Who meets the standards? A multidimensional Approach[1]

Antonio Villar

Pablo de Olavide University and *Ivie*

Abstract

We consider here the evaluation of the performance of a society with respect to a given set of targets. We provide a characterization of an intuitive evaluation formula that consists of the mean of the shares of the achievements in the targets. The criterion so obtained permits one not only to endogenously determine who meets the standards and who does not, but also to quantify the degree of fulfillment. An empirical illustration is provided, considering the compliance of the European Union Stability and Growth Pact.

Key-words: Meeting the standards, Bonus/Malus criterion, multidimensional targets, generalized additive monotonicity.

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Contact address: avillar@upo.es

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1 Introduction

Consider an organization consisting of several units whose performance is to be evaluated with respect to a vector of targets or reference values previously set. This is a very general scenario that may appear under many different formats. Depending on the problem under consideration, those targets may represent absolute values (objectives to be reached or admissible levels externally given), relative performance thresholds (points in the actual distribution of achievements), or a mixture of them (as in the EU "convergence criteria" for the monetary union, that involved absolute thresholds concerning deficit and public debt, and relative thresholds, concerning inflation and interest rates). We can think that the purpose of the evaluation is the allocation of some resources among those who qualify (productivity bonuses, financial aids, or social transfers, say) and/or prestige or recognition (e.g. identifying high quality institutions). The evaluation procedure itself may be conceived as a simple dichotomous criterion (determining who meets the standards as a bonus/malus classification), it may attempt at providing quantitative estimates of the overall degree of fulfilment (e.g. the evaluation of individual or collective achievements), or something in between (e.g. classification in different categories).

We shall refer to the organization as a *society* and to the incumbent units as *agents*. The key feature of the problem is the existence of a society with many agents whose performance is to be evaluated with respect to a given set of multidimensional targets, to be called *standards*. Note that in some cases meeting the standards may imply values below the thresholds (e.g. when we consider the case of financial aid, or insurance premiums).

Deciding who meets the standards in a multidimensional scenario is not immediate. Two extreme positions can be considered. On the one hand, there is the most demanding interpretation by which meeting the standards means achieving all target values simultaneously. On the other hand, there is the other extreme interpretation according to which achieving some target is a sufficient criterion. Each of those polar views makes the decision on who meets the standards rather trivial. The drawback is that in both cases we may find very unfair outcomes, as we can be treating equally highly different performances. The difficult problem is, of course, how to handle the intermediate cases. That is, when agents in the society exceed some of the prescribed targets but fail to reach some others (a relevant case in practice and a usual source of conflicts). The bottom line is whether we admit or not compensations among achievements, both across dimensions and across agents, and what kind of compensations should be considered.

The following examples illustrate different situations in which this problem arises.

Example 1 *A firm consisting of a number of different branches fixes some objectives for the year in order to reward those employees in the branches that comply. Those objectives may refer to total sales, profits, consumers' satisfaction, shares of the local markets, etc.*

Example 2 The *European Central Bank is willing to publicly acknowledge those private banks that are "good". For that it establishes a series of performance standards in terms of the variables that measure solvency, liquidity, efficiency and risk. Those banks that meet the standards get a star.*

Example 3 *A University is providing new chairs to reinforce those Departments that exhibit a better research performance. To do so it fixes some relative thresholds (quantiles) with respect to their achievements concerning publications in refereed journals, publications in the very top journals, research funds raised, and patents obtained. Those Departments that meet the standards will have an extra chair.*

Those examples illustrate three different situations that correspond to the problem under consideration. They involve differences in the type of evaluation required. In Example 1 the firm typically needs not only a criterion to decide which branches deserve the productivity bonus but also a measure of the degree of success, in order to differentiate the incentives received by those branches that qualify. Examples 2 and 3 basically require a dichotomous classification (a bonus/malus case). The difference is that in Example 2 the standards are fixed externally (absolute thresholds) whereas in Example 3 the standards are relative to the actual performance. In Examples 1 and 2 we can also consider the question of whether some specific objectives have been reached (e.g. are total sales of the company good enough?, Is the European banking system efficient?). This type of question is meaningless in Example 3.

We approach this evaluation problem here by using the intuition and principles that are applied for the analysis of development, inequality and poverty. Roughly speaking development measures allow to estimate the achievements, the targets play a similar role to the poverty thresholds, and inequality enters the picture as measuring the degree of substitutability among the achievements. Therefore, even though we shall not discuss any particular model, our background references are Alkire and Foster (2008), Bourguigon and Chakravarty (2003), Chakravarty (2003), Foster, Greer and Thorbecke (1984), Foster, López-Calva, and Székely (2005), Goerlich and Villar (2009), Herrero, Martínez and Villar (2010), Seth (2009), (2010), Tsui (2002), and Villar (2010).

