An introduction to high-order well-balanced numerical schemes for hyperbolic systems with source terms.

C. Parés

Universidad de Málaga

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Planning

Outline of the cours

1. Introduction.
2. A scalar linear balance law.
3. Nonconservative hyperbolic systems.
4. High order methods.
5. Well balancing.
6. Generalized Hydrostatic Reconstruction.

Introduction: PDE systems

System of balance laws

$$\begin{cases} U_t + F(U)_x = S(U)\sigma_x, & x \in \mathbb{R},\ t > 0 \\ U(x,0) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

- $U : \mathbb{R} \times [0, \infty) \to \mathcal{O} \subset \mathbb{R}^N$, $\mathcal{O}$ open and convex;
- $\sigma : \mathbb{R} \to \mathbb{R}$ is known smooth function;
- $F : \mathcal{O} \to \mathbb{R}^N$;
- $S : \mathcal{O} \to \mathbb{R}^N$.

Objective:

To design efficient high-order well-balanced shock-capturing methods for PDE equations of this type.

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Introduction: shallow water model

Hyperbolic Shallow Water Model

$$U_t + F(U)_x = S(U)H_x,$$

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$$U = \begin{bmatrix} h \\ q \end{bmatrix}, \quad F(U) = \begin{bmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2}gh^2 \end{bmatrix}, \quad S(U) = \begin{bmatrix} 0 \\ gh \end{bmatrix}.$$

- h: thickness of the layer;
- q: discharge;
- H: depth function;
- g: gravity acceleration.

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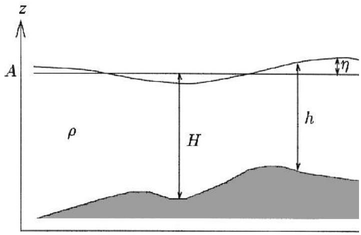

Introduction: well-balancing

- In general, a numerical scheme is said to be well-balanced if it captures correctly the smooth stationary solutions of the system, or at least a family of them.
- Numerical schemes which are not well-balanced may produce spurious oscillations when approaching equilibria or near equilibrium solutions.
- These oscillations tend to 0 as the mesh is refined. Nevertheless, the well-balanced property is important for lower order schemes or even for high order schemes in some particular applications.
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Introduction: PDE systems

System of balance laws with nonconservative products

$$\begin{cases} U_t + F(U)_x = B(U)U_x + S(U)\sigma_x, & x \in \mathbb{R},\ t > 0 \\ U(x,0) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

- $B : \mathcal{O} \mapsto \mathbb{R}^{N \times N}$.

- All the methods and the results concerning their well-balanced properties are valid for this more general family of systems. But the numerical approximation of these systems has some specifical difficultes that will not be discussed here.
- **Examples:** two layer shallow-water models, Saint-Venant-Exner models, turbidity currents models, two layer Savage-Hutter models, multiphase flow models, etc...

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A scalar linear balance law: Formulation

- Let us consider the linear scalar equation:

$$u_t + u_x = u.$$

- The smooth stationary solutions are:

$$u(x) = Ce^x,$$

being C an arbitrary constant.

- **Problem:** design a numerical scheme that preserves all the stationary solutions.
- The upwind numerical scheme:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left(u_{i-1}^n - u_i^n\right) + \Delta t u_{i-1}^n$$

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A scalar linear balance law: Equations

N. cells	L^1 error	order
4	0.0830	-
8	0.0473	0.8121
16	0.0239	0.9848
32	0.0118	1.0229
64	0.0058	1.0215
128	5 0.0029	1.0048

Table: Error in L^1 norm for the upwind scheme with the initial condition $w(x, 0) = e^x$ at time $t = 1$. CFL=0.9.

A scalar linear balance law: Reformulation

- We rewrite the Cauchy problem:

$$\begin{cases} u_t + u_x = u, \\ u(x,0) = u_0(x) \end{cases}$$

as follows:

$$\begin{cases} u_t + u_x = u\sigma_x, \\ \sigma_t = 0, \\ u(x,0) = u_0(x), \\ \sigma(x,0) = x. \end{cases}$$

- In matrix form:

$$\begin{cases} w_t + A(w)\cdot w_x = 0, \\ w(x,0) = w_0(x), \end{cases}$$

where:

$$w = \begin{bmatrix} u \\ \sigma \end{bmatrix}, \quad A(w) = \begin{bmatrix} 1 & -u \\ 0 & 0 \end{bmatrix}, \quad w_0(x) = \begin{bmatrix} u_0(x) \\ x \end{bmatrix}.$$

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A scalar linear balance law: eigenvalues and Riemann invariants

- We consider the problem:

$$w_t + A(w)w_x = 0$$

$$w = \begin{bmatrix} u \\ \sigma \end{bmatrix}, \quad A(w) = \begin{bmatrix} 1 & -u \\ 0 & 0 \end{bmatrix}.$$

- The eigenvalues of the system are $\lambda_1 = 0$, $\lambda_2 = 1$. The characteristic fields are:

$$R_1 = \begin{bmatrix} u \\ 1 \end{bmatrix}, \quad R_2(w) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

.

