value for $\overline{\theta}$ (and thus $\widehat{\theta}(\overline{U}) = \overline{\theta}$), Lemma 3 now describes the corresponding equilibrium support of offers $[\underline{U}, \overline{U}]$ and fully characterises equilibrium $\widehat{\theta}(.)$ over that support.

Lemma 3. For any equilibrium value $\overline{\theta} \in (0, 1)$, a Market Equilibrium implies $\widehat{\theta}(U)$ is given by the implicit function

$$\frac{\sqrt{(1 - \overline{\theta})}}{2(\phi + \delta - \rho)} \int_{\widehat{\theta}}^{\overline{\theta}} \frac{1}{(1 - \theta')^{1/2} [\theta']^{\sigma}} d\theta' = [\overline{U} - U] \tag{24}$$

for all $U \in [\underline{U}, \overline{U}]$ where $\overline{U}, \underline{U}$ are uniquely determined by

$$\frac{[\overline{\theta}]^{1-\sigma}}{1-\sigma} = [\phi - \rho(1-\sigma)]\overline{U} + \delta[\overline{U} - \underline{U}] \tag{25}$$

$$\frac{\sqrt{(1-\overline{\theta})}}{2(\phi+\delta-\rho)} \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{(1-\theta')^{1/2}\,[\theta']^{\sigma}} d\theta' = [\overline{U} - \underline{U}], \tag{26}$$

and $\underline{\theta} \equiv \widehat{\theta}(\underline{U})$, the lowest piece rate paid in the market, is given by

$$(1-\underline{\theta}) = \left[\frac{\phi+\delta-\rho+\lambda}{\phi+\delta-\rho}\right]^2 (1-\overline{\theta}). \tag{27}$$

Proof is in the Appendix.

As we only consider equilibria where $\theta \geq 0$ does not bind, (27) implies we need only consider $\overline{\theta} \geq 1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2$. For any $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$, it is trivial to show a solution to the above equations always exists, is unique, is continuous in $\overline{\theta}$ and implies $0 < \underline{\theta} < \overline{\theta}$ and $\underline{U} < \overline{U}$.

Given (24) describes the solution for equilibrium $\widehat{\theta}$, Theorem 2 now gives equilibrium $\widehat{\Pi}(.)$. All that remains is to determine equilibrium F. It is convenient to define the surplus function

$$S(U) = \int_U^{\overline{U}} [1 - F(U')]dU'.$$

Noting that $U^U = \underline{U}$ in a Market Equilibrium, (17) implies equilibrium S is determined by the linear differential equation

$$\frac{[\widehat{\theta}]^{1-\sigma}}{1-\sigma} + \widehat{\theta}^{-\sigma}\left[1 - \widehat{\theta} + [\rho - \phi - \delta + \lambda\frac{dS}{dU}]\widehat{\Pi}\right] = [\delta + \phi - \rho(1-\sigma)]U - \delta\underline{U} - \lambda S \tag{28}$$

for all $U \in [\underline{U}, \overline{U}]$ with initial value $S(\overline{U}) = 0$. As $\widehat{\Pi} > 0$,this linear differential equation contains no singularities. Thus S is uniquely determined by backward iteration from $\overline{U}$, using the solutions above for $\widehat{\theta}, \widehat{\Pi}, \underline{U}$ with initial value $S(\overline{U}) = 0$. Of course the surplus function uniquely determines F. The final step, then, is to note a Market Equilibrium also requires $U^U = \underline{U}$ where U^U is given by (2).

Theorem 3. [Existence and Characterization]. The necessary and sufficient conditions for a Market Equilibrium with $\underline{\theta} > 0$, is a $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$ with

(A) the support of offers $[\underline{U}, \overline{U}]$ (and corresponding $\underline{\theta}$) given by (25)-(27), where over that support,

(B) $\widehat{\theta}(.)$ is given by (24), $\widehat{\Pi}(.)$ is given by (23), $S(.)$ is the solution to the initial value problem (28) with $S(\overline{U}) = 0$;

and U^U, given by (2), satisfies $U^U = \underline{U}$. Furthermore such a Market Equilibrium exists for any $\sigma \geq 1$.