The paper is organized as follows. Section 2 contains the basic model. We present there the key assumptions and the essential ideas of this contribution by means of a simple and intuitive evaluation function: the arithmetic mean of the shares of the achievements in the targets. The axioms we use for that are rather standard: *weighted anonymity* (the names of the agents to not matter), *neutrality* (all dimensions are equally important), *normalization* (the inf of the index is zero and it is equal to one when all achievements coincide with the targets), *linear homogeneity* (doubling all the achievements without changing the targets implies doubling the value of the index), and *additive monotonicity* (an increase in a particular achievement determines an increase in the evaluation function that depends positively on the size of that increment in that achievement). Section 3 provides an illustration of this approach by analyzing the performance of the countries in the Eurozone, regarding the EU Stability and Growth Pact. This exercise points out the convenience of a more flexible evaluation model, allowing for different degrees of substitutability between dimensions. That venue is dealt with in Section 4, where we characterize the uniparametric family of generalized means by weakening additive monotonicity to *generalized additive monotonicity* (the changes in the function due to an increase in a particular achievement are governed by a power function). Some final comments are gathered in Section 5.

2 The basic model

Let $N = \{1,2,...,n\}$ denote a *society* with n *agents* and let $K = \{1,2,...,k\}$ be a set of *characteristics*. Each characteristic corresponds to a variable that approximates one relevant dimension of the performance of the society under consideration. A *realization* is a matrix $Y = \{y_{ij}\}$ with n rows, one for each agent, and k columns, one for each dimension. The entry $y_{ij} \in \mathbb{R}$ describes the value of variable j for agent i. Therefore, $\mathbb{R}^{nk}$ is the space of realization matrices and we assume implicitly that all dimensions can be approximated quantitatively by real numbers.

There is a parameter vector of reference values $\mathbf{z} \in \mathbb{R}^k_{++}$ that describes the standards fixed for the different dimensions. Those standards may be set externally (absolute thresholds) or may depend on the data of the realization matrix itself (relative thresholds, such as a fraction of the median or the mean value). We shall not discuss here how those thresholds are set, even though the importance of that choice is more than evident.

Finally, to deal with agents of different sizes (e.g. families, firms, regions, countries), there is a vector $\boldsymbol{\rho} \in \mathbb{R}^n_{++}$ that tells us the weights with which the different agents enter into the evaluation.

An *evaluation problem*, or simply a *problem,* is a triple $(Y,\mathbf{z},\boldsymbol{\rho})$. We denote by $N_M(Y,\mathbf{z},\boldsymbol{\rho}),\ N_{DM}(Y,\mathbf{z},\boldsymbol{\rho}) \subset N$ the sets of agents that *meet* and that *do not meet* the standards.

2.1 Measuring the achievements

In order to evaluate the overall achievements of the society with a realization matrix Y, relative to the reference vector $\mathbf{z}$, and a weighting vector ρ, we define a function φ : $\mathbb{R}^{nk} \times \mathbb{R}^k_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}$ that associates to each problem $(Y,\mathbf{z},\boldsymbol{\rho})$ a real value $\varphi(Y,\mathbf{z},\boldsymbol{\rho})$ that provides a measure of its performance. This function is determined by a set of intuitive and reasonable properties that we introduce next.

The first property we consider, *weighted anonymity*, establishes that all weighted agents are treated alike. That is, if we permute agents' realization vectors together with their associated weights, the evaluation does not change. To formalize this idea we let $(Y,\boldsymbol{\rho})$ denote the $n \times (k+1)$ matrix resulting from adding vector ρ as an additional column to matrix Y.

- **Weighted Anonymity**: Let $(Y,\mathbf{z},\rho) \in \mathbb{R}^{nk} \times \mathbb{R}^{k}_{++} \times \mathbb{R}^{n}_{++}$ and let $p^{R}(Y,\rho)$ denote a permutation of the rows of the enlarged matrix (Y,ρ). Then, $\varphi(Y,\mathbf{z},\rho) = \varphi(p^{R}(Y,\rho),\mathbf{z})$.

The second property, *neutrality*, says that all dimensions are equally important. That is:

- **Neutrality**: Let $(Y,\mathbf{z},\rho) \in \mathbb{R}^{nk} \times \mathbb{R}^{k}_{++} \times \mathbb{R}^{n}_{++}$ and let $p^{C}(Y)$ denote a permutation of the columns of Y. Then, $\varphi(Y,\mathbf{z},\rho) = \varphi(p^{C}(Y),\mathbf{z},\rho)$.

The third property, *normalization*, makes the value of the index equal to zero when $Y = 0$ (i.e. when $y_{ij} = 0$, for all i,j) and makes it equal to $\sum_{i \in N} \rho_i$ when $Y = Z$ (i.e. when $y_{ij} = z_j$ for all i,j). Formally:[2]

- **Normalization**: $\varphi(0,\mathbf{z},\rho) = 0$, $\varphi(Z,\mathbf{z},\rho) = \sum_{i \in N} \rho_i$.

