Second-best Efficiency of Allocation Rules: Strategy-proofness and Single-peaked Preferences with Multiple Commodities*

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Abstract

This paper studies the design of a strategy-proof resource allocation rule in economies with perfectly divisible multiple commodities and single-peaked preferences. It is known that *the uniform rule* is the unique allocation rule satisfying *strategy-proofness*, *Pareto efficiency*, and *anonymity* if the number of commodity is only *one* and preferences are single-peaked (Sprumont (1991)). However, if the number of commodities is greater than one, the situation drastically changes and a trade-off between *strategy-proofness* and *Pareto efficiency* arises. The generalized uniform rule in multiple-commodity settings is still strategy-proof, but *not* Pareto efficient. In this paper, we first investigate the existence problem of second-best efficient rules, where a strategy-proof rule is second-best efficient if in the class of all strategy-proof rules, there is *no* other strategy-proof rule that gives a "better" outcome than the considered rule in terms of Pareto domination for all preference profiles. We show that in an n person and m good setup for any strategy-proof rule, there exists a second-best efficient rule that Pareto dominates the former. In

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the proof of the theorem, Zorn's Lemma plays an important role. Furthermore, by relaxing the requirement of Pareto efficiency, we show a possibility theorem, there is an egalitarian rational (consequently, non-dictatorial) strategy-proof rule that is second-best efficient although there is no egalitarian rational strategy-proof rule that is Pareto efficient. Second, we show that the generalized uniform rule is second-best efficient and give a new characterization of the generalized uniform rule with second-best efficiency in a two person and *m* good setup. That is, it is the unique rule satisfying the three axioms of the second-best efficiency, weak peak-onliness, and egalitarian rationality. We also show that the three axioms are mutually independent.

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1 Introduction

Ever since Sprumont (1991), resource allocation in economies with single-peaked preferences has been studied by many authors. If the number of commodity is only one, Sprumont (1991) presented a characterization of a resource allocation rule that satisfies three axioms: *strategy-proofness*, *Pareto efficiency*, and *anonymity*. Strategy-proofness means that announcing their true preferences is a dominant strategy for all agents in the game of stating their preferences. Pareto efficiency requires that any allocation obtained by the rule is Pareto efficient with respect to the reported preference relations. Anonymity says that the rule is independent of the "names" of the agents.

Sprumont proposes a resource allocation rule called *the uniform rule*. [1] Under the uniform rule, the same amount of a single divisible good is alloted to everyone except people whose peaks are small enough if excess demand exists or large enough if excess supply exists. [2]

Sprumont's theorem essentially depends on the assumption that there is only one commodity. If the number of goods is greater than one, we may naturally extend the uniform rule. The extended rule is referred to as *the generalized uniform rule*. [3] The generalized uniform rule is defined by applying the single good uniform rule commodity by commodity.

It is easy to show that the generalized uniform rule is strategy-proof. [4] However, as shown in Example 1.1 below, the rule *violates* Pareto efficiency.

[1] The rule was originally introduced by Benassy (1982).

[2] A single-peaked preference of one good may have several interpretations. One possible interpretation is the "fixed price economy" interpretation. In this interpretation, the peak of a preference is interpreted as a "Walrasian demand" under a fixed price. Given a preference profile, if the total number of all peak points is greater (or smaller) than the amount of a good to be alloted, it is said that there is an excess demand (or an excess supply). The fixed price interpretation is just for expository convenience. The result obtained by this paper may be applicable to many different interpretations.

[3] See Amorós (2002).

[4] It also satisfies anonymity.

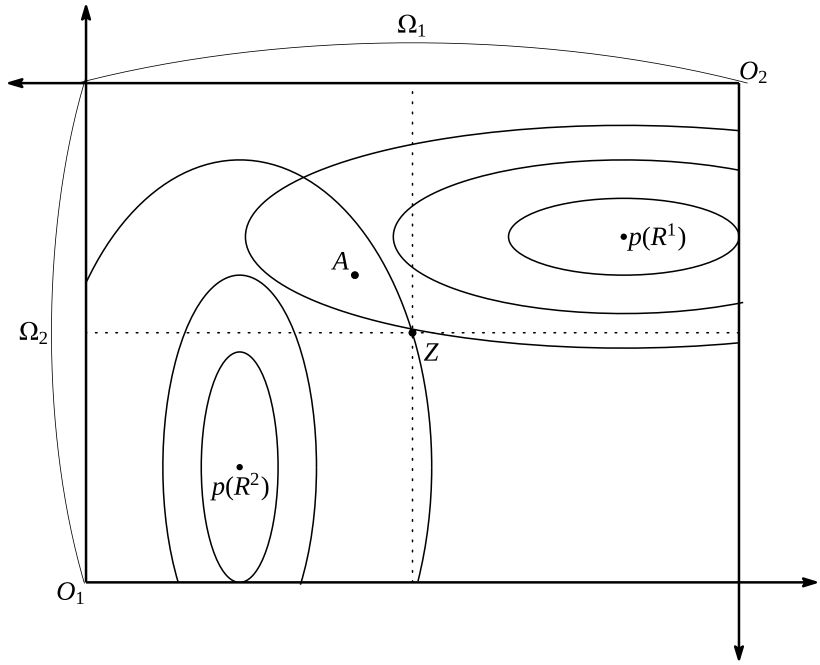

Figure 1

EXAMPLE 1.1. *There are two agents and two goods. Let Ω_1 and Ω_2 be the amounts of goods 1 and 2. Figure 1 is an Edgeworth Box. In Figure 1, $p(R^1)$ and $p(R^2)$ designate the peaks of Mr. 1 and Mr. 2's preferences, respectively. The middle point $Z = (\Omega_1/2, \Omega_2/2)$ is the allocation where equal amounts of goods are alloted to each agent. Since for each good $j = 1, 2$, both agents have peaks greater than $\Omega_j/2$, the generalized uniform rule assigns equal amounts of goods to both. This allocation is given by Z. However, if their indifference curves through Z can be drawn as in Figure 1, there is room for Pareto improvement. (For example, A is better than Z for both agents.)*

The literature on strategy-proofness in economic environments has uncovered a tension between strategy-proofness and Pareto efficiency. For example, Hurwicz (1972) shows that in pure exchange economies with two agents and two goods, no rule is strategy-proof, Pareto-efficient, and individually rational. Zhou (1991) proves that in pure exchange economies with two agents and the number of goods greater than or equal to two, no rule is strategy-proof, Pareto efficient, and non-dictatorial. [5] Serizawa (2002) shows that in pure exchange economies with n agents and m goods, no rule is strategy-proof, Pareto-efficient, and individually rational. Finally, Serizawa and Weymark (2003) show that in n person m goods pure exchange economies, strategy-proof and Pareto efficient rules cannot guarantee a minimum consumption: for arbitrarily small $\varepsilon > 0$, there exists a preference profile such that under the rule and the preference profile, there is an agent whose consumption has an Euclidean norm smaller than ε.

[5] See also Kato and Ohseto (2002).

The same kind of trade-off between strategy-proofness and Pareto efficiency exists in economies with single-peaked preferences and multiple commodities. [6]

One way of escaping this trade-off is to drop or replace an axiom. In fact, Amorós (2002) resolved this difficulty along these lines. That is, he replaced Pareto efficiency with the axiom of *same-sidedness*. This axiom requires that for each good, the amount of the good received by everyone is located on the same side of the agent's own peak. [7] His main theorem says that if the number of agents is *two* and the number of goods is greater than or equal to two, *the generalized uniform rule is the unique rule satisfying envy-freeness, strategy-proofness, and same-sidedness.* [8]

Since same-sidedness is a straightforward extension of Sprumont's efficiency concept, [9] Amorós' characterization theorems may be understood as a multiple-good version of Sprumont's characterization.

In the present paper, however, we study some problems concerning efficiency and strategy-proofness in a multiple-good setup from a much different point of view, and propose a new direction of research. Since the trade-off between strategy-proofness and Pareto efficiency (except when dictatorship is allowed) is inevitable in this setup, we mainly consider strategy-proofness. More precisely, we consider *the set* $\mathit{\Gamma}_{\mathrm{SP}}$ *of all strategy-proof rules*. We ask the following question: are there any strategy-proof rules that are not Pareto-dominated by any other strategy-proof rules? To answer this question, we propose two concepts of second-best efficiency. The first, which is called *weak second-best efficiency among strategy-proof rules* (WSESP), is the weaker version. That is, a strategy-proof rule f_0 is *WSESP,* if for any strategy-proof rule f_1 that Pareto-dominates f_0, then f_0 Pareto-dominates f_1. The second concept, which is called *strong second-best efficiency among strategy-proof rules* (SSESP), is the stronger version. That is, a strategy-proof rule f_0 is *SSESP,* if for any strategy-proof rule f_1 which Pareto-dominates f_0, then f_0 is equal to f_1.

In the proof of the main results (Theorems 2.1, 2.2, 3.1, and 3.2), the *option set* of each agent plays an important role. [10] For the i-th agent, given a preference profile $R^{-i} = (R^1, \cdots, R^{i-1}, R^{i+1}, \cdots, R^n)$ of the other agents, the option set of the i-th agent under a rule f is the set of his consumption bundles that may be assigned to him by rule f if the other agents announce the preference profle R^{-i}. In the set $\mathit{\Gamma}_{\mathrm{SP}}$ of all strategy-proof rules, we prove that for any rules f and $g \in \mathit{\Gamma}_{\mathrm{SP}}$, f Pareto-dominates g *if and only if* the option set of g is included in that of f for all agents (Lemma 4.4). Thus, in the

[6] We see an analogy of the special case of a theorem in Serizawa (2002) in a single-peaked preference environment in Section 3. Amorós (2002) presented that there is no strategy-proof, Pareto efficient, and non-dictatorial rule in a two person setup.

[7] That is, for each good, if the quantity of the good received by an agent is greater than or equal to his own peak amount, then the quantities of the good received by the remaining people should be greater than or equal to their own peak amounts, and *vice versa*. If the number of commodity is one, same-sidedness is equivalent to Pareto efficiency. However, if the number of commodity is greater than one, same-sidedness does not necessarily imply Pareto efficiency. Amorós (2002) called the axiom *Condition E* (CE).

[8] Amorós (2002) Theorem 2. In his Theorem 3, he replaces envy-freeness with *weak anonymity*. Recently, Morimoto, Serizawa, and Ching (2008) extended Amorós' result considerably.

[9] As noted earlier, same-sidedness is equivalent to Pareto efficiency if the number of commodity is one. Sprumont (1991) assumed Pareto efficiency because some properties of same-sidedness are required in his proof. Hence, in multiple-good economies, only same-sidedness is required for extending Sprumont's characterization.

[10] The notion of the option set was originally introduced by Barberà and Peleg (1990).

domain of Γ_{SP}, the relationship of Pareto-domination is equivalent to the relationship of set theoretic inclusion. In the proofs of the main theorems, this observation plays an essential role. For example, in Theorem 2.1, we prove the existence of a Pareto-undominated rule (a WSESP rule). In the proof of the theorem, it is enough to show that there exists a *maximal* option set with respect to the inclusion relationship in the collection of option sets. This is because by observation, a Pareto-undominated rule is a rule that gives the maximal element of the set of option sets under the inclusion relationship.