Proof. By construction these are necessary conditions for a Market Equilibrium. Given any such solution, then by construction all optimal contracts which offer $U_0 \in [\underline{U}, \overline{U}]$ yield the same steady state flow profit. Consider now any deviating contract. Clearly, a suboptimal contract which offers $U_0 \in [\underline{U}, \overline{U}]$ yields lower profit. Further any contract which offers value $U_0 < \underline{U}$ yields zero profit as $U^U = \underline{U}$ and all workers reject such an offer. Finally any contract which offers $U_0 > \overline{U}$ attracts no more workers than an optimal contract which offers $\overline{U}$ while the latter contract earns strictly greater profit per hire. As no deviating contracts exist which yield greater profit, a solution to the above conditions identifies a Market Equilibrium.

We now establish existence of a solution when $\sigma \geq 1$. Given an arbitrary value for $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$, let $\widetilde{F}(.|\overline{\theta})$ denote the solution for F implied by solving parts A,B of Theorem 3 above. Further define $\widetilde{U}^U(\overline{\theta})$ as the solution for U^U where

$$\phi U^U = \frac{b^{1-\sigma}}{1-\sigma} + \lambda \int_{U^U}^{\overline{U}} [1 - \widetilde{F}(U_0|\overline{\theta})] dU_0. \tag{29}$$

Thus $\widetilde{U}^U(\overline{\theta})$ is the optimal reservation piece rate U^U of unemployed workers given offer distribution $\widetilde{F}(.|\overline{\theta})$. Let $\underline{U}(\overline{\theta})$ denote the lowest value contract offered in the market, as determined in Theorem 3A. A Market Equilibrium requires finding a $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$ such that $\widetilde{U}^U(\overline{\theta}) = \underline{U}(\overline{\theta})$. Simple continuity arguments now establish existence.

First note that as $\overline{\theta} \to 1$, (27) implies $\underline{\theta} \to 1$. As all piece rates θ paid must then lie in an arbitrarily small neighborhood around one, frictions ($\lambda < \infty$) and $b < 1$ imply $\widetilde{U}^U < \underline{U}$.

Instead consider the limit $\overline{\theta} \to 1 - [(\phi + \delta - \rho)/(\phi + \delta - \rho + \lambda)]^2$ which, by (27), implies $\underline{\theta} \to 0$. $\sigma \geq 1$ implies the flow payoff by accepting the lowest $\underline{\theta}$ offer, $\underline{\theta}^{1-\sigma}/(1-\sigma)$, becomes unboundedly negative in this limit. (25) and (27) imply $\underline{U} \to -\infty$ as $\underline{\theta} \to 0$. But (29) implies $\phi\widetilde{U}^U > \frac{b^{1-\sigma}}{1-\sigma}$ and thus $\widetilde{U}^U > \underline{U}$ in this limit.

As the solutions for $\widehat{\theta}(.), \widehat{\Pi}(.), S(.)$ and $\underline{U}$ are all continuous in $\overline{\theta}$, continuity now implies a $\overline{\theta} \in (1 - \left[\frac{\phi+\delta-\rho}{\phi+\delta-\rho+\lambda}\right]^2, 1)$ exists where $\widetilde{U}^U = \underline{U}$ and so identifies a Market Equilibrium. This completes the proof of Theorem 3.

Note the existence proof does not consider the case $\sigma < 1$. If $\sigma < 1$, a Market Equilibrium might instead find the constraint $\theta \geq 0$ binds on the optimal contract. Theorem 3 establishes this does not occur if $\sigma \geq 1$. Alternatively, as argued in B/C, one could consider $0 < \sigma < 1$ but then restrict attention to b sufficiently large that $\theta \geq 0$ does not bind.