The fourth property, *linear homogeneity*, says that a proportional change in the realization values, with unchanged targets and weights, results in the same proportional change in the index. That is,

- **Linear homogeneity**: For all $(Y,\mathbf{z},\rho) \in \mathbb{R}^{nk} \times \mathbb{R}^{k}_{++} \times \mathbb{R}^{n}_{++}$, all $\lambda > 0$, $\varphi(\lambda Y,\mathbf{z},\rho) = \lambda \varphi(Y,\mathbf{z},\rho)$.

Our last property, *additive monotonicity,* establishes conditions on the behaviour of the function when agent i's achievement in the jth dimension changes from y_{ij} to $y'_{ij} > y_{ij}$. It requires the change of φ to be a monotone function of the increment in the variable, conditional on the corresponding threshold level. Formally:

- **Additive Monotonicity**: Let $Y, Y' \in \mathbb{R}^{nk}$ be such that $y'_{ij} > y_{ij}$, $y_{ht} = y'_{ht}$, for all $h \neq i$, all $t \neq j$. Then,
$$\varphi(Y',\mathbf{z},\rho) - \varphi(Y,\mathbf{z},\rho) = b_{ij}\left[\left(y'_{ij} - y_{ij}\right), z_j, \rho_i\right]$$
for some function $b_{ij} : \mathbb{R}^{3}_{++} \to \mathbb{R}_{++}$, increasing in its first argument.

Note that this monotonicity requirement is cardinal in nature and involves a separability feature of the overall index.

The following result shows that all those requirements yield an evaluation

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[2] Generally $\sum_{i \in N} \rho_i = 1$. Yet we need to define this property in general terms in order to be able to use it in the characterization result.

function that corresponds to the arithmetic mean of the weighted shares of the achievements in the targets. Formally:

Theorem 1: *An index* $\varphi : \mathbb{R}^{nk} \times \mathbb{R}^{k}_{++} \times \mathbb{R}^{n}_{++}$ *satisfies weighted anonymity, neutrality, normalization, linear homogeneity, and additive monotonicity, if and only if it takes the form:*

$$\varphi(Y,\mathbf{z},\boldsymbol{\rho}) = \frac{1}{k}\sum_{j\in K}\sum_{i\in N}\rho_i \frac{y_{ij}}{z_j} \qquad [1]$$

Moreover, those properties are independent.

<u>Proof</u>

(i) The function in [1] satisfies all those properties. We prove now the converse.

Let $(Y,\mathbf{z},\boldsymbol{\rho}) \in \mathbb{R}^{nk} \times \mathbb{R}^{k}_{++} \times \mathbb{R}^{n}_{++}$. By additive monotonicity and normalization we can write:

$$\varphi\left(\begin{pmatrix} y_{11} & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{pmatrix}, \mathbf{z},\boldsymbol{\rho}\right) = \varphi\big(0,\ \mathbf{z},\boldsymbol{\rho}\big) + b_{11}(y_{11},z_1,\rho_1) = b_{11}(y_{11},z_1,\rho_1)$$

$$\varphi\left(\begin{pmatrix} y_{11} & y_{12} & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{pmatrix}, \mathbf{z},\boldsymbol{\rho}\right) = \varphi\left(\begin{pmatrix} y_{11} & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{pmatrix}, \mathbf{z},\boldsymbol{\rho}\right) + b_{12}(y_{12},z_2,\rho_1)$$

$$= b_{11}(y_{11},z_1,\rho_1) + b_{12}(y_{12},z_2,\rho_1)$$

$$.....$$

$$.....$$

$$\varphi(Y,\mathbf{z},\boldsymbol{\rho}) = \sum_{j\in K}\sum_{i\in N} b_{ij}\left(y_{ij},z_j,\rho_i\right)$$

By weighted anonymity and neutrality, $b_{ij}(.) = b(.)$ for all $i \in N$, all $j \in K$. Let now $[\mathbf{1},\mathbf{1},...,\mathbf{1}]a$ denote a uniform matrix whose generic element is a and take $\mathbf{z} = \mathbf{1}s$, $\boldsymbol{\rho} = \mathbf{1}p$ for some positive scalars s,p where $\mathbf{1}$ is the unit vector in the corresponding space. In this special case we get:

$$\varphi([\mathbf{1},...,\mathbf{1}]a,\ \mathbf{1}s,\mathbf{1}p) = knb(a,s,p)$$

$$\Rightarrow b(a,s,p) = \frac{\varphi([\mathbf{1},...,\mathbf{1}]a,\ \mathbf{1}s,\ \mathbf{1}p)}{kn}$$

Therefore, we conclude:

$$\varphi(Y,\mathbf{z},\boldsymbol{\rho}) = \frac{1}{kn}\sum_{j\in K}\sum_{i\in N}\varphi([\mathbf{1},...,\mathbf{1}]y_{ij},\ \mathbf{1}z_j,\ \mathbf{1}\rho_i) \qquad [2]$$