This paper consists of five sections. In Section 2, we present the model and describe our axioms. Moreover, we examine the existence of strategy-proof, efficient, and equitable allocation rule in an n person and m good setup. In Section 3, a new characterization of the generalized uniform rule is given in a two person and m good setup. In Section 4, we give the proofs of the main results. Section 5 concludes.

2 Second-Best Efficiency of Resource Allocation Rules

2.1 Single-peaked Preferences with Multiple Commodities

Let $N = \{1, \ldots, n\}$ be the set of individuals. Let $M = \{1, \cdots, m\}$ be the set of commodities. All commodities are perfectly divisible. The bundle $\Omega = (\Omega_1, \cdots, \Omega_m) \in \mathbb{R}^m_{++}$ denotes a social endowment of the commodities. [11] Let $B = \{\mathbf{x} = (x^1, \cdots, x^n) \in (\mathbb{R}^m_+)^n \mid \sum_{i=1}^n x^i = \Omega\}$ denote the set of feasible allocations. We do not allow free disposal. The preferences of each individual are given by a complete, transitive, continuous and strictly convex binary relation on $\prod_{j=1}^m [0, \Omega_j]$. [12]

DEFINITION 2.1. *A preference R is single-peaked if there exists $p(R) \in \prod_{j=1}^m [0, \Omega_j]$ such that for all $x, x' \in \prod_{j=1}^m [0, \Omega_j] (x \neq x')$,*

$$\left[\forall j \in M, x'_j \leq x_j \leq p_j(R) \vee p_j(R) \leq x_j \leq x'_j\right] \Rightarrow x\, P(R)\, x'.$$

Let $\mathcal{R}$ be the set of single-peaked preferences. We call each element of $\mathcal{R}^N$ a preference profile, or simply a profile. For each profile $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$, each $i \in N$, the subprofile obtained by removing i's preference is denoted by $\mathbf{R}^{-i}$. That is, $\mathbf{R}^{-i} = (R^1, \cdots, R^{i-1}, R^{i+1}, \cdots, R^n)$. It is convenient to write the profile $(R^1, \cdots, R^{i-1}, \hat{R}^i, R^{i+1}, \cdots, R^n)$ as $(\hat{R}^i; \mathbf{R}^{-i})$. A function from $\mathcal{R}^N$ to B is called *a rule*. Let Γ denotes the set of all rules.

2.2 Pareto Efficiency and Second-Best Efficiency : Impossibility vs. Possibility Results in a Multiple Commodity Setting

In this subsection, we introduce our axioms. First, one cannot be better off by misreporting one's preference. Let f be our generic notation of rule.

[11] The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural and real numbers, respectively. Let $\mathbb{R}_+$ be the set of non-negative numbers and let $\mathbb{R}_{++}$ be the set of positive real numbers.

[12] For each preference R, $P(R)$ and $I(R)$ denote the asymmetric part of R and the symmetric part of R, respectively.

Strategy-proofness (SP): for all $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$*, all* $i \in N$*, and all* $\hat{R}^i \in \mathcal{R}$*,* $f^i(\mathbf{R})\, R^i\, f^i(\hat{R}^i; \mathbf{R}^{-i})$.

Let Γ_{SP} denote the set of all *strategy-proof* rules. The following axiom reflects the idea that everyone should not be worse off than under equal division.

Egalitarian rationality (ER) [13] *: for all* $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$ *and all* $i \in N$*,* $f^i(\mathbf{R})\, R^i\, \frac{\Omega}{n}$.

Next, at the selected allocation, no one can be better off without making someone worse off.

Pareto efficiency : for all $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$*, there is no* $\mathbf{x} = (x^1, \cdots, x^n) \in B$ *such that i)* $x^i R^i f^i(\mathbf{R})$ *for all* $i \in N$*; and ii)* $x^{i_0} P^{i_0} f^{i_0}(\mathbf{R})$ *for some* $i_0 \in N$.

The following proposition is an impossibility result about rules satisfying the axioms above. It follows immediately from a powerful impossibility theorem by Serizawa (2002) or Serizawa and Weymark (2003). The proof of Proposition 2.1 is given in the Appendix.

PROPOSITION 2.1. *Suppose that* $m \geq 2$*. Then, no rule satisfies strategy-proofness, Pareto efficiency and egalitarian rationality.*

Note that with only one commodity, the impossibility described in Proposition 2.1 does not hold because the uniform rule satisfies the stated axioms [14](See Sprumont (1991) and Ching (1994)). Proposition 2.1 describes a difference between the environment with only one commodity and environments with more than one commodities.

DEFINITION 2.2. *The binary relation dom on* Γ *is defined as follows; for all* $f, g \in \Gamma$

$$f \text{ dom } g \;\Leftrightarrow \forall \mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N, \forall i \in N,\, f^i(\mathbf{R})\, R^i\, g^i(\mathbf{R}).^{15}$$

That is, f dom g means that for any preference profiles, the allocation selected by f is better than (or indifferent to) the one selected by g for all agents. By using the dom relation, we obtain the following equivalent expression of *Pareto efficiency*.

Pareto efficiency : for all $g \in \Gamma$*,* g *dom* $f \Rightarrow f$ *dom* g*.*

In words, *Pareto efficiency* requires a rule to be a maximal element of Γ preordered by dom. The following two axioms are the second-best efficiency concepts we adopt in this paper. The first one requires that a rule is a maximal element of Γ_{SP} preordered by dom. The second one is more demanding. It requires that a rule is a maximal element of Γ_{SP} preordered by dom and there is no other rule that is welfare-equivalent to it.

[13] This axiom is sometimes referred to as *the equal division lower bound*.

[14] The rule also satisfies anonymity that is stronger than *ER* under strategy-proofness.

[15] Note that dom is reflexive and transitive. Hence it is a preorder on Γ. But it is not an order on Γ in general. If f dom g and g dom f, then we say f and g are equivalent with respect to the welfare.

Weak second-best efficiency among strategy-proof rules (WSESP) : $f \in \Gamma_{SP}$ *and for all* $g \in \Gamma_{SP}$, g *dom* $f \Rightarrow f$ *dom* g.

Strong second-best efficiency among strategy-proof rules (SSESP) : $f \in \Gamma_{SP}$ *and for all* $g \in \Gamma_{SP}$, g *dom* $f \Rightarrow f = g$.[16]

The following theorem guarantees that for any *SP* rule f, we have a *WSESP* rule that dominates f.

THEOREM 2.1. *For any* $f \in \Gamma_{SP}$*, there exists a rule* $f_0 \in \Gamma_{SP}$ *such that* f_0 *satisfies weak second-best efficiency among strategy-proof rules and* f_0 *dom* f.

As a consequence of relaxing the efficiency condition, we obtain the following possibility result in contrast to Proposition 2.1.

THEOREM 2.2. *There exists a strategy-proof rule satisfying weak second-best efficiency among strategy-proof rules and egalitarian rationality.*

3 A New Characterization of the Generalized Uniform Rule

3.1 The Generalized Uniform Rule

One of the purposes of this paper is to characterize the following rule which is known as *the generalized uniform rule*.

DEFINITION 3.1. *The generalized uniform rule* $U : \mathcal{R}^N \to B$ *is the rule defined by the following; for all* $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$*, all* $j \in M$*, all* $i \in N$*,*

$$U^i_j(\mathbf{R}) = \begin{cases} \min\{p_j(R^i), \lambda_j(\mathbf{R})\} & \text{if } \sum_{i=1}^n p_j(R^i) \geq \Omega_j, \\ \max\{p_j(R^i), \mu_j(\mathbf{R})\} & \text{if } \sum_{i=1}^n p_j(R^i) \leq \Omega_j, \end{cases}$$

where $\lambda_j(\mathbf{R})$ *solves the equation* $\Omega_j = \sum_{i=1}^n \min\{p_j(R^i), \lambda_j(\mathbf{R})\}$ *and* $\mu_j(\mathbf{R})$ *solves the equation* $\Omega_j = \sum_{i=1}^n \max\{p_j(R^i), \mu_j(\mathbf{R})\}$.

3.2 A Characterization of the Generalized Uniform Rule

The following theorem shows an interesting property of the generalized uniform rule.

THEOREM 3.1. *Suppose that* $n = 2$*. The generalized uniform rule satisfies strong second-best efficiency among strategy-proof rules (SSESP).* [17]

[16] Sasaki (2003) first introduced *SSESP*. He called this condition Λ_{SP}-efficiency.

[17] Theorem 3.1 is first proved in Sasaki (2003). In the next section, we provide an alternative proof of the theorem.

In the previous subsection, we point out that for any $n \geq 2$ there exists a rule satisfying *WSESP* and *ER*. Since the generalized uniform rule satisfies *ER*, by Theorem 3.1 it is one of the rules satisfying *WSESP* and *ER*.

Now we ask whether there exists a rule satisfying *WSESP* and *ER* other than the generalized uniform rule. The answer is yes. An example of a rule satisfying *WSESP* and *ER* other than the generalized uniform rule is provided in Example 3.3. However, Theorem 3.2 shows that if we impose an auxiliary condition called *weak peak-onliness* in addition to *WSESP* and *ER*, then the generalized uniform rule is the unique rule that satisfies these three axioms.

Weak peak-onliness (WP) [18] *: for all* $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$*, all* $i \in N$*, all* $\tilde{R}^i \in \mathcal{R}$*,*

$$p(R^i) = p(\tilde{R}^i) \Rightarrow f^i(\mathbf{R}) = f^i(\tilde{R}^i; \mathbf{R}^{-i}).$$

THEOREM 3.2. *Suppose that* $n = 2$*. The generalized uniform rule is the only rule that satisfies weak second-best efficiency among strategy-proof rules (WSESP), egalitarian rationality (ER) and weak peak-onliness (WP).*

3.3 Independence of Axioms in Theorem 3.2

In this subsection we show that Theorem 3.2 is a tight characterization. That is, dropping any one of three axioms leads to other rules.

EXAMPLE 3.1. *An example of a rule that satisfies both ER and WP, but not WSESP is the equal division rule, E, defined as follows: for all* $\mathbf{R} \in \mathcal{R}^N$*,* $E(\mathbf{R}) = (\frac{\Omega}{2}, \frac{\Omega}{2})$*. Obviously E satisfies ER and WP but E not WSESP because U dom E but E does not dominate U.*

EXAMPLE 3.2. *Examples of rules that satisfy both WP and WSESP, but not ER are the priority rules. Let* $D^{(i)}$ *be the priority rule in which individual i has the priority defined as follows: for all* $\mathbf{R} = (R^1, R^2) \in \mathcal{R}^N$*,* $D^{(i)i}(\mathbf{R}) = p(R^i)$*. Since* $D^{(i)}$ *is SP and Pareto efficient, it satisfies WSESP. It is also clear that* $D^{(i)}$ *satisfies WP. However, clearly* $D^{(i)}$ *does not satisfy ER.*

EXAMPLE 3.3. *An example of a rule that satisfies both ER and WSESP, but not WP is* f_0 *below. Let f be the rule defined as follows. For all* $\mathbf{R} = (R^1, R^2) \in \mathcal{R}^N$*,*

$$f(\mathbf{R}) = \begin{cases} (\Omega, 0) & \text{if } \Omega\, R^1 \frac{\Omega}{2} \text{ and } 0\, R^2 \frac{\Omega}{2}, \\ (\frac{\Omega}{2}, \frac{\Omega}{2}) & \text{otherwise.} \end{cases}$$

Obviously f satisfies SP and ER.