6 Discussion and a Numerical Example.

6.1 Ex-ante worker heterogeneity.

It is straightforward to show the results obtained above extend directly to ex-ante heterogeneous workers, where each new entrant has productivity y_i drawn from some population distribution A with support $[\underline{y}, \overline{y}]$. In a Market Equilibrium with CRRA and competition in piece rate contracts, all workers use the same search strategies. Thus the unemployment rate and distributions of experience is the same for all types and as described above. The only difference to the analysis is that the constant profit condition becomes

$$\Omega^*(U_0) = \lambda\widehat{\Pi}(U_0) \int_{\underline{y}}^{\overline{y}} \left[\begin{array}{c} \overline{ue} \int_{x=0}^{\infty} y_i e^{\rho x} dN(x) \\ +(1-\overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} y_i e^{\rho x} dH(x, U') \end{array} \right] dA(y_i) = \overline{\Omega} \text{ for all } U_0 \in [\underline{U}, \overline{U}] \tag{30}$$

Let $\mu = \int_{\underline{y}}^{\overline{y}} y_i dA(y_i)$ denote average entrant productivity. Solving the constant profit condition then reduces to

$$\widehat{\Pi}(U_0) \left[\begin{array}{c} \overline{ue} \int_{x=0}^{\infty} e^{\rho x} dN(x) \\ +(1-\overline{ue}) \int_{U'=\underline{U}}^{U_0} \int_{x=0}^{\infty} e^{\rho x} dH(x, U') \end{array} \right] = \frac{\overline{\Omega}}{\lambda\mu} \text{ for all } U_0 \in [\underline{U}, \overline{U}].$$

Thus replacing y_0 with μ implies the previous analysis applies.

6.2 Equilibrium Wage Outcomes.

If y_i denotes the initial productivity of worker i, then the equilibrium wage earned by this worker after x years experience, with tenure τ, at firm j offering piece rate contract $\theta_j(.)$ is:

$$\log w_{ij}(x,\tau) = \log y_i + \log \theta_j(0) + \rho x + \log \frac{\theta_j(\tau)}{\theta_j(0)}.$$

The observed wage thus depends on the worker fixed effect ($\log y_i$), the firm fixed effect ($\log \theta_j(0)$ which describes firm j's starting piece rate paid to new hires), experience effect x and the tenure effect at firm j. The econometrician's problem then is to separately identify the (firm specific) tenure effect $\log[\theta_j(\tau)/\theta_j(0)]$ from the firm fixed effect $\log \theta_j(0)$. Below we shall argue these effects are negatively correlated - firms which pay low starting salaries will, in equilibrium, have large tenure effects.

As firm j chooses the contract $\theta_j(.)$ optimally, (10) implies optimal wage growth within any employment spell is given by

$$\frac{d}{dt} \log w(x,\tau) = \frac{\lambda F'(U)\Pi}{\sigma \theta^{\sigma}}. \tag{31}$$

Given the previous wage equation, it is perhaps surprising that ρ does not appear in this latter expression. This reflects that an optimal contract smooths the worker's entire future wage (consumption) profile against the increased risk that the worker is poached by a near competitor. The numerical example below finds that wage growth within the employment spell is greatest at firms which offer the lowest value contracts (those firms whose starting piece rates are closest to $\underline{\theta}$) and only for relatively short tenures. This occurs as, by (31),

(i) along the baseline piece rate scale, firms which pay low piece rates enjoy relatively high continuation profit Π per worker. Higher per worker profit implies stronger tenure effects: the firm more quickly raises wages with tenure to reduce the likelihood of a quit to a near competitor. As wages paid become more "competitive", in the sense of closer to marginal product, the more slowly wages increase over time.

(ii) A Market Equilibrium finds most starting offers are concentrated around $\underline{\theta}$ (see the numerical examples below). As in B/C, there is a mass of firms which offer starting pay rate $\theta = \underline{\theta}$. These firms only attract unemployed workers and extract maximal search rents. Offering a starting payrate θ_0 slightly above $\underline{\theta}$ is also advantageous as it is likely to poach a new hire at firms in the mass point, and such hires still generate a large expected profit. Of course relatively intense competition for such employees leads firms in the mass point to raise wages relatively quickly with tenure.

Before turning to the numerical examples, however, we further point out that a Market Equilibrium rules out negative tenure effects. Putting $U = \overline{U}$ and noting Theorem 2 finds $\widehat{\Pi}(\overline{U}) = (1-\overline{\theta})/(\phi+\delta-\rho)$, then inspection establishes that $(\overline{\theta}, \overline{U}, \widehat{\Pi}(\overline{U}))$ is the stationary point of the differential equations describing the baseline piece rate scale; i.e. there is no upper baseline piece rate scale (or instead $\theta^{sh}(.) = \overline{\theta}$).