Let now $f : \mathbb{R}^{3}_{++} \to \mathbb{R}_{++}$ be given by: $f(y_{ij},z_j,\rho_i) := \varphi([\mathbf{1},...,\mathbf{1}]y_{ij},\ \mathbf{1}z_j,\ \mathbf{1}\rho_i)$. By linear homogeneity and normalization, taking $y_{ij} = z_j$ and $\lambda = \frac{y_{ij}}{z_j}$, we have:

$$f\left(\frac{y_{ij}}{z_j}z_j, z_j, \rho_i\right) = n\rho_i \frac{y_{ij}}{z_j}$$

Therefore:

$$\varphi(Y,\mathbf{z},\boldsymbol{\rho}) = \frac{1}{k}\sum_{j\in K}\sum_{i\in N}\rho_i\left(\frac{y_{ij}}{z_j}\right)$$

(ii) To separate the properties let us consider the following indices, for $\rho_i = 1/n$ for all i :

(ii,a) $\varphi^A(Y,\mathbf{z},\mathbf{1}\frac{1}{n}) = \sum_{j\in K}\sum_{i\in N}\frac{y_{ij}}{z_j}$. It satisfies weighted anonymity, neutrality, linear homogeneity, and additive monotonicity but not normalization.

(ii,b) $\varphi^B(Y,\mathbf{z},\mathbf{1}\frac{1}{n}) = \min_{i\in N}\left\{\frac{y_{ij}}{z_j}\right\}$. It satisfies weighted anonymity, neutrality, linear homogeneity, and normalization but not additive monotonicity.

(ii,c) $\varphi^C(Y,\mathbf{z},\mathbf{1}\frac{1}{n}) = \sum_{j\in K}\sum_{i\in N}\omega_i\left(\frac{y_{ij}}{z_j}\right)$, with $\sum_{i\in N}\omega_i = 1$ and $\omega_i \neq 1/n$ for some i. It satisfies neutrality, normalization, linear homogeneity, and additive monotonicity but not weighted anonymity.

(ii,d) $\varphi^D(Y,\mathbf{z},\mathbf{1}\frac{1}{n}) = \sum_{i\in N}\sum_{j\in K}\delta_j\left(\frac{y_{ij}}{z_j}\right)$, with $\sum_{j\in K}\delta_j = 1$ and $\delta_j \neq 1/k$ for some j. It satisfies weighted anonymity, normalization, linear homogeneity, and additive monotonicity but not neutrality.

(ii,e) $\varphi^E(Y,\mathbf{z},\mathbf{1}\frac{1}{n}) = \frac{1}{kn}\sum_{j\in K}\sum_{i\in N}\left(\frac{y_{ij}}{z_j}\right)^2$. It satisfies weighted anonymity, neutrality, normalization, and additive monotonicity but not linear homogeneity.

Q.e.d.

This theorem tells us that assuming weighted anonymity, neutrality, normalization, linear homogeneity, and additive monotonicity amounts to measure social performance as the arithmetic mean of the weighted relative achievements across agents.

It is interesting to observe that, under our assumptions, we have an implicit estimation of the performance of agent i with respect to dimension j, $\pi_{ij}(Y,\mathbf{z},\boldsymbol{\rho})$, that is given by the evaluation of a fictitious society with a uniform realization matrix $[\mathbf{1},...,\mathbf{1}]y_{ij}$, a uniform reference vector $\mathbf{1}z_j$, and a uniform weighting system $\mathbf{1}\rho_i$ (see equation $[2]$). That is,

$$\pi_{ij}(\mathbf{y}_i,\mathbf{z},\boldsymbol{\rho}) = \varphi([\mathbf{1},...,\mathbf{1}]y_{ij},\ \mathbf{1}z_j,\ \mathbf{1}\rho_i) \qquad [3]$$

This allows us to estimate the *overall contribution of an agent*, by simply computing:

$$\vartheta_i(Y,\mathbf{z},\boldsymbol{\rho}) = \frac{1}{k}\sum_{j\in K}\varphi([\mathbf{1},...,\mathbf{1}]y_{ij},\ \mathbf{1}z_j,\ \mathbf{1}\rho_i)$$
$$= \frac{n\rho_i}{k}\sum\nolimits_{j\in K}\frac{y_{ij}}{z_j} \qquad [4]$$

that is, as $n\rho_i$ times the arithmetic mean of all her relative achievements.

Similarly, we can have a measure of the overall success of society in achieving a particular target j, that is given by:[3]

$$\beta_j(Y,\mathbf{z},\boldsymbol{\rho}) = \sum_{i\in N}\varphi([\mathbf{1},...,\mathbf{1}]y_{ij},\ \mathbf{1}z_j)$$
$$= \sum\nolimits_{i\in N}\rho_i\frac{y_{ij}}{z_j} \qquad [5]$$

2.2 *The agents who meet the standards and the targets that have been reached*

Let us now consider the question of who meets the standards and whether we can consider that a given target has been collectively achieved. In our model those problems are solved endogenously by the very formula that measures the overall performance.