First we show that U does not dominate f. By Lemma 4.6 of section 4.2, there exist $\tilde{R}^1, \tilde{R}^2 \in \mathcal{R}$ *such that* $p(\tilde{R}^1) = p(\tilde{R}^2) = (\Omega_1, \cdots, \Omega_{m-1}, 0)$ *and* $\Omega\, P(\tilde{R}^1)\frac{\Omega}{2}$ *and* $0\, P(\tilde{R}^2)\frac{\Omega}{2}$*. Then* $f(\tilde{R}^1, \tilde{R}^2) = (\Omega, 0)$*. Since* $U(\tilde{R}^1, \tilde{R}^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$*, U does not dominate f.*

[18] This axiom is sometimes referred to as *own peak only*.

By Theorem 2.1, there exists $f_0 \in \Gamma_{SP}$ such that f_0 is WSESP and f_0 dom f. Since f satisfies ER and dom is transitive, f_0 satisfies ER. $f_0^1(\tilde{R}^1, \tilde{R}^2)\tilde{R}^1 \Omega\ P(\tilde{R}^1)\frac{\Omega}{2}$ and $f_0^2(\tilde{R}^1, \tilde{R}^2)\tilde{R}^2\ 0\ P(\tilde{R}^2)\frac{\Omega}{2}$ because f_0 dom f. Since $U(\tilde{R}^1, \tilde{R}^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$, $f_0 \neq U$. This means that f_0 does not satisfy WP, because if f_0 satisfied WP, then by Theorem 3.2 $f_0 = U$, a contradiction.

By Example 3.1, 3.2 and 3.3, we have shown that *the three axioms in Theorem 3.2 are mutually independent.*

4 Proofs

4.1 Proofs of Theorems 2.1 and 2.2

To proveTheorems 2.1 and 2.2, we need to use some facts of metric spaces. Let $(\mathbf{X}, d)$ be a metric space.

DEFINITION 4.1. *We define the function $\bar{d} : \mathbf{X} \times (2^{\mathbf{X}} \setminus \{\emptyset\}) \to \mathbb{R}$ by $\bar{d}(x, A) = \inf\{d(x, a) \mid a \in A\}$ for all $x \in \mathbf{X}$ and all $A \in 2^{\mathbf{X}} \setminus \{\emptyset\}$.* [19]

DEFINITION 4.2. *We define $\mathcal{K}(\mathbf{X})$ to be the set of all nonempty compact subsets in $(\mathbf{X}, d)$.* [20]

DEFINITION 4.3. *Let d_H be the function from $\mathcal{K}(\mathbf{X}) \times \mathcal{K}(\mathbf{X})$ to $\mathbb{R}$ defined by*

$$d_H(A, B) = \max\{\max_{a \in A} \bar{d}(a, B), \max_{b \in B} \bar{d}(b, A)\}.$$

It is well-known that $(\mathcal{K}(\mathbf{X}), d_H)$ is also a metric space. The metric d_H is referred to as Hausdorff metric.

REMARK 4.1. *Let $A, B \in \mathcal{K}(\mathbf{X})$. If $A \subseteq B$, $\bar{d}(a, B) = 0$ for all $a \in A$. Hence, $d_H(A, B) = \max_{b \in B} \bar{d}(b, A)$ for all $A, B \in \mathcal{K}(\mathbf{X})$ with $A \subseteq B$.*

LEMMA 4.1. *Suppose that $\mathcal{S} \subseteq \mathcal{K}(\mathbf{X})$ is given, where $\mathcal{S}$ is totally ordered by $\subseteq$.* [21] *Let $C = cl_d(\cup_{S \in \mathcal{S}} S)$*[22]*. Then*

$$\forall S_1, S_2 \in \mathcal{S}, [\{\exists x_0 \in C \text{ s.t. } \bar{d}(x_0, S_1) > \bar{d}(x_0, S_2)\} \Rightarrow S_1 \subseteq S_2].$$

Proof. Suppose the contrary. That is, $S_1 \not\subseteq S_2$. Then, since $\mathcal{S}$ is totally ordered by $\subseteq$, $S_2 \subsetneq S_1$. And

$$\begin{aligned} \bar{d}(x_0, S_1) &= \min\{d(x_0, y) \mid y \in S_1\} \\ &\leq \min\{d(x_0, y) \mid y \in S_2\} \ (\because S_2 \subsetneq S_1) \\ &= \bar{d}(x_0, S_2). \end{aligned}$$

This is a contradiction. □

[19] Note that $\bar{d}(\cdot, A)$ is a continuous function from $\mathbf{X}$ to $\mathbb{R}$ when we fix $A \in 2^{\mathbf{X}} \setminus \{\emptyset\}$ arbitrarily.

[20] Note that $\bar{d}(x, A) = \min\{d(x, a) \mid a \in A\}$ when we restrict the domain of $\bar{d}$ within $\mathbf{X} \times \mathcal{K}(\mathbf{X})$.

[21] In general, $(\mathbf{Z}, \geq)$ is an *ordered set* if $\geq$ is a reflexive, transitive and anti-symmetric binary relation on $\mathbf{Z}$, where $\geq$ is reflexive if for all $x \in \mathbf{Z}$, $x \geq x$, and $\geq$ is anti-symmetric if for all $x, y \in \mathbf{Z}$, if $x \geq y$ and $y \geq x$, then $x = y$. An ordered set $(\mathbf{Z}, \geq)$ is *total* if for all $x, y \in \mathbf{Z}$, $x \geq y$ or $y \geq x$.

[22] Note that $\mathrm{cl}_d(\cup_{S \in \mathcal{S}} S)$ denotes the closure of $\cup_{S \in \mathcal{S}} S$ in $(\mathbf{X}, d)$

LEMMA 4.2. *Suppose that* $(\mathbf{X}, d)$ *is a compact metric space. Suppose also that* $\mathcal{S} \subseteq \mathcal{K}(\mathbf{X})$ *is given, where* $\mathcal{S}$ *is non empty and totally ordered by* $\subseteq$*. Let* $C = cl_d(\cup_{S\in\mathcal{S}} S) \in \mathcal{K}(\mathbf{X})$*. Then there exists a sequence of compact sets in* $\mathcal{S}$ *that converges to* C *with respect to* d_H*.*

Proof. If $C \in \mathcal{S}$, then the conclusion is trivial. We consider the case $C \notin \mathcal{S}$ below.
Step 1 $\forall S' \in \mathcal{S}, \exists S'' \in \mathcal{S}$ s.t. $d_H(S'', C) < \frac{d_H(S',C)}{2}$.

Let $S' \in \mathcal{S}$. Since d_H is a metric and $S' \neq C$, $d_H(S', C) > 0$. Let $d_H(S', C) = \eta$. Let $H = \{x \in C \mid \overline{d}(x, S') \geq \frac{\eta}{2}\}$. The set H is not empty because $d_H(S', C) = \eta$ implies that $\overline{d}(x^*, S') = \eta$ for some $x^* \in C$. Also H is a closed set because it is the inverse image of continuous function $\overline{d}(\cdot, S')$ with respect to $[\frac{\eta}{2}, +\infty)$. Hence, H is compact because $H \subseteq C \in \mathcal{K}(\mathbf{X})$.

For the open cover $\{B(x, \frac{\eta}{4})\} \mid x \in H\}$, we can find finite points $x_1, \cdots, x_h \in H$ such that $H \subseteq \cup_{i=1}^{h} B(x_i, \frac{\eta}{4})$. [23] Since $x_1, \cdots, x_h \in H \subseteq \mathrm{cl}_d(\cup_{S\in\mathcal{S}} S)$,

$$\forall i \in \{1, \cdots, h\}, \exists S_i \in \mathcal{S} \text{ s.t. } S_i \cap B(x_i, \frac{\eta}{4}) \neq \emptyset.$$

Since $S_1, \cdots, S_h$ are totally ordered by $\subseteq$, there is a the maximal one. We call it S''.

In the following, we show $d_H(S'', C) < \frac{\eta}{2}$. To this end, because of Remark 4.1, it is sufficient to show that $\overline{d}(x, S'') < \frac{\eta}{2}$ for all $x \in C$.

Let $x \in C$. First, suppose that $x \in H$. Then

$$\exists i_x \in \{1, \cdots, h\} \text{ s.t. } x \in B(x_{i_x}, \frac{\eta}{4}).$$

Since $S'' \cap B(x_{i_x}, \frac{\eta}{4}) \neq \emptyset$, we can pick $y \in S'' \cap B(x_{i_x}, \frac{\eta}{4})$. By the triangle inequality with respect to d,

$$\begin{aligned} d(x, y) &\leq d(x, x_{i_x}) + d(x_{i_x}, y) \\ &< \tfrac{\eta}{4} + \tfrac{\eta}{4} \\ &= \tfrac{\eta}{2}. \end{aligned}$$

By the definition of $\overline{d}$, we have $\overline{d}(x, S'') < \frac{\eta}{2}$.

Note that we have shown that for $x^* \in H$,

$$\overline{d}(x^*, S') = \eta > \frac{\eta}{2} > \overline{d}(x^*, S'').$$

Hence, $S' \subseteq S''$ by Lemma 4.1.

Next suppose that $x \notin H$. Then

$$\begin{aligned} \tfrac{\eta}{2} &> \overline{d}(x, S') \ (\because x \notin H) \\ &= \min\{d(x, y) \mid y \in S'\} \\ &\geq \min\{d(x, y) \mid y \in S''\} \ (\because S' \subseteq S'') \\ &= \overline{d}(x, S''). \end{aligned}$$

This completes the proof of step 1.
Step 2

[23] Note that $B(x, \varepsilon)$ denotes the open ball centered at x with a radius ε.

By Step 1 and the axiom of choice, we have a function $\Phi : \mathcal{S} \to \mathcal{S}$ such that

$$\forall S \in \mathcal{S}, d_H(\Phi(S), C) < \frac{d_H(S, C)}{2}.$$

Let $S \in \mathcal{S}$. Let $S_1 \equiv S$. For each $k \geq 2$, let $S_k \equiv \Phi(S_{k-1})$. Since $d_H(S_k, C) < \frac{d_H(S,C)}{2^{k-1}}$ for each $k \in \mathbb{N}$, $\{S_k\}_{k\in\mathbb{N}}$ satisfies $d_H(S_k, C) \to 0$ as $k \to +\infty$. □

REMARK 4.2. *Note that the sequence obtained in Lemma 4.2 satisfies*

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots.$$

DEFINITION 4.4. *Let $f \in \mathit{\Gamma}$ and $i \in N$. For each $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$,*

$$B^{f^i}_{\mathbf{R}^{-i}} \equiv \{x \in \prod_{j=1}^{m}[0, \Omega_j] \mid \exists R^i \in \mathcal{R} \text{ s.t. } f^i(R^i; \mathbf{R}^{-i}) = x\}.$$

We call $B^{f^i}_{\mathbf{R}^{-i}}$ the option set of individual i under f and $\mathbf{R}^{-i}$.