6.3 A Numerical Example.

We use numerical examples to gain some qualitative insights on how tenure effects and learning-by-doing effects interact on equilibrium wage outcomes. Using a year as the reference unit of time we use the following parameter values:

$\phi = 0.025$ (workers have a 40 year expected working lifetime)

$\delta = 0.055$ (job destruction rate of 5.5% p.a.)

$\rho = 0.01$ (productivity growth 1% through learning-by-doing)

$y_0 = 1$ (identical labour market entrants)

$b = 0.7$ (home productivity is 70% of workplace productivity),

$\sigma = 2$ (degree of relative risk aversion).

The above numbers are relatively standard (see Burdett et al (2009) for a full discussion). We consider two separate values for λ. For the U.S., Jolivet et. al. (2006) estimate job separation rate $\delta = 0.055$ and $\lambda_1 = 0.15$;i.e. employed workers receive outside offers every 6.7 years. As that paper identifies λ_1 by assuming no wage tenure contracts, this estimate of λ_1 is downward biased. Indeed in Stevens (2004) with risk neutral workers, employed workers receive outside offers at rate λ but never quit which would lead the the econometrician to infer $\widehat{\lambda} = 0$. In our main example we consider $\lambda = 2$ so that employed workers on average receive two outside offers each year. As we assume employed and unemployed receive job offers at the same rate, this also implies an average duration of unemployment equal to 6 months. For reasons that will become clear, we refer to this latter case as the low frictions case. $\lambda = 0.15$ is termed the high frictions case.

The low frictions case ($\lambda = 2$). Solving the conditions of Theorem 3 for the above parameter values finds $F(\underline{U}) = 0.21$;i.e., a Market Equilibrium implies 21% of all firms offer reservation starting piece rate $\underline{\theta}= 0.30$. Note this reservation starting piece rate is far below $b = 0.7$. This low value occurs for two reasons. First there is the foot-in-the-door effect as described in B/C: although the worker might start employment at a very low pay rate, he/she anticipates rapid promotion to higher payrates. This promotion mechanism lowers the unemployed worker's reservation piece rate (and raises firm profit). Second experience is valuable and the low reservation piece rate reflects the investment value of employment.

Although the theory determines the distribution of starting payoffs, $F = F(U)$, it is empirically more interesting to describe instead the distribution of starting piece rates denoted $F_\theta(\theta)$, where $F_\theta(\widehat{\theta}) = F(U)$. Figure 2 describes the density of F_θ for $\theta > \underline{\theta}$.

Figure 2 around here.

Along with the mass point at $\underline{\theta}$, most firms offer relatively low starting piece rates. Of course relatively intense competition for employees on low piece rates implies tenure effects are steep. Figure 3 describes the corresponding baseline piece rate scale.

In the low frictions case, tenure effects are very steep for workers on low pay rates. This reflects that employed workers regularly receive outside offers. After 4 years tenure, a worker who started on piece rate $\underline{\theta}$ will be enjoying a wage close to marginal product. Of course in those intervening 4 years, the worker expects to receive 8 outside offers and is likely to quit to a better job offer.

A useful consistency check is to define $1-G(U)$ as the number of employed workers with current piece rate value no lower than U. As $G(\underline{U}) = \overline{ue}$ (and $\overline{ue}$ is given by lemma 1), standard turnover arguments imply $G(U)$ evolves according

$$\frac{dG}{dU} = \frac{\phi + \delta - [\phi + \delta + \lambda[1 - F]]\, G}{\dot{U}},$$

where $\dot{U} = \dot{U}(U)$ is given by (22). The consistency check is that $G \to 1$ as $U \to \overline{U}$. Doing this not only finds all is well, but one can use the computed $G(.)$ to infer the density of piece rates paid across all employed workers. Figure 4 describes that density.

Figure 4 here.

With low frictions, most currently employed workers enjoy a wage close to marginal product. This occurs as employed workers regularly receive outside offers and competition for employed workers imply tenure effects are steep. At these high pay-rates, however, tenure effects are almost zero. Suppose then the econometrican were to estimate a standard wage equation of the form

$$\log w_{ij}(x, \tau) = \log y_i + \log \theta_j(0) + \rho x + g\tau.$$

This equation includes a worker fixed effect, a firm fixed effect but, crucially, assumes the return to tenure (g) varies neither between firms and between workers within a firm. As average tenure effects across the whole working population are small, the estimated "average" tenure effect, g, would be close to zero. Nevertheless marginal tenure effects are large for certain workers: the recently unemployed who currently earn low piece rates.