First note that, by definition, an agent with $y_{ij} > z_j$, for all j, certainly meets the standards. Take the case in which $y_{hj} = z_j$, for all $j \in K$, some $h \in N$. According to equation $[4]$, the overall performance this agent is given by: $\vartheta_h\left((Y_{-h},\mathbf{z}),\mathbf{z},\boldsymbol{\rho}\right) = n\rho_h$ (where $(Y_{-h},\mathbf{z})$ describes a matrix whose h-th row is precisely $\mathbf{z}$). Therefore, we can establish that an individual meets the standards if and only if: $\vartheta_i(Y,\mathbf{z},\boldsymbol{\rho}) \geq n\rho_i$. From that it follows immediately that the set $N_M(Y,\mathbf{z},\boldsymbol{\rho})$ of agents who meet the standards is given by:

$$N_M(Y,\mathbf{z},\boldsymbol{\rho}) = \left\{ i \in N \ \ / \ \ \frac{1}{k}\sum\nolimits_{j\in K}\frac{y_{ij}}{z_j} \geq 1 \right\} \qquad [6]$$

(note that we allow for the existence of agents in N_M with achievements below the target in some dimension, provided they are compensated with over compliance in other dimensions).

Expression $[6]$ permits one to directly identify the set of those who meet the standards in the k-dimensional space in which we plot on $\mathbb{R}^k$ all agents' vectors of relative achievements, $\mathbf{y}_i(\mathbf{z}) = (y_{i1}/z_1,\ y_{i2}/z_2,\ ...,\ y_{ik}/z_k)$, for all $i \in N$.

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[3] Computing the success in a given dimension makes sense when the thresholds are externally given and may not be meaningful when they correspond to functions of the actual values of the realization matrix.

Indeed, the set $N_M(Y,\mathbf{z},\boldsymbol{\rho})$ is given by all those agents whose vectors of relative achievements are above the hyperplane defined by $\sum_{j\in K} y_{ij}(\mathbf{z}) = k$.

When the reference values $\mathbf{z}\in\mathbb{R}^k_{++}$ are externally given (i.e. they correspond absolute thresholds), we can also consider whether a specific objective has been reached by society. According to equation [5], objective j is achieved provided $\beta_j(Y,\mathbf{z},\boldsymbol{\rho}) \geq \sum_{i\in N}\rho_i = \beta_j\left(\left(Y^{-j},\mathbf{1}z_j\right),\mathbf{z},\boldsymbol{\rho}\right)$, where $(Y^{-j},\mathbf{1}z_j)$ describes a matrix whose jth column is equal to z_j in all entries. Therefore, the set of objectives that have been collectively achieved are those that satisfy the following condition:

$$\sum_{i\in N}\rho_i \frac{y_{ij}}{z_j} \geq \sum_{i\in N}\rho_i\ , \quad j\in K \qquad [7]$$

Remark: *This framework permits one to compare the evaluation criterion presented in Theorem 1 with the two extreme cases mentioned in the Introduction. On the one hand, the case in which an agent meets the standards when she attains all targets simultaneously (that is,* $\min_{j\in K}\left(\frac{y_{ij}}{z_j}\right)\geq 1$). *On the other hand, the case in which an agent meets the standards when she attains some target (that is,* $\max_{j\in K}\left(\frac{y_{ij}}{z_j}\right)\geq 1$). *The same applies to the fulfilment of individual targets in society.*

3. Intermezzo: The European Union Stability and Growth Pact

Let us apply our evaluation formula to a relevant problem: the EU Stability and Growth Pact (SGP). Mind that our aim here is to illustrate the extent and limits of our approach by means of a real-life example, rather than contributing to the substance of the discussion on the fulfilment of that pact.

The SGP is an agreement among the 16 members of the European Union that take part in the Eurozone, to facilitate and maintain the stability of the Economic and Monetary Union. It involves setting reference values for some key public finance variables and aims at enforcing fiscal discipline after the monetary union (member states adopting the euro have to meet the Maastricht convergence criteria, and the SGP ensures that they continue to observe them). The basic reference values are two:

(a) An annual budget deficit no higher than 3% of GDP.

(b) A national debt lower than 60% of GDP.

Let us take those values as the thresholds applicable to evaluate the performance of the states in the Eurozone, ignoring all implementation issues and the re-interpretations and refinements introduced later. Table 1 below provides the data on budget deficit and national debt for the 16 countries in the

Eurozone, between 2006 and 2009. The question is to determine which countries do satisfy those criteria and which do not.