Let $\tau(R, Y) = \{x \in Y \mid \forall y \in Y, xRy\}$ for all $Y \subseteq \prod_{j=1}^{m}[0, \Omega_j]$, all $R \in \mathcal{R}$. That is, $\tau(R, Y)$ denotes the best consumptions on $Y \subseteq \prod_{j=1}^{m}[0, \Omega_j]$ with respect to $R \in \mathcal{R}$. [24]

LEMMA 4.3. *Let $f \in \mathit{\Gamma}$.*

$$f \in \mathit{\Gamma}_{SP} \Leftrightarrow \forall \mathbf{R} \in \mathcal{R}^N, \forall i \in N, f^i(\mathbf{R}) \in \tau(R^i, B^{f^i}_{\mathbf{R}^{-i}}).$$

Proof. Obvious. □

LEMMA 4.4. *Let $f, g \in \mathit{\Gamma}_{SP}$.*

$$f \text{ dom } g \Leftrightarrow \forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, B^{g^i}_{\mathbf{R}^{-i}} \subseteq B^{f^i}_{\mathbf{R}^{-i}}.$$

Proof. ($\Rightarrow$) Let $i \in N, \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$. Let $x \in B^{g^i}_{\mathbf{R}^{-i}}$. Let $R^i \in \mathcal{R}$ be such that $\tau(R^i, \prod_{j=1}^{m}[0, \Omega_j]) = x$. By Lemma 4.3, $g^i(R^i; \mathbf{R}^{-i}) = x$. Since f dom g, $f^i(R^i; \mathbf{R}^{-i})\ R^i\ g^i(R^i; \mathbf{R}^{-i})$. Hence, $f^i(R^i; \mathbf{R}^{-i}) = x$. This means $x \in B^{f^i}_{\mathbf{R}^{-i}}$.
($\Leftarrow$) Let $i \in N, \mathbf{R} \in \mathcal{R}^N$. By Lemma 4.3, $f^i(\mathbf{R}) \in \tau(R^i, B^{f^i}_{\mathbf{R}^{-i}})$ and $g^i(\mathbf{R}) \in \tau(R^i, B^{g^i}_{\mathbf{R}^{-i}})$. Since $B^{g^i}_{\mathbf{R}^{-i}} \subseteq B^{f^i}_{\mathbf{R}^{-i}}$, $f^i(\mathbf{R})\ R^i\ g^i(\mathbf{R})$. □

For all $f, g \in \mathit{\Gamma}_{\mathrm{SP}}$, $f \sim g$ if and only if f dom g and g dom f. Note that $\sim$ is an equivalence relation on $\mathit{\Gamma}_{\mathrm{SP}}$. For each $f \in \mathit{\Gamma}_{\mathrm{SP}}$, $[f]$ denotes the equivalence class of f in the quotient set $\mathit{\Gamma}_{\mathrm{SP}}/\sim$.

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[24] If $f \in \mathit{\Gamma}_{\mathrm{SP}}$, then $B^{f^i}_{\mathbf{R}^{-i}}$ is closed set in $\prod_{j=1}^{m}[0, \Omega_j]$ for all $i \in N$, $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$. To see this, fix $i \in N$ and $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, then we have a one agent social choice function from $\mathcal{R}$ to $\prod_{j=1}^{m}[0, \Omega_j]$ induced by f. Note that $B^{f^i}_{\mathbf{R}^{-i}}$ is the range of the one agent social choice function. Applying Proposition 5 in Le Breton and Weymark (1999), we have the conclusion. See also Barberà and Peleg (1990) .

DEFINITION 4.5. *The binary relation Dom on $\Gamma_{SP}/\sim$ is defined as follows : for all $[f],[g] \in \Gamma_{SP}/\sim$*

$$[f]\ Dom\ [g] \ \Leftrightarrow\ f\ dom\ g.$$ [25]

DEFINITION 4.6. *Let $(\mathbf{Z}, \geq)$ be an ordered set. We say that $(\mathbf{Z}, \geq)$ is inductive if every non empty totally ordered subset of $\mathbf{Z}$ has an upper bound.*

The following theorem shows an important property of an inductive ordered set.

Zorn's Lemma *Let $(\mathbf{Z}, \geq)$ be an ordered set which is inductive and let x_0 be an element of $\mathbf{Z}$. Then*

$$\exists a \in \mathbf{Z} \text{ s.t. } a \geq x_0 \wedge \forall x \in \mathbf{Z}, \neg(x > a).$$

LEMMA 4.5. *The ordered set $\left(\Gamma_{SP}/\sim, Dom\right)$ is inductive.*

Proof. Let $\mathcal{S} \subseteq \Gamma_{\text{SP}}/\sim$. Suppose that $\left(\mathcal{S}, \text{Dom}\ |_{\mathcal{S}\times\mathcal{S}}\right)$ is totally ordered. We construct a rule $F : \mathcal{R}^N \to B$ and show that $[F]$ is an upper bound of $\mathcal{S}$.

For any $i \in N$, $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, $C^{\mathbf{R}^{-i}} = \text{cl}_d\left(\cup_{[f]\in\mathcal{S}} B^{f^i_{\mathbf{R}^{-i}}}\right)$.

Step 1

In this step, we show that

$$\exists\{[f_k]\}_{k\in\mathbb{N}} \text{ in } \mathcal{S} \text{ s.t. } \forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, B^{f^i_{k\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}. \tag{1}$$ [26]

By Lemma 4.2, for each $i \in N$, $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$,

$$\exists\{[f_{ik}]\}_{k\in\mathbb{N}} \text{ in } \mathcal{S} \text{ s.t. } B^{f^i_{ik\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}.$$

Note that by Remark 4.2, for any $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{R}^N$,

$$\begin{array}{c} B^{f^1_{11\mathbf{R}^{-1}}} \subseteq B^{f^1_{12\mathbf{R}^{-1}}} \subseteq B^{f^1_{13\mathbf{R}^{-1}}} \subseteq \cdots, \\ B^{f^2_{21\mathbf{R}^{-2}}} \subseteq B^{f^2_{22\mathbf{R}^{-2}}} \subseteq B^{f^2_{23\mathbf{R}^{-2}}} \subseteq \cdots, \\ \vdots \\ B^{f^n_{n1\mathbf{R}^{-n}}} \subseteq B^{f^n_{n2\mathbf{R}^{-n}}} \subseteq B^{f^n_{n3\mathbf{R}^{-n}}} \subseteq \cdots. \end{array}$$

For each $k \in \mathbb{N}$, we have the greatest among $[f_{1k}], [f_{2k}], \cdots, [f_{nk}]$ with respect to Dom because $\{[f_{1k}], [f_{2k}], \cdots, [f_{nk}]\}$ is totally ordered by Dom. Define $[f_k]$ by it. We show that $\{[f_k]\}_{k\in\mathbb{N}}$ satisfies (1).

Let $i \in N$, $\mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}$, $k \in \mathbb{N}$. Because f_k dom f_{ik}, $B^{f^i_{ik\mathbf{R}^{-i}}} \subseteq B^{f^i_{k\mathbf{R}^{-i}}}$, we obtain $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{k\mathbf{R}^{-i}}}) \leq d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{ik\mathbf{R}^{-i}}})$. Since $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{ik\mathbf{R}^{-i}}}) \to 0$ as $k \to +\infty$, $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{k\mathbf{R}^{-i}}}) \to 0$ as $k \to +\infty$.

[25] Note that Dom is an order on $\Gamma_{\text{SP}}/\sim$.

[26] $\{B^{f^i_{k\mathbf{R}^{-i}}}\}_{k\in\mathbb{N}}$ is a sequence in $(\mathcal{K}(\prod_{j=1}^m[0,\Omega_j]), d_H)$. $B^{f^i_{k\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}$ means the convergence of the sequence with the d_H metric.

Step 2

In this step, we define F. We can pick a representative f_k for each equivalence class $[f_k]$ ($\because$ Axiom of choice.). Later on, we fix $\{f_k\}_{k\in\mathbb{N}}$. For each $\mathbf{R} \in \mathcal{R}^N$, we obtain the sequence $\{f_k(\mathbf{R})\}_{k\in\mathbb{N}}$ in B. Since B is compact, there exists a convergent subsequence $\{f_{\mathbf{R}(k)}(\mathbf{R})\}_{k\in\mathbb{N}}$ [27]. Now we define $F : \mathcal{R}^N \to B$ as follows : for each $\mathbf{R} \in \mathcal{R}^N$, $F(\mathbf{R}) = \lim_{k\to+\infty} f_{\mathbf{R}(k)}(\mathbf{R})$. Obviously, $F \in \Gamma$.

Step 3 $\forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, B^{F^i_{\mathbf{R}^{-i}}} = C^{\mathbf{R}^{-i}}$.

First we show $B^{F^i_{\mathbf{R}^{-i}}} \subseteq C^{\mathbf{R}^{-i}}$. Let $x^i \in B^{F^i_{\mathbf{R}^{-i}}}$. Then

$$\exists \hat{R}^i \in \mathcal{R} \text{ s.t. } F^i(\hat{R}^i; \mathbf{R}^{-i}) = x^i.$$

Let $\hat{\mathbf{R}} = (\hat{R}^i; \mathbf{R}^{-i})$. By the definition of F, $F(\hat{\mathbf{R}}) = \lim_{k\to+\infty} f_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}})$. For all $k \in \mathbb{N}$, by the *strategy-proofness* of $f_{\hat{\mathbf{R}}(k)}$, $f^i_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}}) \in \tau(\hat{R}^i, B^{f^i_{\hat{\mathbf{R}}(k)\mathbf{R}^{-i}}}) \subseteq B^{f^i_{\hat{\mathbf{R}}(k)\mathbf{R}^{-i}}} \subseteq C^{\mathbf{R}^{-i}}$. Hence, $\{f^i_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}})\}_{k\in\mathbb{N}}$ is a convergent sequence in $C^{\mathbf{R}^{-i}}$. Since $C^{\mathbf{R}^{-i}}$ is closed in $\prod_{j=1}^m [0, \Omega_j]$, $x^i \in C^{\mathbf{R}^{-i}}$.