The high frictions case ($\lambda = 0.15$).

A Market Equilibrium implies 27% of all firms offer the reservation starting piece rate, which is now slightly higher at $\underline{\theta}$= 0.34. The higher reservation piece rate (relative to the low frictions case) reflects the reduced foot-in-the-door effect: promotion rates are much reduced when frictions are high. Figure 3 plots the baseline piece rate scale for the high frictions case. As employed workers are much less likely to receive an outside offer, the marginal tenure effect at low pay rates is significantly weaker than before. The reduced competition for employed workers also implies $\overline{\theta}$ is now a significant distance from marginal product. Further as promotion and quit rates are low, it is no surprise that steady state finds a large number of employed workers now earn payrates significantly below $\overline{\theta}$. Nevertheless the same qualitative insights hold: marginal tenure effects are largest for workers earning currently low piece rates while average tenure effects across the entire working population may be relatively small.

7 Conclusion

Aside from Becker (1974) in a perfectly competitive framework and Bagger et al (2008) in a frictional framework, we are not aware of an alternative equilibrium framework where wages depend endogenously on both experience and tenure effects. A central contribution of this paper is to provide a framework within which such wage effects might be formally identified.

As tenure effects are firm specific, a major econometric problem is to separately identify the firm specific return to tenure from the firm fixed effect. The theory strongly suggests these will be negatively correlated: firms which pay the lowest starting salaries will have the steepest tenure effects. The model also demonstrates clearly why a mispecified wage equation, one which estimates the "average" return to tenure, finds the average return to tenure is small even though marginal tenure effects can be large for new hires. The identification arguments presented in Dustmann and Meghir (2005) would seem the most promising way forward: first to identify ρ using re-employment wage data for those workers laid-off through plant closure, then use firm level wage data to infer the underlying baseline piece rate scale.

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8 Appendix.

Proof of Theorem 1. Let $\Phi(t) = e^{\rho t}\psi(t|\theta(.))$. The firm's optimal contract solves

$$\max_{\theta(.)\geq 0} \int_0^\infty \Phi(t)[1-\theta(t)]dt \tag{32}$$

where on-the-job learning and optimal job search by employees implies

$$\dot{\Phi} = [\rho - \phi - \delta - \lambda(1 - F(U))]\Phi \tag{33}$$

$$\frac{dU}{dt} = [\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0. \tag{34}$$

and starting values

$$\psi(0) = 1; U(0) = U_0. \tag{35}$$

Define the Hamiltonian

$$\begin{aligned} H &= \Phi(t)[1-\theta(t)] + \mu_\Phi[\rho - \phi - \delta - \lambda(1-F(U))]\Phi \\ &\quad + \mu_U \left[[\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1 - F(U_0)]dU_0 \right]. \end{aligned}$$

Whenever the corner constraint $\theta \geq 0$ is not binding, the Maximum Principle implies optimal $\theta(.)$ satisfies the first order condition

$$\frac{\partial H}{\partial \theta} = -\Phi(t) - \mu_U \theta(t)^{-\sigma} = 0$$

and μ_Φ, μ_U evolve according to the differential equations

$$\frac{d\mu_\Phi}{dt} = -[1-\theta(t)] - \mu_\Phi[\rho - \phi - \delta - \lambda(1-F(U))]$$

$$\frac{d\mu_U}{dt} = -\mu_\Phi \lambda F'(U)\Phi - \mu_U \left[\delta + \phi + \lambda[1-F(U)] - \rho(1-\sigma)\right]$$

with $\Phi(t), U$ given by the differential equations stated above. No discounting implies the additional constraint $H = 0$ (**e.g.** p298, Leonard and Long (1992)) and so we also have

$$
\begin{aligned}
0 &= \Phi(t)[1-\theta(t)] + \mu_\Phi[\rho - \phi - \delta - \lambda(1-F(U))]\Phi \\
&\quad + \mu_U \left[[\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1-F(U_0)]dU_0 \right].
\end{aligned}
$$