Table 1: Public Debt and deficit in the Euro Area (2006-2009)								
Country	Deficit 2006	Debt 2006	Deficit 2007	Debt 2007	Deficit 2008	Debt 2008	Deficit 2009	Debt 2009
Belgium	-0,3	88,1	0,2	84,2	1,2	89,8	6	96,7
Germany	1,6	67,6	-0,2	65	0	66	3,3	73,2
Greece	3,6	97,8	5,1	95,7	7,7	99,2	13,6	115,1
Spain	-2	39,6	-1,9	36,2	4,1	39,7	11,2	53,2
France	2,3	63,7	2,7	63,8	3,3	67,5	7,5	77,6
Ireland	-3	24,9	-0,1	25	7,3	43,9	14,3	64
Italy	3,3	106,5	1,5	103,5	2,7	106,1	5,3	115,8
Cyprus	1,2	64,6	-3,4	58,3	-0,9	48,4	6,1	56,2
Luxemboourg	-1,4	6,5	-3,6	6,7	-2,9	13,7	0,7	14,5
Malta	2,6	63,7	2,2	61,9	4,5	63,7	3,8	69,1
Netherlands	-0,5	47,4	-0,2	45,5	-0,7	58,2	5,3	60,9
Austria	1,5	62,2	0,4	59,5	0,4	62,6	3,4	66,5
Portugal	3,9	64,7	2,6	63,6	2,8	66,3	9,4	76,8
Slovenia	1,3	26,7	0	23,4	1,7	22,6	5,5	35,9
Slovakia	3,5	30,5	1,9	29,3	2,3	27,7	6,8	35,7
Finland	-4	39,7	-5,2	35,2	-4,2	34,2	2,2	44
Average	1,3	68,3	0,6	66	2	69,4	6,3	78,7
Source: Eurostat (Euroindicators 2010)								

This table suggests two alternative ways of interpreting the evaluation problem. On the one hand, we may consider that satisfying the performance criteria means meeting the standards every single year. In that case we would have four separate evaluation problems. On the other hand, one may also consider the evaluation for the whole period, as the performance of the countries is affected by the economic cycle. In that case we treat deficits and debt data corresponding to different years as different variables. Table 2 provides the summary data of the countries' performance under the two evaluation approaches; each cell in the table corresponds to the value of $\frac{1}{k}\sum_{j\in K}\frac{y_{ij}}{z_j}$ (see equation [6]), for each of the years considered and for the whole period.

The data show that, according to the criterion in [6], there are only two countries that meet the standards year by year between 2006 and 2009: Luxembourg and Finland. There are 7 additional countries that satisfy the criteria when considering the whole period: Germany, Spain, Cyprus, Netherlands, Austria, Slovenia and Slovakia. Greece and Italy are the two countries that do not meet the standards in any of the years considered.

Table 2: Relative Public Debt and deficit in the Euro Area (2006-2009)									
Country	Deficit 2006	Debt 2006	Deficit 2007	Debt 2007	Deficit 2008	Debt 2008	Deficit 2009	Debt 2009	Global
Belgium	-0,10	1,47	0,07	1,40	0,40	1,50	2,00	1,61	1,04
Germany	0,53	1,13	-0,07	1,08	0,00	1,10	1,10	1,22	0,76
Greece	1,20	1,63	1,70	1,60	2,57	1,65	4,53	1,92	2,10
Spain	-0,67	0,66	-0,63	0,60	1,37	0,66	3,73	0,89	0,83
France	0,77	1,06	0,90	1,06	1,10	1,13	2,50	1,29	1,23
Ireland	-1,00	0,42	-0,03	0,42	2,43	0,73	4,77	1,07	1,10
Italy	1,10	1,78	0,50	1,73	0,90	1,77	1,77	1,93	1,43
Cyprus	0,40	1,08	-1,13	0,97	-0,30	0,81	2,03	0,94	0,60
Luxembоourg	-0,47	0,11	-1,20	0,11	-0,97	0,23	0,23	0,24	-0,21
Malta	0,87	1,06	0,73	1,03	1,50	1,06	1,27	1,15	1,08
Netherlands	-0,17	0,79	-0,07	0,76	-0,23	0,97	1,77	1,02	0,60
Austria	0,50	1,04	0,13	0,99	0,13	1,04	1,13	1,11	0,76
Portugal	1,30	1,08	0,87	1,06	0,93	1,11	3,13	1,28	1,34
Slovenia	0,43	0,45	0,00	0,39	0,57	0,38	1,83	0,60	0,58
Slovakia	1,17	0,51	0,63	0,49	0,77	0,46	2,27	0,60	0,86
Finland	-1,33	0,66	-1,73	0,59	-1,40	0,57	0,73	0,73	-0,15
Average	0,43	1,14	0,20	1,10	0,67	1,16	2,10	1,31	1,01
Source: Eurostat (Euroindicators 2010)									

Let us now consider whether the European Union Stability and Growth Pact has been fulfilled collectively concerning the deficit and debt objectives, along the years analyzed in tables 1 and 2. If we let the weight of each country be given by its relative GDP, then the average value that appears in the last line of table 2 corresponds precisely to the list of values of equation [5]. We observe that, taking the two objectives together, there is only one year in which the Eurozone did not satisfy the criteria of the SGP. Yet the deviation was bad enough as to conclude that for the whole Eurozone and the whole period, the pact has not been fulfilled (as $\varphi(.) = 1.01$). Looking at each objective individually, year by year, we observe that the Eurozone failed to satisfy the deficit target in 2008 and 2009, and failed to satisfy the debt target in all the years considered. As for the overall fulfilment of deficit and debt targets in the whole period, we find that the debt target has been reached whereas the Eurozone obviously failed to achieve the deficit target.