Next we show $C^{\mathbf{R}^{-i}} \subseteq B^{F^i_{\mathbf{R}^{-i}}}$. Let $x^i \in C^{\mathbf{R}^{-i}}$. Let $R^i_d \in \mathcal{R}$ be a preference represented by the utility function u_d defined by $u_d(z) = -\|x^i - z\|$. Let $\mathbf{R}_d = (R^i_d; \mathbf{R}^{-i})$. We show that $F^i(\mathbf{R}_d) = x^i$ by contradiction. Suppose that $x^i \neq F^i(\mathbf{R}_d) = \lim_{k\to+\infty} f^i_{\mathbf{R}_d(k)}(\mathbf{R}_d)$. Then, without loss of generality, we may assume

$$\exists \varepsilon > 0, \exists K_1 \in \mathbb{N} \text{ s.t. } \forall \ell \geq K_1, f^i_{\mathbf{R}_d(\ell)}(\mathbf{R}_d) \notin B(x^i, \varepsilon). \tag{2}$$

Since $B^{f^i_{k\mathbf{R}^{-i}}} \to C^{\mathbf{R}^{-i}}$, there exists $K_2 \in \mathbb{N}$ such that $d_H(C^{\mathbf{R}^{-i}}, B^{f^i_{K_2\mathbf{R}^{-i}}}) < \varepsilon$. By Remark 4.1, $B^{f^i_{K_2\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon) \neq \emptyset$. Furthermore, by Remark 4.2,

$$\forall \ell \geq K_2, B^{f^i_{\ell\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon) \neq \emptyset. \tag{3}$$

Let L be the positive integer such that $L = \max\{K_1, K_2\}$. By (2) and (3),

$$f^i_{\mathbf{R}_d(L)}(\mathbf{R}_d) \notin B(x^i, \varepsilon) \text{ and } B^{f^i_{\mathbf{R}_d(L)\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon) \neq \emptyset.$$

Now let $y \in B^{f^i_{\mathbf{R}_d(L)\mathbf{R}^{-i}}} \cap B(x^i, \varepsilon)$ and let $\tilde{R}^i_d \in \mathcal{R}$ such that $\tau(\tilde{R}^i_d, \prod_{j=1}^m [0, \Omega_j]) = \{y\}$. Then by Lemma 4.3, $f^i_{\mathbf{R}_d(L)}(\tilde{R}^i_d; \mathbf{R}^{-i}) = y$. This implies that $f_{\mathbf{R}_d(L)}$ is manipulable by i at $(R^i_d; \mathbf{R}^{-i})$ via $\tilde{R}^i_d$, a contradiction. Hence $F^i(\mathbf{R}_d) = x^i$. This implies $x^i \in B^{F^i_{\mathbf{R}^{-i}}}$.

Step 4 The rule F is strategy-proof.

Let $\mathbf{R} \in \mathcal{R}^N$, $i \in N$. First we prove that

$$\forall k \in \mathbb{N}, F^i(\mathbf{R})\ R^i\ f^i_k(\mathbf{R}). \tag{4}$$

[27] $\mathbf{R}(\cdot)$ is an operator from $\mathbb{N}$ to $\mathbb{N}$ creating a subsequence. In general, there may exist more than one convergent subsequences. In this case, let us choose an arbitrary convergent subsequence, and define it as $\{f_{\mathbf{R}(k)}(\mathbf{R})\}_{k\in\mathbb{N}}$.

Because for all $k \in \mathbb{N}$, $B^{f^i_{k\mathbf{R}^{-i}}} \subseteq B^{f^i_{k+1\mathbf{R}^{-i}}}$, $f^i_{k+1}(\mathbf{R})\ R^i\ f^i_k(\mathbf{R})$. Since R^i is continuous and $F^i(\mathbf{R}) = \lim_{k\to+\infty} f^i_{\mathbf{R}(k+1)}(\mathbf{R})$, we obtain (4).

Next, we prove $F \in \Gamma_{\mathrm{SP}}$ by contradiction. Suppose the contrary, that is

$$\exists x^i \in B^{F^i_{\mathbf{R}^{-i}}} \text{ s.t. } x^i P(R^i) F^i(\mathbf{R}).$$

Since $x^i \in B^{F^i_{\mathbf{R}^{-i}}}$, $F^i(\hat{R}^i; \mathbf{R}^{-i}) = x^i$ for some $\hat{R}^i \in \mathcal{R}$. Let $\hat{\mathbf{R}} = (\hat{R}^i; \mathbf{R}^{-i})$. By the definition of F,

$$x^i = \lim_{k\to+\infty} f^i_{\hat{\mathbf{R}}(k)}(\hat{\mathbf{R}}). \tag{5}$$

Since $x^i \in \mathrm{SUC}(R^i, F^i(\mathbf{R}))$ [28] and R^i is continuous, $B(x^i, \varepsilon) \subseteq \mathrm{SUC}(R^i, F^i(\mathbf{R}))$ for some $\varepsilon > 0$. By (5), for sufficiently large $L \in \mathbb{N}$, $f^i_{\hat{\mathbf{R}}(L)}(\hat{\mathbf{R}}) \in \mathrm{SUC}(R^i, F^i(\mathbf{R}))$. By (4), $\mathrm{SUC}(R^i, F^i(\mathbf{R})) \subseteq \mathrm{SUC}(R^i, f^i_{\hat{\mathbf{R}}(L)}(\mathbf{R}))$. This implies that $f^i_{\hat{\mathbf{R}}(L)}(\hat{\mathbf{R}}) \in \mathrm{SUC}(R^i, f^i_{\hat{\mathbf{R}}(L)}(\mathbf{R}))$. This means that $f^i_{\hat{\mathbf{R}}(L)}$ is manipulable by i at $\mathbf{R} = (R^i; \mathbf{R}^{-i})$ via $\hat{R}^i$. This contradicts the *strategy-proofness* of $f^i_{\hat{\mathbf{R}}(L)}$. Hence $F \in \Gamma_{\mathrm{SP}}$.

By *Step 4*, $[F] \in \Gamma_{\mathrm{SP}}/\sim$. By *Step 3* and the definition of $C^{\mathbf{R}^{-i}}$,

$$\forall [f] \in \mathcal{S}, [F] \text{ Dom } [f].$$

Hence $[F]$ is an upper bound of $\mathcal{S}$. □

Proof of Theorem 2.1. By Lemma 4.5 and Zorn's lemma, for any $f \in \Gamma_{\mathrm{SP}}$, there exists $f_0 \in \Gamma_{\mathrm{SP}}$ such that f_0 dom f and $[f_0]$ is a maximal element of $\left(\Gamma_{\mathrm{SP}}/\sim, \mathrm{Dom}\right)$. □

Proof of Theorem 2.2. Let E be the rule defined by

$$\forall \mathbf{R} \in \mathcal{R}^N, \forall i \in N, E^i(\mathbf{R}) = \frac{\Omega}{n}.$$

Obviously E satisfies *SP* and *ER*. By Theorem 2.1, there exists a rule E_0 which satisfies *WSESP* and E_0 dom E. Since dom is transitive, E_0 satisfies *ER*. □

4.2 Proofs of Theorems 3.1 and 3.2

First, we introduce an useful lemma provided in Amorós (2002).

LEMMA 4.6. *(Amorós (2002)) If $x^*, x', x'' \in \prod_{j=1}^m [0, \Omega_j]$ satisfy*

$$\left[\exists j \in M \text{ s.t. } |x^*_j - x''_j| < |x^*_j - x'_j|\right] \vee \left[\exists j \in M \text{ s.t. } (x^*_j - x''_j)(x^*_j - x'_j) < 0\right],$$

then there exists $R \in \mathcal{R}$ such that $p(R) = x^*$ and $x'' P(R) x'$.

For the purpose of reference, we prepare an equivalent expression of Lemma 4.6.

[28] Let $\mathrm{UC}(R, x) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid yRx\}$, $\mathrm{SUC}(R, x) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid yP(R)x\}$ and $\mathrm{LC}(R, x) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid xRy\}$ for all $x \in \prod_{j=1}^m [0, \Omega_j]$, and all $R \in \mathcal{R}$.

LEMMA 4.7. *If* $x^*, x', x'' \in \prod_{j=1}^m [0, \Omega_j] (x' \neq x'')$ *satisfy*

$$\neg[\forall j \in M, x^*_j \leq x'_j \leq x''_j \vee x''_j \leq x'_j \leq x^*_j],$$

then there exists $R \in \mathcal{R}$ *such that* $p(R) = x^*$ *and* $x'' P(R) x'$.

Before we prove Theorem 3.1, we show the following lemma about the shape of the option set. Note that Lemma 4.8 holds in the environment with more than two individuals.

LEMMA 4.8. *Suppose that* f *satisfies SP and WP. Then*

$$\forall i \in N, \forall \mathbf{R}^{-i} \in \mathcal{R}^{N\setminus\{i\}}, \forall j \in M, \exists a_j, b_j \in [0, \Omega_j] \text{ s.t. } B^{f^i_{\mathbf{R}^{-i}}} = \prod_{j=1}^m [a_j, b_j].$$

Proof. Fix $i \in N$ and $R^{-i} \in \mathcal{R}^{N\setminus\{i\}}$ arbitrarily. Proof is done by two steps.

Step 1 The option set $B^{f^i_{R^{-i}}}$ is convex.

To this end, suppose the contrary. That is, we assume

$\exists \hat{v}, \hat{w} \in B^{f^i_{R^{-i}}}, \exists \lambda \in (0, 1)$ s.t. $\lambda \hat{v} + (1 - \lambda)\hat{w} \notin B^{f^i_{R^{-i}}}$.

Obviously

$\exists v, w \in B^{f^1_{R^2}}, \forall \lambda \in (0, 1)$ s.t. $\lambda v + (1 - \lambda) w \notin B^{f^1_{R^2}}$.

Let $\tilde{x} = \frac{1}{2}v + \frac{1}{2}w$ and let $\tilde{R} \in \mathcal{R}$ be a preference that satisfies $p(\tilde{R}) = \tilde{x}$.

Case 1. $f^1(\tilde{R}, R^2) = v \vee f^1(\tilde{R}, R^2) = w$

For each $\mathbf{d} = (d_1, \cdots, d_m) \in \prod_{j=1}^m \{e_j, -e_j\}$ and each $y \in \prod_{j=1}^m [0, \Omega_j]$, $E(y, \mathbf{d}) = \{z \in \prod_{j=1}^m [0, \Omega_j] \mid \exists \gamma_1, \cdots, \gamma_m \in \mathbb{R}_+ \text{ s.t. } z = y + \sum_{j=1}^m \gamma_j d_j\}$, where e_j denotes the m-dimensional vector in which jth coordinate is 1 and other coordinates are 0. Without loss of generality, we may assume $f^1(\tilde{R}, R^2) = v$. Suppose that for $\mathbf{d} = (d_1, \cdots, d_m), \mathbf{d}' = (d'_1, \cdots, d'_m) \in \prod_{j=1}^m \{e_j, -e_j\}$, $v \in E(p(\tilde{R}), \mathbf{d})$ and $w \in E(p(\tilde{R}), \mathbf{d}')$. Suppose also that $\mathbf{d}, \mathbf{d}'$ satisfy that

$$\forall j \in M, \left[v \in E(p(\tilde{R}), (-d_j, \mathbf{d}_{-j})) \vee w \in E(p(\tilde{R}), (-d'_j, \mathbf{d}'_{-j})) \Rightarrow d_j = d'_j\right], \quad (6)$$

where $(-d_j, \mathbf{d}_{-j}) = (d_1, \cdots, d_{j-1}, -d_j, d_{j+1}, \cdots, d_m)$ and $(-d'_j, \mathbf{d}'_{-j})$ is defined in the same manner. Obviously, there exists $j' \in M$ such that $d_{j'} = -d'_{j'}$ because $p(\tilde{R})$ is the midpoint of v and w. Since $v_{j'} < \tilde{x}_{j'} < w_{j'}$ or $w_{j'} < \tilde{x}_{j'} < v_{j'}$, by Lemma 4.6 there exists $\hat{R}^1 \in \mathcal{R}$ such that $p(\hat{R}^1) = \tilde{x}$ and $wP(\hat{R}^1)v$. Hence $f^1(\hat{R}^1, R^2) \neq v$ by Lemma 4.3. Since f satisfies *WP*, this is a contradiction.