- The integral curves of the characteristic fields are, respectively:

$$ue^{-\sigma} = constant, \quad \sigma = constant.$$

- The Riemann invariants are, respectively:

$$ue^{-\sigma}, \quad \sigma.$$

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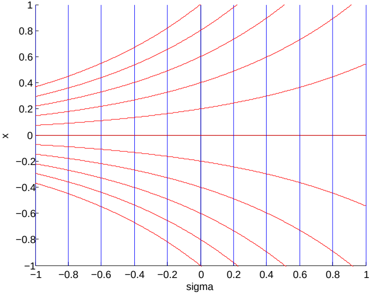

A scalar linear balance law: Riemann problems

- The solution of the Riemann problem:

$$\begin{cases} w_t + A(w)w_x = 0, \\ w(x,0) = \begin{cases} w_l & \text{if } x < 0, \\ w_r & \text{if } x > 0, \end{cases} \end{cases}$$

is

$$w(x,t) = \begin{cases} w_l & \text{if } x < 0, \\ w^* & \text{if } 0 < x < t, \\ w_r & \text{if } x > t, \end{cases}$$

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A scalar linear balance law: Godunov method

- Once the exact solutions of the Riemann problem are known, the Godunov method can be applied.
- For simplicity let us consider computing cells $I_i = [x_{i-1/2}, x_{i+1/2}]$ with constant size Δx. Let x_i denote the center of I_i.
- The initial cell averages are:

$$w_i^0 = \begin{bmatrix} u_i^0 \\ \sigma_i^0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} u_0(x)\, dx \\ \end{bmatrix}$$

- Godunov method writes as follows:

$$\begin{aligned} u_i^{n+1} &= u_i^n + \frac{\Delta t}{\Delta x} \left(e^{\Delta x} u_{i-1}^n - u_i^n \right) \\ \sigma_i^{n+1} &= \sigma_i^n. \end{aligned}$$

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A scalar linear balance law: Godunov method

- The method is exactly well-balanced in the following sense: if it is applied to an initial condition given by the **point values** at the center of the cells of a stationary solution, i.e.

$$u_i^0 = Ce^{x_i},$$

then:

$$u_i^1 = u_i^0 + \frac{\Delta t}{\Delta x}\left(e^{\Delta x} u_{i-1}^0 - u_i^0\right)$$

- Observe that every two pair of adjacent values of the initial conditions $Ce^{x_{i-1}}$, Ce^{x_i} belong to the same integral curve of the first linearly degenerate field and thus the exact solution of the Riemann problems is a stationary contact discontinuity.

- Notice that this property is not satisfied by the **cell averages** of a stationary solution. If the initial conditions are given by the cell averages of a stationary solution, this solution is preserved up to second order.

- Therefore, although the derivation of the Godunov method is based on cell-averages, the method is only exactly well-balanced when applied to point values of a stationary solution. This point will be further discussed in Section 5.

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- Therefore, although the derivation of the Godunov method is based on cell-averages, the method is only exactly well-balanced when applied to point values of a stationary solution. This point will be further discussed in Section 5.

A scalar linear balance law: Godunov method

- The method is exactly well-balanced in the following sense: if it is applied to an initial condition given by the **point values** at the center of the cells of a stationary solution, i.e.

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- Notice that the last term is a first order approximation of the source term.
- The solution of the Riemann problem for the augmented system consists of three constant states linked by two contact discontinuities. This is not the case for the Riemann problem corresponding to the original formulation

$$\begin{cases} u_t + u_x = u, \\ u(x,0) = \begin{cases} u_L & \text{if } x < 0; \\ u_R & \text{if } x > 0. \end{cases} \end{cases}$$

In this case, there is only one wave traveling at speed λ connecting two states that are no constant but exponentially growing.

- The passage through the augmented problem may be understood as an approximation of the source terms by a **Dirac's comb**: see Gosse Math. Comp. 2002. To advance in time from t^n to t^{n+1}, the equation is approached by:

$$u_t + u_x = \sum_i u_{i+1/2}^n \delta_{x=x_{i+1/2}},$$

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Weak solutions: reformulation of the system

- Let us first rewrite the general problem:

$$U_t + F(U)_x = S(U)\sigma_x, \quad x \in \mathbb{R},\ t > 0.$$

by adding the artificial unknown σ and the associated equation:

$$\sigma_t = 0.$$

- The Cauchy problem can be written as follows:

$$\begin{cases} W_t + \mathcal{A}(W)W_x = 0, & x \in \mathbb{R},\ t > 0, \\ W(x,0) = W_0(x), & x \in \mathbb{R}, \end{cases}$$

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$$W = \begin{bmatrix} U \\ \sigma \end{bmatrix}, \quad W_0(x) = \begin{bmatrix} U_0(x) \\ \sigma(x) \end{bmatrix}, \quad \mathcal{A}(W) = \begin{pmatrix} J(U) & -S(U) \\ 0 & 0 \end{pmatrix},$$

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Weak solutions: reformulation of the system

- In nonconservative form the shallow water system reads as follows:

$$W_t + \mathcal{A}(W)W_x = 0,$$

$$W = \begin{bmatrix} h \\ q \\ H \end{bmatrix}, \qquad \mathcal{A}(W) = \begin{bmatrix} 0 & 1 & 0 \\ gh - q^2/h^2 & 2q/h & -g \\ 0 & 0 & 0 \end{bmatrix}$$

where q is the mass-flux, h the thickness of the water layer, H the depth function, and g, the gravity.

Weak solutions: motivation

- Let us describe the general strategy to derive high-order numerical methods for general hyperbolic systems of the form

$$W_t + \mathcal{A}(W)W_x = 0.$$

- The computational domain is split into cells. By simplicity, we consider uniform meshes:

$$I_i = [x_{i-1/2}, x_{i+1/2}], \quad x_{i+1/2} - x_{i-1/2} = \Delta x, \quad \forall i.$$

- We denote by $W_i(t)$ the approximation of the cell averages of the exact solution provided by the semi-discrete numerical scheme:

$$W_i(t) \cong \bar{W}_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} W(x,t)\, dx.$$

- At every instant t, the numerical scheme produces a piecewise constant approximation of $W(\cdot, t)$.
- To design the numerical methods, we shall first obtain the system of equations satisfied for the cell averages of the sought weak solution $\bar{W}_i(t)$, and then an approximate system will be derived to obtain their approximations.

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Weak solutions: families of paths

- A smooth solution of the system satisfies the equality:

$$\frac{d}{dt}\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W(x,t)\,dx\right) = -\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{A}(W(x,t))W_x(x,t)\,dx.$$

- The solution W may develop discontinuities even for smooth initial conditions. In this case, the integral of the last equality has to be defined: Dirac masses should appear at the discontinuities but the mathematics of the problem are not enough to determine their weights.