4. A more flexible formulation

The additive structure of the evaluation function φ in Theorem 1 implies a particular trade-off between the different achievements, as the evaluation only depends on the sum of the agent's relative realizations but not on their distribution. So each agent can substitute any relative realization for another one at a constant rate no matter the level at which this happens. Similarly, the relative achievements of an agent in a given dimension can be substituted by those of another one, also at a constant rate. One might be willing to consider evaluation criteria in which the dispersion of the relative achievements is also taken into account and/or the degree of substitutability may change with the levels of the relative achievements.

The additive structure of φ depends essentially on the monotonicity property that we have assumed (additive monotonicity). Other monotonicity properties can be considered, that imply changes in the evaluation formula that allow the distribution of relative realizations to play a role. A case in point is that of *ratio monotonicity* [Villar (2010)] that leads to the geometric mean of the relative achievements.[4] That is, $\varphi(Y,\mathbf{z}) = \prod_{i\in N}\prod_{j\in K}\left(\frac{y_{ij}}{z_j}\right)^{1/nk}$, for the case of symmetric agents (i.e. $\rho_i = 1/n$ for all i).

Both monotonicity properties can be regarded as particular cases of a more general principle, that we call *Generalized Additive Monotonicity*, defined as follows:

- **Generalized Additive Monotonicity**: Let $Y, Y' \in \mathbb{R}^{nk}_{++}$ be such that $y'_{ij} > y_{ij}$, $y_{ht} = y'_{ht}$, for all $h \neq i$, all $t \neq j$. Then,
$$\varphi(Y',\mathbf{z},\boldsymbol{\rho}) = \left[\left[\varphi(Y,\mathbf{z},\boldsymbol{\rho})\right]^{1/q} + b_{ij}\left[\left(y'_{ij} - y_{ij}\right), z_j, \rho_i\right]\right]^q$$
for some function b_{ij} : $\mathbb{R}^3_{++} \to \mathbb{R}_{++}$, increasing in its first argument, some $q \in \mathbb{R}$.

Additive monotonicity trivially corresponds to the case $q = 1$. Allowing for different values of q provides more flexibility on the type of response of the evaluation function. Note, however, that we restrict the domain of realization matrices to those with strictly positive entries. This implies modifying the normalization requirement by making the value of the index tend to zero when $Y \to 0$ (i.e. when $y_{ij} \to 0$, for all i, j). We call this modified property *"normalization"* (with inverted commas).[5]

Combining generalized additive monotonicity with weighted anonymity, neutrality, "normalization" and linear homogeneity we obtain the family of generalized means of order $\alpha = 1/q$. Formally:

Theorem 2: *A function* φ : $\mathbb{R}^{(n+1)(k+1)}_{++} \to \mathbb{R}_+$ *satisfies weighted anonymity, neutrality, "normalization", linear homogeneity, and generalized additive monotonicity, if and only if it takes the form:*

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[4] Ratio monotonicity is defined as follows: Let $Y, Y' \in \mathbb{R}^{nk}_{++}$ be such that $y'_{ij} > y_{ij}$, $y_{ht} = y'_{ht}$, for all $h \neq i$, all $t \neq j$. Then, $\frac{\varphi(Y',\mathbf{z})}{\varphi(Y,\mathbf{z})} = g_i\left[\frac{y'_{ij}}{y_{ij}}, z_j\right]$ for some function g_i : $\mathbb{R}^2_{++} \to \mathbb{R}_{++}$, increasing in its first argument.

[5] This property can be formally defined as follows: $\lim_{Y\to 0} \varphi(Y,\mathbf{z},\boldsymbol{\rho}) = 0$, $\varphi(Z,\mathbf{z},\boldsymbol{\rho}) = 1$.

$$\varphi(Y,\mathbf{z})=\begin{cases}\left[\frac{n^{\alpha}}{nk}\sum_{i\in N}\sum_{j\in K}\left(\rho_i\frac{y_{ij}}{z_j}\right)^{\alpha}\right]^{1/\alpha} & for\ \ \alpha\neq 0\\ \prod_{i\in N}\prod_{j\in K}\left(\frac{y_{ij}}{z_j}\right)^{\rho_i/k} & for\ \ \alpha=0\end{cases} \quad [6]$$

Moreover, those properties are independent.

(We omit the proof as it is practically a replica of that of Theorem 1)

Remark: *Theorem 1 is not a particular case of Theorem 2 because the domain on which the evaluation functions is defined is different (it is* $\mathbb{R}_{++}^{(n+1)(k+1)}$ *is Theorem 2 and* $\mathbb{R}^{nk}\times\mathbb{R}_{++}^{k}\times\mathbb{R}_{++}^{n}$ *in Theorem 1).*

The set of those who meet the standards is now given by all agents whose vectors of relative realizations, $\mathbf{y}_i(\mathbf{z})\in\mathbb{R}_{++}^{k}$, are above the hypersurface defined by $\sum_{j\in K}\left(y_{ij}/z_j\right)^{\alpha}=k$. Therefore, choosing α (the elasticity of substitution) amounts to fix the bonus/malus frontier. In particular, $\alpha\to-\infty$ (resp. $\alpha\to+\infty$) corresponds to the extreme case in which an agent meets the standards when she is above the targets in *all* dimensions simultaneously (resp. above *some* target). As for the intermediate cases, we find two of special relevance: the *arithmetic mean*, associated to the value $\alpha=1$, discussed in the former section, and the *geometric mean*, associated to the value $\alpha=0$.