Case 2. $f^1(\tilde{R}, R^2) \neq v \wedge f^1(\tilde{R}, R^2) \neq w$

Let $c = f^1(\tilde{R}, R^2)$. If

$$[\exists j \in M \text{ s.t. } |\tilde{x}_j - v_j| < |\tilde{x} - c_j| \vee (\tilde{x} - v_j)(\tilde{x} - c_j) < 0]$$

or

$$[\exists j \in M \text{ s.t. } |\tilde{x}_j - w_j| < |\tilde{x} - c_j| \vee (\tilde{x} - w_j)(\tilde{x} - c_j) < 0],$$

then by Lemma 4.6,

$$\left[\exists \tilde{R}_v \in \mathcal{R} \text{ s.t. } p(\tilde{R}_v) = \tilde{x} \wedge vP(\tilde{R}_v)c\right] \vee \left[\exists \tilde{R}_w \in \mathcal{R} \text{ s.t. } p(\tilde{R}_w) = \tilde{x} \wedge wP(\tilde{R}_b)c\right].$$

This contradicts the fact that $c = f^1(\tilde{R}, R^2)$ and f is weakly peak-only. Hence

$$[\forall j \in M,\ |\tilde{x}_j - v_j| \geq |\tilde{x} - c_j| \ \wedge\ (\tilde{x} - v_j)(\tilde{x} - c_j) \geq 0]$$

and

$$[\forall j \in M,\ |\tilde{x}_j - w_j| \geq |\tilde{x} - c_j| \ \wedge\ (\tilde{x} - w_j)(\tilde{x} - c_j) \geq 0].$$

Suppose that $\mathbf{d}, \mathbf{d}' \in \prod_{j=1}^m \{e_j, -e_j\}$ satisfy $v \in E(p(\tilde{R}), \mathbf{d})$, $w \in E(p(\tilde{R}), \mathbf{d}')$ and the condition (6) in *Case 1*. Fix $j \in M$ arbitrarily. If $d_j = -d'_j$, then $c_j = \tilde{x}_j$ because $(\tilde{x} - v_j)(\tilde{x} - c_j) \geq 0$ and $(\tilde{x} - w_j)(\tilde{x} - c_j) \geq 0$. If $d_j = d'_j$, then v_j and w_j can be represented $v_j = \tilde{x}_j + \lambda d_j$ and $w_j = \tilde{x}_j + \lambda' d_j$ for some λ, λ'. Since $\tilde{x} = \frac{1}{2}v + \frac{1}{2}w$, $\lambda = \lambda'$. Hence $v_j = w_j = \tilde{x}_j$. Since $|\tilde{x}_j - v_j| \geq |\tilde{x} - c_j|$, $|\tilde{x} - c_j| = 0$. Hence $c_j = \tilde{x}_j$. We have shown that $c = \tilde{x}$. Since $\tilde{x} \notin B^{f^1_{R^2}}$ and $c \in B^{f^1_{R^2}}$, a contradiction.

Step 2 $\forall j \in M, \exists a_j, b_j \in [0, \Omega_j]$ s.t. $B^{f^1_{R^2}} = \prod_{j=1}^m [a_j, b_j]$.

For each $j \in M$, let Pr_j denote the projection with respect to jth coordinate. Since Pr_j is continuous and $B^{f^1_{R^2}}$ is compact, $\mathrm{Pr}_j(B^{f^1_{R^2}}) \subseteq [0, \Omega_j]$ is compact. Let $a_j = \min \mathrm{Pr}_j(B^{f^1_{R^2}})$ and $b_j = \max \mathrm{Pr}_j(B^{f^1_{R^2}})$. We show that $\prod_{j=1}^m \{a_j, b_j\} \subseteq B^{f^1_{R^2}}$ by contradiction. We only show that $(b_1, \cdots, b_m) \in B^{f^1_{R^2}}$. We can prove other cases in the same manner. Suppose the contrary. That is, $(b_1, \cdots, b_m) \notin B^{f^1_{R^2}}$. Let $\tilde{\tilde{R}} \in \mathcal{R}$ be a preference that satisfies $p(\tilde{\tilde{R}}) = (b_1, \cdots, b_m)$. Let $h \in B^{f^1_{R^2}}$ satisfy $f^1(\tilde{\tilde{R}}, R^2) = h$. Since $h \neq (b_1, \cdots, b_m)$,

$$\exists j' \in M \text{ s.t. } h_{j'} < b_{j'}.$$

Since $b_{j'} = \max \mathrm{Pr}_j(B^{f^1_{R^2}})$, there exists $h' \in B^{f^1_{R^2}}$ such that $h'_{j'} = b_{j'}$. Hence, by Lemma 4.6,

$$\exists R^1 \in \mathcal{R} \text{ s.t. } p(R^1) = (b_1, \cdots, b_m) \wedge h' P(R^1) h$$

because $|b_{j'} - h'_{j'}| < |b_{j'} - h_{j'}|$. However this implies that

$$f^1(R^1, R^2) \neq h.$$

This is a contradiction because f satisfies *weak peak-onliness*. □

Proof of Theorem 3.1. Let U be the generalized uniform rule. Note that U satisfies weak peak-onliness and strategy-proofness. We concludes by contradiction. Suppose that there exists $g \in \Gamma_{\mathrm{SP}}$ such that g dom U but $U \neq g$. By Lemma 4.4, $B^{U^1_{R^2}} \subseteq B^{g^1_{R^2}}$ and $B^{U^2_{R^1}} \subseteq B^{g^2_{R^1}}$ for all $(R^1, R^2) \in \mathcal{R}^N$. By the hypothesis,

$$\exists (R^1, R^2) \in \mathcal{R}^N \text{ s.t. } B^{U^1_{R^2}} \subsetneq B^{g^1_{R^2}} \vee B^{U^2_{R^1}} \subsetneq B^{g^2_{R^1}}. \tag{7}$$

If not, then $B^{U^1_{R^2}} = B^{g^1_{R^2}}$ and $B^{U^2_{R^1}} = B^{g^2_{R^1}}$ for all $(R^1, R^2) \in \mathcal{R}^N$. Note that $B^{U^1_{R^2}}$ is a direct product of closed intervals by Lemma 4.8. This implies by single-peakedness that $\#\tau(R^1, B^{U^1_{R^2}}) = 1$ for all $(R^1, R^2) \in \mathcal{R}^N$. Then, by Lemma 4.3 we have $U^1(R^1, R^2) = g^1(R^1, R^2)$ for all $(R^1, R^2) \in \mathcal{R}^N$. By the feasibility condition, $U^2(R^1, R^2) = g^2(R^1, R^2)$ for all $(R^1, R^2) \in \mathcal{R}^N$. Hence, $U = g$, a contradiction. Hence (7) holds. Without loss of generality, suppose that there exists $R^2 \in \mathcal{R}$ such that $B^{U^1_{R^2}} \subsetneq B^{g^1_{R^2}}$.

We have $\tilde{x} \in \prod_{j=1}^m [0, \Omega_j]$ such that $\tilde{x} \in B^{g^1_{R^2}}$ and $\tilde{x} \notin B^{U^1_{R^2}}$. Let $\tilde{R} \in \mathcal{R}$ satisfy $p(\tilde{R}) = \tilde{x}$. By Lemma 4.8, for each $j \in M$, there exist $a_j, b_j \in [0, \Omega_j]$ such that $B^{U^i_R} = \prod_{j=1}^m [a_j, b_j]$. Then for each $j \in M$, one of the following three holds;

$$\begin{aligned} &\text{(i)}\ \ \tilde{x}_j < a_j\ \ (\Leftrightarrow \Omega_j - a_j < \Omega_j - \tilde{x}_j), \\ &\text{(ii)}\ \ \tilde{x}_j \in [a_j, b_j]\ \ (\Leftrightarrow \Omega_j - \tilde{x}_j \in [\Omega_j - b_j, \Omega_j - a_j]), \\ &\text{(iii)}\ \ b_j < \tilde{x}_j\ \ (\Leftrightarrow \Omega_j - \tilde{x}_j < \Omega_j - b_j). \end{aligned}$$

Let $y \in B^{U^1_{R^2}}$ be defined by the following; for each $j \in M$, $y_j = a_j$ if (i) holds, $y_j = \tilde{x}_j$ if (ii) holds and $y_j = b_j$ if (iii) holds. Obviously, $\tau(\tilde{R}, B^{U^1_{R^2}}) = \{y\}$. Hence,

$$U^1(\tilde{R}, R^2) = y \ \text{ and } \ g^1(\tilde{R}, R^2) = \tilde{x}.$$

Now let us consider the allotment for individual 2. For all $Y \subseteq \prod_{j=1}^m [0, \Omega_j]$, define $sym(Y) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid \exists x \in Y \text{ s.t. } y = \Omega - x\}$. Let $R^* \in \mathcal{R}$ be such that $p(R^*) = \Omega - p(R^2)$. Then, by the definition of the generalized uniform rule, $\Omega - p(R^2) = U^1(R^*, R^2)$. Hence, $\Omega - p(R^2) \in B^{U^1_{R^2}}$. Hence $p(R^2) \in sym(B^{U^1_{R^2}}) = \prod_{j=1}^m [\Omega_j - b_j, \Omega_j - a_j]$. By the definition of y and $\tilde{x}$, for each $j \in M$,

$$\begin{aligned} \text{(i)} \ &\Rightarrow\ \Omega_j - y_j = \Omega_j - a_j, \\ \text{(ii)} \ &\Rightarrow\ \Omega_j - y_j = \Omega_j - \tilde{x}_j, \\ \text{(iii)} \ &\Rightarrow\ \Omega_j - y_j = \Omega_j - b_j. \end{aligned}$$

Since $p_j(R^2) \in [\Omega_j - b_j, \Omega_j - a_j]$ for all $j \in M$,

$$\forall j \in M, p_j(R^2) \leq \Omega_j - y_j \leq \Omega_j - \tilde{x}_j \ \vee \ \Omega_j - \tilde{x}_j \leq \Omega_j - y_j \leq p_j(R^2).$$

Because $\tilde{x} \neq y$, $\Omega - \tilde{x} \neq \Omega - y$. By the single-peakedness of R^2, $(\Omega - y)P(R^2)(\Omega - \tilde{x})$. Feasibility implies that

$$U^2(\tilde{R}, R^2) = \Omega - y \ \text{ and } \ g^2(\tilde{R}, R^2) = \Omega - \tilde{x}.$$

However, this contradicts that g dom U. □

LEMMA 4.9. *Suppose that f satisfies SP, ER and WP. Then U dom f.*

Proof. We show that $B^{f^i_R} \subseteq B^{U^i_R}$ for all $i \in N$ and all $R \in \mathcal{R}$. Then we obtain the conclusion by Lemma 4.4. Without loss of generality, suppose that $i = 1$. Let $R^2 \in \mathcal{R}$. Then *ER* and the feasibility condition imply that $B^{f^1_{R^2}} \subseteq sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))$.