While $\theta > 0$ along the optimal path, optimality implies $\mu_U = -\Phi/(\theta^{-\sigma})$. Substituting out μ_U in the previous expression implies

$$
\begin{aligned}
0 &= [1-\theta(t)] + \mu_\Phi[\rho - \phi - \delta - \lambda(1-F(U))] \qquad \text{(H=0)} \\
&\quad - \frac{1}{\theta^{-\sigma}} \left[[\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1-F(U_0)]dU_0 \right].
\end{aligned}
$$

Now integrating the linear differential equation for μ_Φ yields:

$$
\mu_\Phi(t) = \int_t^\infty e^{-\int_t^s [\delta+\phi-\rho+\lambda(1-F(U(\tau))]d\tau}(1-\theta(s))ds + A_0 e^{\int_0^t [\phi+\delta+\lambda(1-F(V(x)))-\rho]dx}
$$

where A_0 is the constant of integration. Denote the first term as the firm's continuation profit $\Pi(t)$ and note $\Pi(t)$ evolves according to (?) stated in the Theorem.

Using

$$
\mu_\Phi(t) = \Pi(t) + A_0 e^{\int_0^t [\phi+\delta+\lambda(1-F(V(x)))-\rho]dx},
$$

to substitute out $\mu_\Phi(t)$ in the $[H=0]$ condition yields

$$
\begin{aligned}
0 &= [1-\theta] + \left[\Pi(t) + A_0 e^{\int_0^t [\phi+\delta+\lambda(1-F(V(x)))-\rho]dx}\right][\rho - \phi - \delta - \lambda(1-F(U))] \qquad (36) \\
&\quad - \frac{1}{\theta^{-\sigma}} \left[[\delta + \phi - \rho(1-\sigma)]U - \frac{[\theta(t)]^{1-\sigma}}{1-\sigma} - \delta U^U - \lambda \int_U^{\overline{U}} [1-F(U_0)]dU_0 \right].
\end{aligned}
$$

A contradiction argument now establishes $A_0 = 0$. As Π and U are uniformly bounded (to be proved), $A_0 \neq 0$ and $\phi > \rho$ implies the second term in (36) grows exponentially as $t \to \infty$. Thus, (36) requires $\theta \to 0$ in this limit. But such a contract with $b > 0$ implies all workers quit at a finite tenure date, which contradicts optimality of the contract. Thus $A_0 = 0$.

Putting $A_0 = 0$ in (36) with some rearranging yields

$$
\begin{aligned}
&\frac{[\theta(t)]^{1-\sigma}}{1-\sigma} + \theta^{-\sigma}[1 - \theta + [\rho - \phi - \delta - \lambda(1-F(U))]\Pi(t)] \\
&= [\delta + \phi - \rho(1-\sigma)]U - \delta U^U - \lambda \int_U^{\overline{U}} [1-F(U_0)]dU_0.
\end{aligned}
$$

Using this to substitute out $\frac{[\theta(t)]^{1-\sigma}}{1-\sigma}$ in (**??**) and now gives:

$$\frac{dU}{dt} = \theta^{-\sigma}\left[1-\theta+[\rho-\phi-\delta-\lambda(1-F(U))]\Pi(t)\right] \quad (37)$$

$$= -\theta^{-\sigma}\frac{d\Pi}{dt}. \quad (38)$$

then yields (9). Connected F implies $(\theta^{\infty}, V^{\infty}, \Pi^{\infty})$ is the unique stationary point of this differential equation system. This completes the proof of Theorem 1.

Proof of Lemma 2: Consider the pool of employed workers who have experience no greater than $x > 0$ and piece rate value no greater than U. Then for $U < U^{\infty}$, the total outflow of workers from this pool, over any instant of time $dt > 0$, is

$$(1-\overline{ue})H(x,U)[\phi+\delta+\lambda(1-F(U)]dt + (1-\overline{ue})\int_{U'=\underline{U}}^{U}\int_{x'=x-dt}^{x}\frac{\partial^2 H}{\partial U \partial x}dU'dx'$$
$$+(1-\overline{ue})\int_{U'=U^s(t-dt)}^{U^s(t)}\int_{x'=0}^{x}\frac{\partial^2 H}{\partial U \partial x}dU'dx' + O(dt^2),$$