- The theory introduced by Dal Maso, LeFloch & Murat J. Math. Pures Appl. 1995 allows one to define this integrand as a measure. To do this, a family of Lipschitz continuous paths $\Phi : [0,1] \times \Omega \times \Omega \to \Omega$ has to be prescribed, which must satisfy certain natural regularity conditions, in particular

$$\Phi(0; W_L, W_R) = W_L, \qquad \Phi(1; W_L, W_R) = W_R,$$

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- According to this definition, given a bounded variation function $V : [a,b] \to \mathbb{R}$, we define:

$$\fint_a^b \mathcal{A}(V(x))V_x(x)\,dx = \int_a^b \mathcal{A}(V(x))V_x(x)\,dx + \sum_l \int_0^1 \mathcal{A}(\Phi(s;V_l^-,V_l^+))\frac{\partial\Phi}{\partial s}(s;V_l^-,V_l^+)\,ds, \quad (1)$$

where V_l^- and V_l^+ represent, respectively, the limits of V to the left and right of its lth discontinuity.

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Weak solutions: jump conditions

- A piecewise smooth function W is a weak solution if, and only if:
  - It is a classical solution in its smoothness regions.
  - Across a discontinuity the following jump condition is satisfied:

$$\xi(W^+ - W^-) = \int_0^1 \mathcal{A}(\Phi(s; W^-, W^+)) \frac{\partial \Phi}{\partial s}(s; W^-, W^+)\, ds,$$

    where ξ is the speed of propagation and $W^\pm$ the limits to the left and to the right.

- For conservative problems, the definition of weak solutions coincides with the usual one regardless of the choice of paths.

- Even if the mathematics of the problem gives some hints concerning the family of paths to be chosen (Muñoz & CP M2AN 2007), in some cases an 'external' amount of information is required to choose the correct paths: for instance, the viscous profiles of a regularized system.

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Weak solutions: system of balance laws

- For system of balance laws, the jump conditions write as follows:

$$\begin{cases} \xi(U^+ - U^-) = F(U^+) - F(U^-) + \displaystyle\int_0^1 S(\Phi_U(s; W^-, W^+))\partial_s \Phi_\sigma(s; W^-, W^+)\, ds; \\ \xi(\sigma^+ - \sigma^-) = 0. \end{cases}$$

where the following notation has been used for the family of paths:

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then:

- A discontinuity such that $\sigma^+ \neq \sigma^-$ is stationary and the limit states satisfy:

$$F(U^+) - F(U^-) + \int_0^1 S(\Phi_U(s; W^-, W^+))\partial_s \Phi_\sigma(s; W^-, W^+)\, ds = 0$$

or, equivalently,

$$\int_0^1 \mathcal{A}(\Phi(s; W^-, W^+)) \frac{\partial \Phi}{\partial s}(s; W^-, W^+)\, ds = 0.$$

Weak solutions: the example revisited

- In the case of the linear scalar balance law, was a family of paths chosen for solving the Riemann problems?
- Yes, implicitly…
- The path connecting to states w_l and w_r is composed by the exponential curve linking w_i to the intermediate state w^* appearing at the Riemann solution and the segment connecting w^* to w_r.

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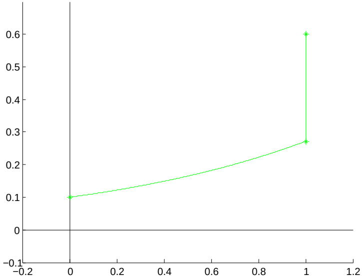

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- Nevertheless, it is possible to choose a different family of paths obtaining thus different solutions for the Riemann problem.
- For instance we can choose the paths:

$$\Phi(s; w_l, w_r) = \begin{cases} [u_l, \quad \sigma_l + 2(\sigma_r - \sigma_l)]^T & \text{if } 0 \leq s \leq 1/2, \\ [u_l + 2(s - 1/2)(u_r - u_l), \quad \sigma_r]^T & \text{if } 1/2 \leq s \leq 1, \end{cases}$$

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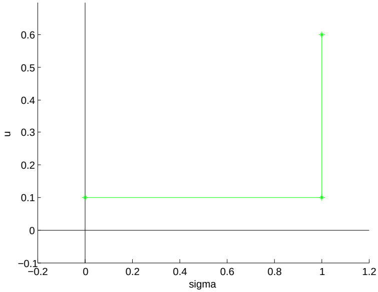

Weak solutions: the example revisited

- The solution of the Riemann problem is:

$$w(x,t) = \begin{cases} w_l & \text{if } x < 0, \\ w^* & \text{if } 0 < x < t, \\ w_r & \text{if } x > t, \end{cases}$$

where now

$$w^* = \begin{bmatrix} (1 + [\sigma])u_l \\ \sigma_r \end{bmatrix}.$$

- The corresponding Godunov method is the upwind scheme

$$u_i^{n+1} = u_i^n + \frac{\Delta x}{\Delta t}(u_i^n - u_{i-1}^n) + \Delta t\, u_{i-1}^n.$$

that solves the stationary solutions only with first order accuracy.