Step 1 $\forall \tilde{R}^2 \in \mathcal{R}, \left[p(\tilde{R}^2) = p(R^2) \text{ and } \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}) \subseteq \mathrm{UC}(R^2, \frac{\Omega}{2}) \Rightarrow B^{f^1_{R^2}} \subseteq sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))\right]$.

Suppose not. We have $\tilde{R}^2 \in \mathcal{R}$ such that $p(\tilde{R}^2) = p(R^2)$, $\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}) \subseteq \mathrm{UC}(R^2, \frac{\Omega}{2})$ and $B^{f^1_{R^2}} \nsubseteq sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))$. Then there exists a consumption bundle x such that $x \in B^{f^1_{R^2}}$ and $x \notin sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))$. Obviously we can take $R_x \in \mathcal{R}$ such that $p(R_x) = x$. By Lemma 4.3, $f(R_x, R^2) = (x, \Omega - x)$. Since $B^{f^1_{\tilde{R}^2}} \subseteq sym(\mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2}))$, then $x \notin B^{f^1_{\tilde{R}^2}}$. Hence, $f(R_x, \tilde{R}^2) \neq (x, \Omega - x)$. Then, f satisfies *WP*, a contradiction.

Step 2 $\forall x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))\backslash B^{U^1_{R^2}}$, $\exists \tilde{R}^2 \in \mathcal{R}$ s.t. $p(\tilde{R}^2) = p(R^2)$ and $\Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$.

For each $j \in M$, define

$$a_j = \begin{cases} \Omega_j - p_j(R^2) & \text{if } \frac{\Omega_j}{2} \leq p_j(R^2), \\ \frac{\Omega_j}{2} & \text{otherwise,} \end{cases} \qquad b_j = \begin{cases} \frac{\Omega_j}{2} & \text{if } \frac{\Omega_j}{2} \leq p_j(R^2), \\ \Omega_j - p_j(R^2) & \text{otherwise,} \end{cases}$$

then $B^{U^1_{R^2}} = \prod_{j=1}^m [a_j, b_j]$. Hence, $sym(B^{U^1_{R^2}}) = \prod_{j=1}^m [\Omega_j - b_j, \Omega_j - a_j]$.

Let $x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))\backslash B^{U^1_{R^2}}$. Note that $x \neq \frac{\Omega}{2}$ because $\frac{\Omega}{2} \in B^{U^1_{R^2}}$. Hence $\Omega - x \neq \frac{\Omega}{2}$. We show the following by contradiction.

$$\neg\Big[\forall j \in M, \frac{\Omega_j}{2} \leq \Omega_j - x_j \leq p_j(R^2) \;\vee\; p_j(R^2) \leq \Omega_j - x_j \leq \frac{\Omega_j}{2}\Big]. \tag{8}$$

Suppose not. Then for all $j \in M$, if $\frac{\Omega_j}{2} \leq \Omega_j - x_j \leq p_j(R^2)$, then $\Omega_j - p_j(R^2) \leq x_j$ and $x_j \leq \frac{\Omega_j}{2}$. This is equivalent to $x_j \in [a_j, b_j]$. Similarly we can show that for all $j \in M$, if $p_j(R^2) \leq \Omega_j - x_j \leq \frac{\Omega_j}{2}$, then $x_j \in [a_j, b_j]$. Hence we have shown that $x \in B^{U^1_{R^2}}$, a contradiction. We have (8).

By Lemma 4.7, there exists $\tilde{R}^2 \in \mathcal{R}$ such that

$$p(\tilde{R}^2) = p(R^2) \text{ and } \frac{\Omega}{2} P(\tilde{R}^2)(\Omega - x).$$

Now we show $B^{f^1_{R^2}} \subseteq B^{U^1_{R^2}}$ by contradiction. Suppose that we have a consumption bundle x in $B^{f^1_{R^2}}$ but not in $B^{U^1_{R^2}}$. Then *ER* and the feasibility condition imply that $x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))$. By Step 2, we have a preference $\tilde{R}^2 \in \mathcal{R}$ such that $p(\tilde{R}^2) = p(R^2)$ and $\Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$. Let R_x be a preference whose peak is x. Then, by Lemma 4.3 and feasibility condition $f(R_x, R^2) = (x, \Omega - x)$. By *WP*, $f(R_x, \tilde{R}^2) = (x, \Omega - x)$. $f^2(R_x, \tilde{R}^2) = \Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$ but f satisfies *ER*, a contradiction. □

Proof of Theorem 3.2. Obviously U satisfies *ER* and *WP*. By Theorem 3.1, U satisfies *SSESP* which is stronger requirement than *WSESP*. Next, we show the converse. Suppose that f satisfies *WSESP*, *WP* and *ER*. By Lemma 4.9, U dom f. Since f satisfies *WSESP*, f dom U. Since U satisfies *SSESP*, $f = U$. □

5 Concluding Remarks

In an environment with multiple commodities and single-peaked preferences, it is known that there is no allocation rule satisfying strategy-proofness, Pareto efficiency, and egalitarian rationality (Proposition 2.1 and Amorós (2002)). In contrast to this well-known fact, in this paper, we proposed second-best efficiency concepts and showed that WSESP is compatible with strategy-proofness and egalitarian rationality in an n person m good economy (Theorem 2.2).

In addition, we showed that in a two person m good economy, the generalized uniform rule is the only rule that satisfies *ER*, *WP*, and *WSESP* (Theorem 3.2). As a conclusion, we present some open questions concerning second-best efficiency.

Whether the generalized uniform rule in economies with n person and m good satisfies *WSESP* is still open.

Furthermore, in pure exchange economies with non-satiated preferences, we may raise a similar question. Barberà and Jackson (1995) investigate some incentive issues of the exchange economies in which the prices are rigid. To study whether their strategy-proof allocation rules are second-best efficient seems to be a very important issue. In fact, Sasaki (2006) shows that fixed-price trading [29] satisfies SSESP in a two person two good pure exchange economy. Extending Sasaki's result to the n person m good setting would be interesting. In addition, Barberà and Jackson (1995) investigate a class of rules called fixed-proportion trading, a generalization of fixed-price trading, and characterize the rules with strategy-proofness and individual rationality. However, the class of fixed-proportional trading rules includes some apparently inefficient rules. (For example, the trivial rule, which is the rule that assigns the initial endowment, is included in the class of fixed-proportion trading rules.) With the second-best efficiency concepts proposed in this paper, we may characterize a subclass of fixed-proportion trading rules.

More generally, as discussed in the *Introduction*, it is known that there are several impossibility theorems suggesting the existence of a trade-off between Pareto efficiency and strategy-proofness in various kinds of resource allocation problems. If the requirement of Pareto efficiency is weakened as was done in the present paper, the same kind of positive results as in this paper may be obtained in different settings.

Appendix

In the appendix, we prove Proposition 2.1. Lemma A.1 says that if the peak of a preference R is Ω, then we can easily find an indifference curve of R that intersects with all rays on $\mathbb{R}^m_+$ from the origin.

LEMMA A.1. *Suppose that $R \in \mathcal{R}$ satisfies $p(R) = \Omega$. Then*

$$\exists x_0 \in \prod_{j=1}^{m} [0, \Omega_j] \backslash \{0\} \; s.t. \forall h \in \{x \in \mathbb{R}^m_+ \mid ||x|| = 1\}, (UC(x_0, R) \cap LC(x_0, R)) \cap \ell(0, h) \neq \emptyset,$$

where $\ell(0, h) = \{\mu h \in \mathbb{R}^m \mid \mu \in \mathbb{R}_+\}$.

Proof. For each $j \in M$, o_j denotes the vector defined by

[29] See Barberà and Jackson (1995).

$$o_{jk} = \begin{cases} \Omega_j & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Let u denote a continuous utility representation of R. Since $\text{co}\{o_1, \cdots, o_m\}$[30] is compact,

$$\exists x_0 \in \text{co}\{o_1, \cdots, o_m\} \text{s.t.} \forall x \in \text{co}\{o_1, \cdots, o_m\}, u(x) \geq u(x_0).$$

Since R is strictly monotonic on $\prod_{j=1}^m [0, \Omega_j]$, $x_0 \neq 0$.

Let h be in $\{x \in \mathbb{R}_+^m \mid \|x\| = 1\}$. Obviously, $\ell(0, h) \cap \text{co}(\{o_1, \cdots, o_m\}) \neq \emptyset$. Pick $y \in \ell(0, h) \cap \text{co}(\{o_1, \cdots, o_m\})$ arbitrarily. Then we have

$$u(y) \geq u(x_0) > u(0).$$

Since $\text{co}(\{0, y\})$ is connected, by the intermediate value theorem,

$$\exists z \in \text{co}(\{0, y\}) \text{ s.t. } u(z) = u(x_0).$$

Obviously, z is an element of $(\text{UC}(x_0, R) \cap \text{LC}(x_0, R)) \cap \ell(0, h)$. □

REMARK A.1. *Note that $(UC(x_0, R) \cap LC(x_0, R)) \cap \ell(0, h)$ is singleton because R is strictly monotonic on $\prod_{j=1}^m [0, \Omega_j]$.*

Lemma A.2 states that if the peak of a preference R is Ω, then we have an utility representation of R whose values on the diagonal line of the consumption set depends only on the distance from the origin.

LEMMA A.2. *Suppose that $R \in \mathcal{R}$ satisfies $p(R) = \Omega$. Then there exists a continuous utility representation u of R that satisfies*

$$\forall x \in co(\{0, \Omega\}),\ u(x) = \|x\|$$

Proof. Obvious. □

Lemma A.3 guarantees that if a single-peaked preference R with $p(R) = \Omega$ is "homothetic" on $\prod_{j=1}^m [0, \Omega_j]$ can be extended to a strictly monotonic homothetic preference on $\mathbb{R}_+^m$. Note that for each strictly monotonic homothetic preference on $\mathbb{R}_+^m$, its restriction on $\prod_{j=1}^m [0, \Omega_j]$ is a "homothetic" single-peaked preference with $p(R) = \Omega$.

LEMMA A.3. *Let $R \in \mathcal{R}$ be a preference such that*

(1) $p(R) = \Omega$,

(2) $\forall x, x' \in \prod_{j=1}^m [0, \Omega_j]$, $\forall \lambda \in \mathbb{R}_+$, $[xI(R)x'$ and $\lambda x, \lambda x' \in \prod_{j=1}^m [0, \Omega_j] \Rightarrow (\lambda x)I(R)(\lambda x')]$.