From a different viewpoint the parameter α may be regarded as an equality coefficient in the following sense: the smaller the value of α the more weight we attach to a more egalitarian distribution of the agents' achievements, both among themselves and with respect to the different dimensions. The case $\alpha=1$ shows no concern for the distribution, as only the sum of the achievements matters (*inequality neutrality*). Values of α smaller than one correspond to inequality aversion. The geometric mean, in particular, penalizes moderately the unequal distribution of the achievements, whereas the extreme case $\alpha\to-\infty$ (resp. $\alpha\to+\infty$) implies caring only about the smallest (resp. the highest) achievement of each agent.

This can be illustrated as follows. Take the evaluation function of a given agent,

$$\vartheta_i^{\alpha}(Y,\mathbf{z},\boldsymbol{\rho})=\left(\frac{n^{\alpha}}{k}\sum_{j\in K}\left(\rho_i\frac{y_{ij}}{z_j}\right)^{\alpha}\right)^{1/\alpha} \quad [8]$$

The parameter α controls the curvature (degree of substitutability among the different dimensions) on an indifference curve, $\vartheta_i^{\alpha}(Y,\mathbf{z},\boldsymbol{\rho})=C$. The smaller the value of α the more difficult to substitute the achievement in one dimension by that in another one. In the limit, no substitution is allowed so that meeting the standards implies surpassing all target levels.

Similarly, assuming that the reference values correspond to absolute thresholds externally given, the evaluation of the global performance with respect to a given target, $j\in K$, is given by:

$$\beta_j(Y,\mathbf{z}) = \left[\frac{n^{\alpha}}{n} \sum_{i \in N} \left(\rho_i \frac{y_{ij}}{z_j} \right)^{\alpha} \right]^{1/\alpha} \qquad [9]$$

The parameter α tells us now about the substitutability between individuals within a given dimension. The higher the value of α the easier to substitute the achievement of one individual by the achievement of another one, and viceversa.

The agents who meet the standards in this case are those who satisfy the following condition:

$$\frac{1}{k} \sum_{j \in K} \left(\frac{y_{ij}}{z_j} \right)^{\alpha} \geq 1 \qquad [10]$$

The set of targets that have been collectively reached are those that satisfy:

$$\sum_{i \in N} \left(\rho_i \frac{y_{ij}}{z_j} \right)^{\alpha} \geq \sum_{i \in N} \rho_i^{\alpha} \qquad [11]$$

5 Final comments

We have provided here a criterion to evaluate the performance of a society with respect to a collection of targets. This criterion materializes in a simple an intuitive formula, a mean of order α of the shares of the realizations in the targets, which has been characterized by means of standard requirements. The order of the mean is a parameter that determines the substitutability between the achievements and therefore the admissible degree of compensation among the various dimensions and the different agents. From this perspective the model can be regarded as producing endogenously a system of *shadow prices* that permits one to aggregate the different dimensions.

We have discussed in some detail the linear case, corresponding to the value $\alpha = 1$. There are good reasons to singularize this special case:

(a) It entails a principle very easy to understand: the arithmetic mean. This aspect may be important when the standards involve incentives, because understanding properly the incentives scheme is usually a necessary condition for its effectiveness.

(b) It permits one to perform the evaluation in the context of poor data. There are many situations in which we only have average values of realizations across agents but not individual data. Since the arithmetic mean of the original data coincides with the mean of the average values, we can apply this procedure even in the absence of rich data.

(c) It allows to handle negative values (e.g. deficit percentages in the

Eurozone).

(d) It fits well in those cases in which it is not clear whether one should penalize or foster diversity. Recall that values of α smaller than 1 penalize progressively the dispersion of the achievements whereas values of α greater than 1 do the contrary. So choosing α above or below unity amounts to promoting the differentiation of the agents' performance (specialization) or the homogeneous behaviour (uniformity). The linear case represents preference neutrality regarding pooling or separating behaviour.

Needless to say there are contexts in which values $\alpha \neq 1$ will be more suitable (e.g. when meeting the standards involves security issues or when similar behaviour is preferable).

We have introduced the notion of weighted anonymity in order to deal with agents of different size or importance. We have kept a symmetric treatment of the objectives, though, because one can apply the appropriate scaling in terms of the vector $\mathbf{z}$ of reference values. A different problem is that of handling targets with different *degrees of priority*, that is, differentiating groups of targets that admit different degrees of substitutability (e.g. a subset of targets all of which have to be fulfilled before any other group of targets be taken into consideration). The analysis of that case is left for future research.

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