- To advance in time from t^n to t^{n+1}, the equation is approached by:

$$u_t + u_x = \sum_i u_i^n \delta_{x = x_{i+1/2}}.$$

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- The natural extension to general systems of balance laws of the family of paths leading to a Godunov method with better well-balanced properties is the following:
- Given two states $W_L = [U_L, \sigma_L]^T$ and $W_R = [U_R, \sigma_R]^T$ the associated Riemann problem is solved by imposing that the Riemann invariants corresponding to the null eigenvalue are preserved though the wave standing at $x = 0$ (the so-called *zero wave*).
- Let us denote by:

$$W_0^- = \begin{bmatrix} U_0^- \\ \sigma_L \end{bmatrix}, \quad W_0^+ = \begin{bmatrix} U_0^+ \\ \sigma_R \end{bmatrix},$$

  the limits to the left and to the right of $x = 0$ of the solution of the Riemann problem.
- Then the path $\Psi(\cdot; W_L, W_R)$ is a parameterization of the curve composed by:
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- Nevertheless:
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High order methods: strategy

- A weak solution satisfies the equality:

$$\frac{d}{dt}\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W(x,t)\,dx\right) = -\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{A}(W(x,t))W_x(x,t)\,dx.$$

- The idea is to approximate this system to obtain the approximations $W_i(t)$.

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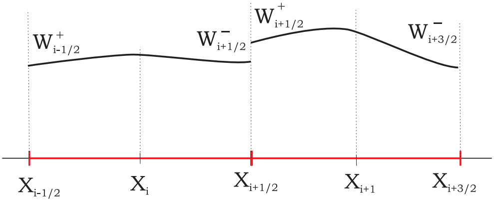
High order methods: reconstruction of states

- We consider a **high-order reconstruction operator** providing an approximation function P_i^t at every cell I_i at every instant t, as well as two reconstructed operator at the inter-cells:

$$\lim_{x \to x_{i-1/2}^+} P_i^t(x) = W_{i-1/2}^+(t), \qquad \lim_{x \to x_{i+1/2}^-} P_i^t(x) = W_{i+1/2}^-(t).$$



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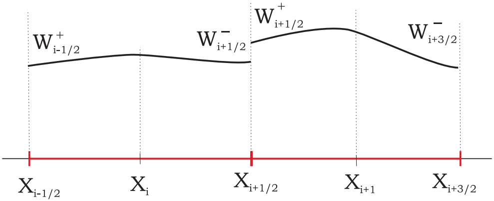

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High order methods: general expression

- We consider semi-discrete numerical scheme of the form (CP SINUM 06):

$$W_i'(t) = -\frac{1}{\Delta x}\left(D^+_{i-1/2} + D^-_{i+1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{A}(P_i^t(x))\frac{d}{dx}P_i^t(x)\,dx\right),$$

where

$$D^\pm_{i+1/2} = D^\pm(W^-_{i+1/2}(t), W^+_{i+1/2}(t)).$$

- $D^\pm(W_L, W_R)$ are two functions satisfying:

$$D^-(W_L, W_R) + D^+(W_L, W_R) = \int_0^1 \mathcal{A}(\Phi(s; W_L, W_R))\frac{\partial \Phi}{\partial s}(s; W_L, W_R)\,ds, \quad \forall W_L, W_R,$$

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High order methods: system of balance laws

- For systems of balance laws, the consistency conditions together with some natural assumption for $D^{\pm}$, imply the existence of a consistent numerical flux $G : \mathcal{O} \times \mathcal{O} \mapsto \mathcal{O}$

$$G(U, U) = F(U)$$

and two functions $S^{\pm} : \Omega \times \Omega \mapsto \mathcal{O}$ satisfying

$$S^{-}(W_L, W_R) + S^{+}(W_L, W_R) = \int_0^1 S(\Phi_U(s; W_L, W_R)) \frac{\partial \Phi_\sigma}{\partial s}(s; W_L, W_R)\, ds, \quad \forall W_L, W_R,$$

$$S^{\pm}\left(\begin{bmatrix} U_L \\ \bar{\sigma} \end{bmatrix}, \begin{bmatrix} U_R \\ \bar{\sigma} \end{bmatrix} \right) = 0,$$

such that

$$\begin{aligned} D^{+}(W_L, W_R) &= \begin{bmatrix} F(U_R) - G(U_L, U_R) - S^{+}(W_L, W_R) \\ 0 \end{bmatrix} \\ D^{-}(W_L, W_R) &= \begin{bmatrix} G(U_L, U_R) - F(U_L) - S^{-}(W_L, W_R) \\ 0 \end{bmatrix}. \end{aligned}$$

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- The numerical scheme can be then rewritten as follows:

$$U_i'(t) = \frac{1}{\Delta x}\left(G_{i-1/2} - G_{i+1/2} + S^{+}_{i-1/2} + S^{-}_{i+1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} S(P^t_{U,i}(x))\,\frac{d}{dx}p^t_{\sigma,i}(x)\,dx \right),$$

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- Notice that, if $S \equiv 0$, the numerical scheme reduces to a conservative method:

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- While in the conservative case the order of the method is given by the accuracy of the reconstruction at the intercells, in the presence of source terms it also depends on the accuracy of the reconstructions at the interior of the cells.

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$$U_i'(t) = \frac{1}{\Delta x}\left(G_{i-1/2} - G_{i+1/2} + S^+_{i-1/2} + S^-_{i+1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} S(P^t_{U,i}(x))\,\frac{d}{dx}p^t_{\sigma,i}(x)\,dx \right),$$

  where

$$G_{i+1/2} = G(U^-_{i+1/2}, U^+_{i+1/2}), \quad S^{\pm}_{i+1/2} = S^{\pm}(W^-_{i+1/2}, W^+_{i+1/2}),$$

$$P_i^t = \left[\begin{array}{c} P^t_{U,i} \\ p^t_{\sigma,i} \end{array} \right].$$

- Notice that, if $S \equiv 0$, the numerical scheme reduces to a conservative method:

$$U_i'(t) = \frac{1}{\Delta x}\left(G_{i-1/2} - G_{i+1/2} \right).$$

- While in the conservative case the order of the method is given by the accuracy of the reconstruction at the intercells, in the presence of source terms it also depends on the accuracy of the reconstructions at the interior of the cells.