Then, there exists $\tilde{R} \subseteq \mathbb{R}_+^m \times \mathbb{R}_+^m$ such that

(1') $\tilde{R}$ is complete[31], transitive[32], continuous[33], strictly monotonic[34] and homothetic[35],

[30] $\text{co}(\{o_1, \cdots, o_m\})$ denotes the convex hull of $\{o_1, \cdots, o_m\}$

[31] For all $x, y \in \mathbb{R}_+^m$, xRy or yRx.

[32] For all $x, y, z \in \mathbb{R}_+^m$, xRy and yRz implies xRz.

[33] For all $x \in \mathbb{R}_+^m$, $\text{UC}(x, R)$ and $\text{LC}(x, R)$ are closed sets in $\mathbb{R}_+^m$.

[34] For all $x, y \in \mathbb{R}_+^m (x \neq y)$, if $x \geq y$, then $xP(R)y$.

[35] For all $x, y \in \mathbb{R}_+^m$, all $\lambda \in \mathbb{R}_+$, $xI(R)y$ implies $(\lambda x)I(R)(\lambda y)$.

(2') $\tilde{R}\mid_{\prod_{j=1}^m[0,\Omega_j]\times\prod_{j=1}^m[0,\Omega_j]} = R$.
Moreover $\tilde{R}$ is unique.

Proof. Let $u : \prod_{j=1}^m[0,\Omega_j] \to \mathbb{R}$ be a continuous utility representation of R whose values on $\text{co}(\{0,\Omega\})$ are distance from the origin. Suppose that the indifference curve that includes $x_0 \in \prod_{j=1}^m[0,\Omega_j]\backslash\{0\}$ intersects with all rays from the origin ($\because$ Lemma A.1). Without loss of generality, we can suppose that $x_0 \in \ell(0,\Omega)$. For each $h \in \{x \in \mathbb{R}_+^m \mid \|x\| = 1\}$, let x_h be the point in $(\text{UC}(x_0,R) \cap \text{LC}(x_0,R)) \cap \ell(0,h)$.[36]

Define $\tilde{u} : \mathbb{R}_+^m \to \mathbb{R}$ by the following. $\tilde{u}(0) = 0$ and for each $x \in \mathbb{R}_+^m\backslash\{0\}$,

$$\tilde{u}(x) = \frac{\|x\|}{\|x_{h(x)}\|}\|x_0\|,$$

where $h(x) = \frac{x}{\|x\|}$. Let $\tilde{R} \subseteq \mathbb{R}_+^m \times \mathbb{R}_+^m$ be the binary relation defined by for all $x, y \in \mathbb{R}_+^m$,

$$x\tilde{R}y \Leftrightarrow \tilde{u}(x) \geq \tilde{u}(y).$$

$\tilde{R}\mid_{\prod_{j=1}^m[0,\Omega_j]\times\prod_{j=1}^m[0,\Omega_j]} = R$ is trivial. $\tilde{R}$ is trivially homothetic, too. $\tilde{R}$ is complete and transitive because $\tilde{u}$ is a real-valued function.

Claim 1. $\tilde{R}$ is continuous.

To this end, we show that $\tilde{u}$ is continuous function. Let $\tilde{\tilde{u}} : \mathbb{R}_+^m \to \mathbb{R}$ be the function such that $\tilde{\tilde{u}}(0) = 0$ and $\tilde{\tilde{u}}(x) = \frac{\|x\|}{\|x_{h(x)}\|}$ for all $x \in \mathbb{R}_+^m\backslash\{0\}$. Obviously, if $\tilde{\tilde{u}}$ is continuous, then $\tilde{u}$ is continuous because $\|x_0\|$ is constant. Hence, we show that $\tilde{\tilde{u}}$ is continuous. Note that h is continuous on $\mathbb{R}_+^m\backslash\{0\}$ since norm is a continuous function. By Remark A.1, the restriction of h on $\text{UC}(x_0,R)\cap\text{LC}(x_0,R)$ is a bijection. Hence, $h\mid_{\text{UC}(x_0,R)\cap\text{LC}(x_0,R)}$ is a homeomorphism between $\text{UC}(x_0,R) \cap \text{LC}(x_0,R)$ and $\{x \in \mathbb{R}_+^m \mid \|x\| = 1\}$.[37] Let $f = h\mid_{\text{UC}(x_0,R)\cap\text{LC}(x_0,R)}$. Let g be the Euclidean norm. Then $\tilde{\tilde{u}} = \frac{g}{g\circ f^{-1}\circ h}$ on $\mathbb{R}_+^m\backslash\{0\}$. Hence, $\tilde{\tilde{u}}\mid_{\mathbb{R}_+^m\backslash\{0\}}$ is continuous. Note that we have shown that $\tilde{\tilde{u}} : \mathbb{R}_+^m \to \mathbb{R}$ is continuous at each $x \in \mathbb{R}_+^m\backslash\{0\}$.[38]

Next we show that $\tilde{\tilde{u}}$ is continuous at 0. Let $\{x_n\}$ be a convergent sequence in $\mathbb{R}_+^m$ whose limit is 0. First, we suppose $\{x_n\}$ is a sequence in $\mathbb{R}_+^m\backslash\{0\}$. Note that

$$\exists\alpha \in \mathbb{R}_+ \text{ s.t. } \forall x \in \text{UC}(x_0,R) \cap \text{LC}(x_0,R), \alpha \geq \|x\|,$$

since $\text{UC}(x_0,R) \cap \text{LC}(x_0,R)$ is compact. Hence, $\tilde{\tilde{u}}(x_n) \to 0$ as $n \to +\infty$.[39] Now we consider the general case. First, we show that $\{\tilde{\tilde{u}}(x_n)\}$ is bounded. Since the sequence is defined on $\mathbb{R}_+$, we show that the sequence is bounded above. Suppose not. Then we have a increasing subsequence $\{\tilde{\tilde{u}}(x_{k(n)})\}$ of $\{\tilde{\tilde{u}}(x_n)\}$ that diverges. Since $\tilde{\tilde{u}}(x) = 0$ if and only if $x = 0$, we can suppose that $\{x_{k(n)}\}$ is in $\mathbb{R}_+^m\backslash\{0\}$. We have shown that

[36] See Remark A.1.

[37] If a bijection from a compact space to a Hausdorff space is continuous, then it is a homeomorphism.

[38] Let $\{x_n\}$ be a convergent sequence in $\mathbb{R}_+^m$ whose limit is $x' \in \mathbb{R}_+^m\backslash\{0\}$. Since every subsequence of a convergent sequence converges to the limit of original sequence, x_n is equal to 0 at most finitely many $n \in \mathbb{N}$. Hence, without loss of generality, we can suppose that $\{x_n\}$ is a convergent sequence in $\mathbb{R}_+^m\backslash\{0\}$. Since we have shown that $\tilde{\tilde{u}}\mid_{\mathbb{R}_+^m\backslash\{0\}}$ is continuous, $\tilde{\tilde{u}}(x_n) \to \tilde{\tilde{u}}(x')$. This proves that $\tilde{\tilde{u}}$ is continuous at every $x' \in \mathbb{R}_+^m\backslash\{0\}$.

[39] Note that $\alpha\|x_n\| \to 0$ as $n \to +\infty$. Note also that $\alpha\|x_n\| \geq \tilde{\tilde{u}}(x_n)$ for all $n \in \mathbb{N}$.

for any convergent sequence on $\mathbb{R}^m_+ \backslash \{0\}$ with $x_{k(n)} \to 0$, $\{\tilde{\tilde{u}}(x_{k(n)})\} \to 0$, a contradiction. Hence, $\{\tilde{\tilde{u}}(x_n)\}$ is bounded. Suppose that $\{\tilde{\tilde{u}}(x_n)\}$ is in $[0, \delta]$, where $\delta > 0$ is a sufficiently large real number. We show that $\{\tilde{\tilde{u}}(x_n)\}$ converges to a number in $[0, \delta]$. Suppose the contrary. Then $\{\tilde{\tilde{u}}(x_n)\}$ has at least two limit points in $[0, \delta]$.[40] Suppose that $\beta, \gamma \in [0, \delta]$ with $\beta \neq \gamma$ are limit points of $\{\tilde{\tilde{u}}(x_n)\}$. Then at least one of β and γ is not 0. Without loss of generality, suppose that β is not 0. Then, we have a subsequence $\{\tilde{\tilde{u}}(x_{k'(n)})\}$ that converges to β. We can easily construct a subsequence $\{\tilde{\tilde{u}}(x_{t(k'(n))})\}$ of $\{\tilde{\tilde{u}}(x_{k'(n)})\}$ that satisfies

$$\forall n \in \mathbb{N}, x_{t(k'(n))} \in \mathbb{R}^m_+ \backslash \{0\},$$

a contradiction. Hence, $\{\tilde{\tilde{u}}(x_n)\}$ converges to a number in $[0, \delta]$. If the limit of $\{\tilde{\tilde{u}}(x_n)\}$ is not equal to 0, we have a contradiction same as above. This completes the proof of claim 1.

Claim 2. $\tilde{R}$ is strictly monotonic.

To this end, we show that $\tilde{u}$ is strictly monotonic. Note that by the definition of $\tilde{u}$, $\tilde{u}$ is strictly monotonic on all rays from the origin. Let $x, x' \in \mathbb{R}^m_+$ satisfy $x \geq x'$ and $x \neq x'$. If $x' = 0$, then $\tilde{u}(x) > \tilde{u}(x')$. We suppose $x' \neq 0$ below. For sufficiently small $\lambda > 0$, $\lambda x, \lambda x' \in \prod_{j=1}^m [0, \Omega_j]$, $\lambda x \geq \lambda x'$ and $x \neq x'$. By (2'), $\tilde{u}(\lambda x) > \tilde{u}(\lambda x') > \tilde{u}(0)$. By the intermediate value theorem, we have $y \in \text{co}(\{0, \lambda x\})$ such that $\tilde{u}(y) = \tilde{u}(\lambda x')$. Since $\tilde{u}$ is strictly monotonic on $\ell(0, x)$, $\lambda x \geq y$ and $\lambda x \neq y$. Hence, $\tilde{u}(x) > \tilde{u}(\frac{y}{\lambda})$ because $x \geq \frac{y}{\lambda}$ and $x \neq \frac{y}{\lambda}$. On the other hand, $\tilde{u}(x') = \tilde{u}(\frac{y}{\lambda})$ since $\tilde{R}$ is homothetic. This completes the proof of claim 2.

The uniqueness of $\tilde{R}$ is obvious by (2'). We complete the proof of Lemma A.3. □

Proof of Proposition 2.1. Obvious from Lemma A.3 and Theorem in Serizawa (2002). □

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[40] $x' \in \mathbb{R}$ is a limit point of a sequence $\{x_n\}$ in $\mathbb{R}$ if there exists a convergent subsequence of $\{x_n\}$ whose limit is x'. Here, we have subsequences that converges to lim sup and lim inf of $\{\tilde{\tilde{u}}(x_n)\}$.

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