where the first term is the number who die, lose their job or quit through receiving a job offer with value greater than U, the second is the number who exit through achieving greater experience, while the third is the number who exit through internal promotion, where $U^s(t) = U$. The inflow is $\lambda\overline{ue}F(U)N(x)dt$ which is the number of unemployed workers with experience no greater than x who receive a job offer with value no greater than U. Setting inflow equal to outflow, using the solution for $\overline{ue}$ in lemma 1 and letting $dt \to 0$ implies H satisfies:

$$H(x,U)[\phi+\delta+\lambda(1-F(U)] + \int_{U'=\underline{U}}^{U}\frac{\partial^2 H}{\partial U \partial x}dU'$$
$$+\dot{U}\int_{x'=0}^{x}\frac{\partial^2 H}{\partial U \partial x}dx' = (\phi+\delta)F(U)N(x).$$

Integrating thus yields

$$H(x,U)[\phi+\delta+\lambda(1-F(U)] + [\frac{\partial H[x,U]}{\partial x} - \frac{\partial H[x,\underline{U}]}{\partial x}]$$
$$+\dot{U}\left[\frac{\partial H[x,U]}{\partial U} - \frac{\partial H[0,U]}{\partial U}\right] = (\phi+\delta)F(U)N(x).$$

But $H(0,U) = H(x,\underline{U}) = 0$ implies $\frac{\partial H[x,\underline{U}]}{\partial x} = \frac{\partial H[0,U]}{\partial U} = 0$ which with the above equation yields the stated solution. This argument but for $U > U^{\infty}$ establishes the same differential equation. This completes the proof of Lemma 2.

Proof of Theorem 2. we begin with two preliminary facts. As $\partial H(0,U)/\partial U = 0$ (by Lemma 2), then putting $x = 0$ in the pde for H implies

$$\frac{\partial H(0,U)}{\partial x} = (\phi+\delta)N_0 F(U).$$

Also using the solution for $N(.)$ straightforward algebra establishes:

$$\int_0^{\infty} e^{\rho x'} dN(x') = \frac{\phi\,(\phi+\delta+\lambda)}{\phi+\delta}\left[\frac{\phi+\delta-\rho}{\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)}\right].$$

we now turn to solving the constant profit condition. The key is to solve for $\int_{x'=0}^{\infty}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU'$ with $H(.)$ as described by lemma 2. First note that as the measure of employed workers with no experience is zero, then

$$\begin{aligned}\int_{x'=0}^{\infty}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU' &= \int_{x'>0}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU' \\ &= \int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx'.\end{aligned}$$

Thus (23) requires solving

$$\widehat{\Pi}(U_0)\left[\frac{\phi(\phi+\delta-\rho)}{\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)} + \frac{\lambda}{\lambda+\phi+\delta}\int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx'\right] = \frac{\overline{\Omega}}{\lambda y_0} \text{ for all } U_0 \in [\underline{U},\overline{U}]. \tag{39}$$

First put $U_0 = \overline{U}$ in (39) and let $\overline{\Pi} = \widehat{\Pi}(\overline{U})$. As, by (21), $H(x,\overline{U}) = 1 - e^{-\frac{\phi(\phi+\delta+\lambda)x}{(\phi+\lambda)}}$ then straightforward algebra establishes

$$\overline{\Pi}\left[\frac{\phi(\phi+\delta-\rho)+\lambda\phi}{\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)}\right] = \frac{\overline{\Omega}}{\lambda y_0}. \tag{40}$$

Now consider $U_0 \in [\underline{U},\overline{U}]$. Using (39) and differentiating wrt U_0 implies

$$\left[\begin{array}{c}\frac{d\widehat{\Pi}}{dU}\left[\frac{\phi(\phi+\delta-\rho)}{\phi(\phi+\delta+\lambda)-\rho(\phi+\lambda)} + \frac{\lambda}{\lambda+\phi+\delta}\int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx'\right] \\ +\widehat{\Pi}\left[\frac{\lambda}{\lambda+\phi+\delta}\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x',U_0)}{\partial x \partial U}dx'\right]\end{array}\right] = 0 \text{ for all } U_0 \in [\underline{U},\overline{U}]. \tag{41}$$