High order methods: examples

Fluctuation functions based on exact Riemann solvers

- Following the principle of Godunov method, a first strategy to define the fluctuation functions consists on finding the exact solution of the Riemann problem associated to (W_L, W_R) (according to the chosen family of paths) and averaging the solutions.
- The corresponding fluctuation functions are:

$$\begin{aligned} D_G^-(W_L, W_R) &= -\int_{-\infty}^{0} \left(V(s; W_L, W_R) - W_L\right) ds; \\ D_G^+(W_L, W_R) &= -\int_{0}^{\infty} \left(V(s; W_L, W_R) - W_R\right) ds; \end{aligned}$$

where $V(x/t; W_L, W_R)$ represents the self-similar solution of the Riemann problem associated to W_L and W_R.

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High order methods: examples

- Under some assumptions on the family of paths, the fluctuation functions can be written on the simplified form:

$$\begin{aligned} D_G^-(W_L, W_R) &= \int_0^1 \mathcal{A}(\Phi(s; W_L, W_0^-)) \frac{\partial \Phi}{\partial s}(s; W_L, W_0^-)\, ds, \\ D_G^+(W_L, W_R) &= \int_0^1 \mathcal{A}(\Phi(s; W_0^+, W_R)) \frac{\partial \Phi}{\partial s}(s; W_0^+, W_R)\, ds, \end{aligned}$$

where $W_0^\pm = V(0^\pm; W_L, W_R)$ are the limits to the right and to left of $x = 0$ of the solution of the Riemann problem (Muñoz & CP M2AN 2007).

- For system of balance laws, the fluctuation functions reduce to:

$$\begin{aligned} D_G^-(W_L, W_R) &= \begin{bmatrix} F(U_0^-) - F(U_L) \\ 0 \end{bmatrix}, \\ D_G^+(W_L, W_R) &= \begin{bmatrix} F(U_R) - F(U_0^+) \\ 0 \end{bmatrix}, \end{aligned}$$

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High order methods: examples

- The corresponding high-order schemes can be written as follows:

$$U_i'(t) = \frac{1}{\Delta x}\left(F(U^+_{0,i-1/2}) - F(U^-_{0,i+1/2}) + \int_{x_{i-1/2}}^{x_{i+1/2}} S(P^t_{U,i}(x))\, \frac{d}{dx} p^t_{\sigma,i}(x)\, dx \right),$$

where $W^{\pm}_{0,i+1/2} = [U^{\pm}_{0,i+1/2}, \sigma^{\pm}_{i+1/2}]^T$ denote the limits to the right and to the left of $x = 0$ of the solution of the Riemann problem corresponding to the reconstructed states $(W^-_{i+1/2}, W^-_{i+1/2})$.

High order methods: examples

An example of fluctuation functions based on approximate Riemann solvers: Roe method

- First, a Roe linearization has to be chosen for the nonconservative system and the chosen family of paths (Toumi JCP 1992), i.e. a function $\mathcal{A}_\Phi : \Omega \times \Omega \mapsto \mathbb{R}^{N \times N}$ such that $\mathcal{A}_\Phi(W_L, W_R)$ has N real different eigenvalues and it satisfies:

$$\mathcal{A}_\Phi(W, W) = \mathcal{A}(W), \quad \forall W;$$

$$\mathcal{A}_\Phi(W_L, W_R)\cdot(W_R - W_L) = \int_0^1 \mathcal{A}(\Phi(s; W_L, W_R)) \frac{\partial \Phi}{\partial s}(s; W_L, W_R)\, ds, \quad \forall W_L, W_R.$$

- The corresponding fluctuation functions are:

$$D_R^{\pm}(W_L, W_R) = \mathcal{A}_\Phi^{\pm}(W_L, W_R) \cdot (W_R - W_L).$$

where $\mathcal{A}_\Phi^{+}(W_L, W_R)$ (resp. $\mathcal{A}_\Phi^{-}(W_L, W_R)$) is the diagonalizable matrix with the same eigenvector basis of $\mathcal{A}_\Phi(W_L, W_R)$ and whose eigenvalues are the positive (resp. negative) part of those of $\mathcal{A}_\Phi(W_L, W_R)$.

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High order methods: examples

- For systems of balance laws, a Roe matrix is given by:

$$\mathcal{A}_\Phi(W_L, W_R) = \left[\begin{array}{c|c} J(U_L, U_R) & \widetilde{S}(W_L, W_R) \\ \hline 0 & 0 \end{array}\right]$$

  where

  - $J(U_L, U_R)$ is a Roe matrix for the homogeneous problem, i.e.

$$J(U_L, U_R) \cdot (U_R - U_L) = F(U_R) - F(U_L),$$

  - $\widetilde{S}(W_L, W_R)$ satisfies:

$$\widetilde{S}(W_L, W_R) = \frac{1}{\sigma_R - \sigma_L} \int_0^1 S(\Phi_U(s; W_L, W_R)) \frac{\partial \Phi_\sigma}{\partial s}(s; W_L, W_R)\, ds,$$

- provided it has real different eigenvalues.

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High order methods: examples

- The corresponding semi-discrete high order scheme can be then written as follows:

$$U_i'(t) = \frac{1}{\Delta x}\left(G_{i-1/2} - G_{i+1/2} + S^+_{i-1/2} + S^-_{i+1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} S(P^t_{U,i}(x))\,\frac{d}{dx}p^t_{\sigma,i}(x)\,dx \right),$$

- where

$$\begin{aligned}
G_{i+1/2} &= \frac{1}{2}\left(F(U^-_{i+1/2}) + F(U^+_{i+1/2})\right) - \frac{1}{2}\left|J_{i+1/2}\right|\left(U^+_{i+1/2} - U^-_{i+1/2}\right),\\
S^\pm_{i+1/2} &= \mathcal{P}^\pm_{i+1/2}\cdot \widetilde{S}_{i+1/2}(\sigma_{i+1} - \sigma_i),\\
J_{i+1/2} &= J(U^-_{i+1/2}, U^+_{i+1/2}),\\
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High order methods: remarks

- An accurate enough quadrature formula is used to compute the integrals appearing in the expression of the methods.
- A standard higher order method is used to discretize the system in time, as the TVD Runge-Kutta schemes (Gottlieb & Shu Math.Comp. 1998).
- An entopy-fix technique must be used for Roe methods.
- The derivation of the numerical scheme in the more general formulation makes easier:
  - To obtain schemes with good stability properties: the presence of source terms and/or nonconservative products do not add any restriction on the CFL condition.
  - To obtain well-balanced methods.
  - To extend some well-known families of conservative schemes to non-conservative problems: FORCE, GFORCE, MUSTA, ADER, DG ... (in collaboration with E.F. Toro, M. Dumbser, A. Hidalgo), central schemes on staggered grids (in collaboration with G. Puppo, G. Russo), Entropy-preserving schemes (in collaboration with S. Mishra, U.Fjordholm), HLLC (in collaboration with E.Fernández Nieto, G. Narbona, E. Fernández Nieto), etc.
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High order methods: remarks

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High order methods: remarks

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- Nevertheless, the choice of paths is strongly related to the well-balanced properties of the scheme.

Well-balancing: stationary solutions

- A system can only have nontrivial steady state solutions if it has at least one linearly degenerate field: if $W(x)$ is a smooth non-trivial stationary solution

$$\mathcal{A}(W(x)) \cdot W'(x) = 0 \quad \forall\, x \in \mathbb{R},$$

then 0 is an eigenvalue of $\mathcal{A}(W(x))$ for every x such that $W'(x) \neq 0$, and $W'(x)$ is an associated eigenvector.

- Therefore, $x \to W(x)$ can be interpreted a parameterization of an arc of an integral curve of a characteristic field whose corresponding eigenvalue vanishes alone the curve. As a consequence, the characteristic field has to be linearly degenerate.
- Let us introduce the set Γ of all the integral curves γ of a linearly degenerate field of $\mathcal{A}(W)$ such that the corresponding eigenvalue vanishes on γ.

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Well-balancing: stationary solutions

- In the particular case of a system of balance laws, the set Γ is composed by the integral curves of the ODE system:

$$\frac{d}{d\sigma}F(U) = S(U).$$

- In the particular case of the shallow water system, Γ is composed by the curves:

$$q = constant, \qquad h + \frac{q^2}{2gh^2} - H = constant.$$

- In the particular case $q = 0$ these curves are straight lines in the h, q, H space:

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Well-balanced schemes: definitions

- Given a curve $\gamma \in \Gamma$:

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A semi-discrete scheme is said to be exactly well-balanced for γ if it solves exactly any smooth stationary solution W such that

$$W(x) \in \gamma \quad \forall\, x$$

in the following sense: the sequence of cell-averages

$$W_i = \frac{1}{\Delta x} \int_{I_i} W(x)\, dx$$

is a stationary solution of the ODE system given by the scheme.

Well-balanced schemes: definitions

- Given a curve $\gamma \in \Gamma$:

Well-balanced fluctuation functions

The fluctuations functions $D^{\pm}$ are said to be exactly well-balanced for γ if

$$D^{\pm}(W_L, W_R) = 0.$$

for any pair of states (W_L, W_R) belonging to γ.

Well-balanced schemes: definitions

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Well-balanced scheme: definitions

A reconstruction operator is said to be well-balanced for γ if, given a vector $\{W_i\}$ defined by

$$W_i = \frac{1}{\Delta x} \int_{I_i} W(x)\, dx,$$

where $W(x)$ is a smooth function taking values on γ, the approximation functions corresponding to this vector satisfy:

$$P_i(x) \in \gamma, \quad \forall x \in [x_{i-1/2}, x_{i+1/2}].$$

Well-balanced schemes: a general result

Theorem

Given a semi-discrete scheme

$$W_i'(t) = -\frac{1}{\Delta x}\left(D^+_{i-1/2} + D^-_{i+1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} \mathcal{A}(P_i^t(x))\frac{d}{dx}P_i^t(x)\,dx\right).$$

if both the reconstruction operator and the fluctuations functions are well-balanced for γ, the numerical scheme is also well-balanced for γ.

Well-balanced fluctuation functions

- Given a curve $\gamma \in \Gamma$, let us suppose that the family of paths satisfies the following property:

  (P_γ) if two states W_L and W_R belong to γ, then the path

  $$s \in [0,1] \to \Phi(s; W_L, W_R)$$

  is a parametrization of the arc of γ linking the states.

- If (P_γ) is satisfied, it can be easily shown that, given two states W_L, W_R belonging to γ of a linearly degenerate field, the contact discontinuity:

  $$W(x,t) = \begin{cases} W_L & \text{if } x < 0 \\ W_R & \text{if } x > 0 \end{cases}$$

  is a weak solution of the system, i.e. the jump conditions are satisfied.

- Notice that the family of paths Ψ described above whose construction is based on the Riemann invariants of the null eigenvalue satisfies (P_γ) for every $\gamma \in \Gamma$.

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Well-balanced fluctuation functions: Godunov method

- If the family of paths satisfies the property (P_γ) the fluctuation functions of the Godunov method are well-balanced.
- Remember that, for system of balance laws, these fluctuation functions can be written as follows:

$$D_G^-(W_L, W_R) = \begin{bmatrix} F(U_0^-) - F(U_L) \\ 0 \end{bmatrix},$$

$$D_G^+(W_L, W_R) = \begin{bmatrix} F(U_R) - F(U_0^+) \\ 0 \end{bmatrix}.$$

  If W_L and W_R belong to γ, then the solution of the Riemann problem is the stationary contact discontinuity and thus:

$$U_0^- = U_L, \quad U_0^+ = U_R.$$

- In particular, the Godunov method based on the family of paths Ψ is well-balanced for every $\gamma \in \Gamma$.
- **Remark:** Notice that the fluctuation functions are applied to the reconstructed states at the inter-cells. Therefore, for high order methods the relevant well-balanced property of the fluctuation functions is related to point values and not to cell averages.