To compute the integral in the second line then, for $x > 0$, partial differentiation wrt x of the pde for H, given by lemma 2, implies

$$\dot{U}\frac{\partial^2 H}{\partial x \partial U} = (\phi+\delta)FN'(x) - \left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right],$$

where $\dot{U} = \dot{U}(U)$ is given by (22). Thus

$$\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x',U_0)}{\partial x \partial U}dx' = \frac{1}{\dot{U}}\int_{x'>0} e^{\rho x'}\left[(\phi+\delta)FN'(x) - \left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right]\right]dx'. \tag{42}$$

Straightforward algebra using the solution for $N(.)$ finds

$$\int_{x'>0} e^{\rho x'}(\phi+\delta)FN'(x)dx = \frac{\phi\lambda\delta\,(\phi+\delta+\lambda)}{(\phi+\lambda)\,[\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)]}F(U)$$

The second term is computed using an appropriate integrating factor:

$$\begin{aligned} & \int_{x'>0} e^{\rho x'}\left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right]dx' \\ = & \int_{x'>0} e^{[\rho-[\phi+\delta+\lambda(1-F)]]x'}\left\{e^{[\phi+\delta+\lambda(1-F)]x'}\left[[\phi+\delta+\lambda(1-F)]\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2}\right]\right\}dx' \\ = & \left[e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}\right]_{0^+}^{\infty} - \int_{x'>0}[\rho-[\phi+\delta+\lambda(1-F)]]e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx' \\ = & [\phi+\delta-\rho+\lambda(1-F)]\int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx' - \frac{\phi(\phi+\lambda+\delta)}{\phi+\lambda}F. \end{aligned}$$

Inserting this solution into (42) now yields a closed form expression for $\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x',U_0)}{\partial x \partial U}dx'$. Using that solution to substitute out $\int_{x'>0} e^{\rho x'}\frac{\partial^2 H(x',U_0)}{\partial x \partial U}dx'$ in (41) then yields a closed form solution for $\int_{x'>0} e^{\rho x'}\frac{\partial H(x',U_0)}{\partial x}dx'$. But this expression is the same as $\int_{x'=0}^{\infty}\int_{U'=\underline{U}}^{U_0} e^{\rho x'}\frac{\partial^2 H(x',U')}{\partial x \partial U'}dx'dU'$. Substituting this closed form solution into (23) and simplifying then yields

$$\widehat{\Pi}^2 = \frac{[\phi\,(\phi+\delta+\lambda)-\rho(\phi+\lambda)]\,\overline{\Omega}}{\phi(\phi+\delta-\rho)[\delta+\phi-\rho+\lambda]\lambda y_0}(1-\widehat{\theta}) \text{ for all } U_0 \in [\underline{U},\overline{U}].$$

Using (40) to substitute out $\overline{\Omega}$ yields

$$\widehat{\Pi}^2 = \frac{\overline{\Pi}}{(\phi+\delta-\rho)}(1-\widehat{\theta}) \text{ for all } U_0 \in [\underline{U},\overline{U}].$$

Finally setting $U_0 = \overline{U}$ implies

$$\overline{\Pi} = \frac{1-\overline{\theta}}{\phi+\delta-\rho}. \tag{43}$$

and combining the last two expressions yields the Theorem. This completes the proof of Theorem 2.

Proof of Lemma 3. Using Theorem 2 to compute $d\widehat{\Pi}/dU$ and using (16) in Claim 1 implies $\widehat{\theta}(.)$ satisfies

$$\frac{\phi(\phi+\delta-\rho)+\lambda\phi}{2\phi(\phi+\delta-\rho)[\delta+\phi-\rho+\lambda]}\sqrt{(1-\overline{\theta})}(1-\widehat{\theta})^{-1/2}\widehat{\theta}^{-\sigma}\frac{d\widehat{\theta}}{dU} = 1$$

with $\widehat{\theta} = \overline{\theta}$ at $U = \overline{U}$. Integrating implies (24). Putting $U = \overline{U}$ in (17) and using (43) in the Appendix implies (25). Noting $\partial H(x, \underline{U})/\partial x = 0$, then putting $U = \underline{U}$ in (39) and using (40), (43) and Theorem 2 yields (27). Finally putting $U = \underline{U}$ in (**??**) implies (26). This completes the proof of Lemma 3.