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$$D_R^{\pm}(W_L, W_R) = \mathcal{A}_\Phi^{\pm}(W_L, W_R) \cdot (W_R - W_L).$$

If W_L and W_R belong to γ, one has:

$$\mathcal{A}_\Phi(W_L, W_R) \cdot (W_R - W_L) = \int_0^1 \mathcal{A}(\Phi(s; W_L, W_R)) \frac{\partial \Phi}{\partial s}(s; W_L, W_R)\, ds = 0.$$

Notice that, because of (P_γ), the integrand vanishes at every point.

- Therefore, 0 is an eigenvalue of $\mathcal{A}_\Phi(W_L, W_R)$ and $(W_R - W_L)$ an associated eigenvector. As a consequence

$$\mathcal{A}_\Phi^{\pm}(W_L, W_R)(W_R - W_L) = 0,$$

and the fluctuation functions vanish.

- A Roe method based on the family of paths Ψ is well-balanced for every $\gamma \in \Gamma$, but such a method would be rather paradoxical: the approximate Riemann solver is based on a family of paths whose computation requires the exact Riemann solver...

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Well-balanced fluctuation functions: Roe method

- In Section 6 an alternative approach to construct families of paths different from Ψ satisfying the property (P_γ).
- In many cases, it is enough if the family of paths satisfying (P_γ) for the curves $\gamma \in \Gamma$ corresponding to the stationary solutions to be preserved. For instance, if the curves to be preserved are straight lines in Ω, it is enough if the family of straight segments

$$\Phi(s; W_L, W_R) = W_L + s(W_R - W_L)$$

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- Moreover, the first order Roe method based on the family of straight segments solves up to second order any smooth stationary solution: CP & Castro M2AN 2004.
- In particular, for the shallow water system, the first order Roe method based on the family of straight segments solves exactly water at rest solutions and up to second order any smooth stationary solution.
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- Coming back to the linear scalar balance law, the Roe scheme based on the family of straight segments can be written as follows:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) + \Delta t \frac{u_{i-1}^n + u_i^n}{2}.$$

- It solves the stationary solutions with second order accuracy:

n. cells	L^1 error	order
15	1.4677	-
30	3.4590e-1	2.08
60	8.4829e-2	2.03
120	2.1053e-2	2.01
240	5.2472e-3	2.01

Table: Error in L^1 norm for the Roe scheme with the initial condition $w(x,0) = e^x$ at time $t = 1$. CFL=0.9.

- **Exercise:** Does this Roe method coincide with the Godunov method based on the family of straight segments?

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Well-balanced schemes: reconstruction operators

- In general, a standard reconstruction operator cannot be expected to be well-balanced for an integral curve γ.
- For particular cases in which the geometry of γ is simple enough is easy to adapt the reconstruction operators to become well-balanced (this is the case for water at rest solutions of the shallow water system).
- First order numerical methods can be interpreted as the particular case corresponding to the piecewise constant reconstruction operator:

$$P_i^t(x) = W_i, \quad \forall x \in [x_{i-1/2}, x_{i+1/2}].$$

- This reconstruction operator is not well-balanced in general: the averages of a function taking values in a curve γ are not in general in γ (unless γ is a straight line). Therefore, first order schemes, as it has been seen with the scalar balance laws, are not well-balanced in general for cell-averages.
- Given a standard reconstruction operator in $\mathbb{R}^N$ of order s

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a well-balanced reconstruction operator for every stationary solution can be constructed as follows:

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$$P_i^t(x) = W_i, \quad \forall x \in [x_{i-1/2}, x_{i+1/2}].$$

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Well-balanced reconstruction operator: a general strategy

1. Look for the stationary solution $W_i^*(x)$ such that:

$$W_i = \frac{1}{\Delta x} \int_{I_i} W_i^*(x)\, dx.$$

2. For $j = i - l, \ldots, i + r$ define V_j by:

$$V_j = W_j - \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} W_i^*(x)\, dx.$$

3. Apply the reconstruction operator to obtain:

$$Q_i = Q_i(x; V_{i-l}, \ldots, V_{i+r}).$$

4. Define the approximation functions by:

$$P_i(x) = W_i^*(x) + Q_i(x).$$

In particular:

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- In general, the fist step can be difficult. For the scalar linear balance law it is easy: it reduces to look for C such that:

$$u_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} Ce^x\,dx.$$

- Following this idea we have implemented a second order extension of the Godunov scheme based on the MUSCL reconstruction that solves exactly any stationary solution: Castro, Gallardo, López & CP SINUM 2008 .

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- It is easy to design reconstruction operators which are well-balanced for water at rest solutions: once the cell averages h_i^n, q_i^n, H_i are known, apply a standard reconstruction operator to the variables q_i^n, H_i, and

$$\eta_i^n = h_i^n - H_i,$$

to obtain the approximation functions $p_{q,i}$, $p_{H,i}$, $p_{\eta,i}$ at the cells. Then define:

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- It is also possible to apply the general methodology to obtain a reconstruction operator which is well-balanced for every stationary solution: López, PhD. Thesis 201.
- The more difficult stage is the first one: given a cell approximation W_i^n, a stationary solution has to be calculated whose average is equal to W_i^n.
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GHR: motivation

- It has been seen that, in order to have well-balanced properties, the chosen family of paths has to satisfy the property (P_γ) for as much as curves $\gamma \in \Gamma$ as possible.

- The family of paths Ψ has this property, but as it has been mentioned its computation can be costly and/or difficult in practice.

- In this final part of the course, a different strategy to obtain families of paths satisfying (P_γ) for a subset $\Gamma_0 \subset \Gamma$ is introduced. This strategy, introduced in Castro, Pardo & CP M3AS 2007 is a generalization of the Hydrostatic Reconstruction technique introduced in Audusse, Bouchut, Bristeau, Klein & Perthame J. Sci. Comp. 2004.

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- Let us suppose that it is possible to associate to every state W a curve C_W in Ω in such a way that:
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- We consider any standard numerical flux $G(U_L, U_R)$ consistent with F and define the fluctuation functions as follows:

$$D^+(W_L, W_R) = \left[\begin{array}{c} F(U_0^+) - G(U_0^-, U_0^+) - \displaystyle\int_0^1 S(P_{U,R}(s)) \frac{\partial}{\partial s} p_{\sigma,R}(s)\, ds \\ \\ 0 \end{array} \right]$$

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$$s \mapsto P_L(s) = \left[\begin{array}{c} P_{U,L}(s) \\ p_{\sigma,L}(s) \end{array} \right], \quad s \mapsto P_R(s) = \left[\begin{array}{c} P_{U,R}(s) \\ p_{\sigma,R}(s) \end{array} \right]$$

are respectively two parameterizations of the arc of C_{W_L} linking W_L and W_0^- and the arc of C_{W_R} linking W_0^+ and W_R.

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GHR: example

- We want to obtain numerical schemes that are well-balanced for water-at-rest solutions, i.e. for the family Γ_0 of curves:

$$q = 0, \qquad h - H = constant.$$

- We associate to every state $W^* = [h^*, q^*, H^*]^T$ the curve C_{W^*} defined by:

$$q = \frac{q^*}{h^*} h, \quad h - H = h^* - H^*,$$

or,

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- Given two states $W_L = [h_L, q_L, H_L]^T$, $W_R = [h_R, q_R, H_R]^T$, we define

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- The properties (P1)-(P5) are satisfied for water at rest equilibria.
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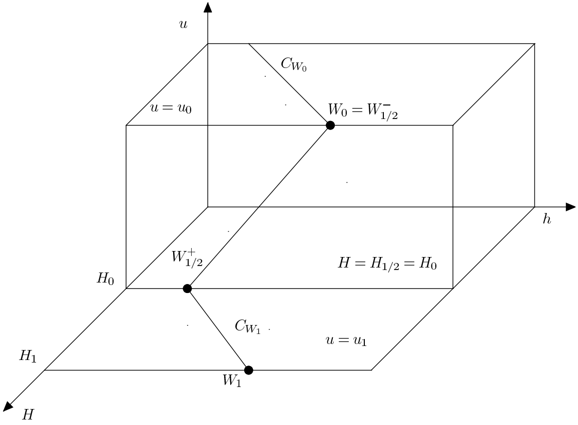

GHR: some remarks

- In practice, in order to preserve the positivity, the reconstructed states are defined as follows:

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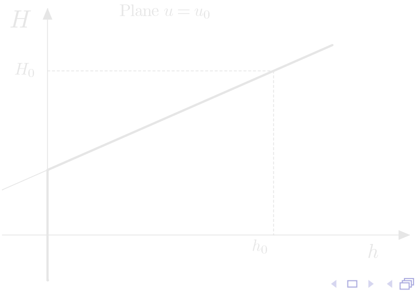
- This modification can be also interpreted in terms of the family of paths: when a cuve C_W touches the axis $h = 0$, it is replaced by this axis from the intersection point.



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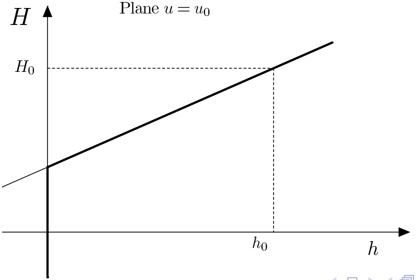

GHR: well-balanced methods

- In order to obtain a numerical scheme which is exactly well-balanced for every stationary solution, we can associate to every state W the integral curve C_W of the linearly degenerate field corresponding to the null eigenvalue passing by W. If, given two states W_L and W_R it is possible to find two states $W_0^{\pm}$ satisfying (P1)-(P5), it would be possible to derive a well-balanced numerical scheme.

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GHR: Applications to the shallow water systems

- Such a choice of intermediate states can be performed for the shallow water system: see Castro, Pardo & CP M3AS 2007.
- Given two states $W_L = [h_L, q_L, H_L]^T$ and $W_R = [h_R, q_R, H_R]^T$, first an adequate intermediate value of the depth has to be chosen. In particular, the value

$$H_0 = \min(H_L, H_R)$$

is always chosen whenever it's possible.
- Two reconstructed states are considered at the intercell $W_0^- = [h_0^-, q_0^-, H_0]^T$ and $W_0^+ = [h_0^+, q_0^+, H_0]^T$ such that:

$$q_0^- = q_L, \quad g(h_0^- - H_0) + \frac{(q_0^+)^2}{2(h_0^-)^2} = g(h_L - H_L) + \frac{q_L^2}{2h_L^2},$$

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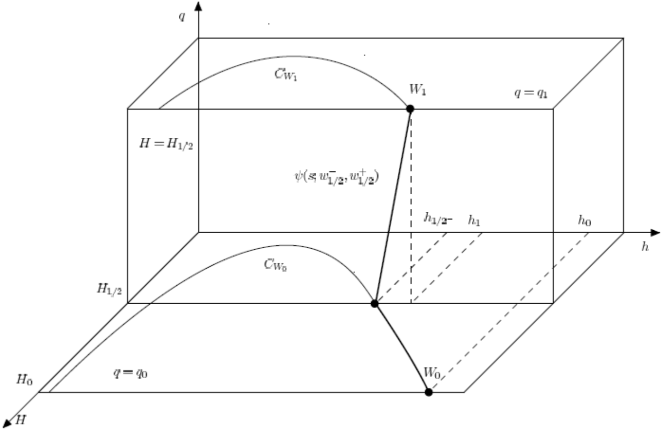

GHR: Applications to the shallow water systems

- We compare two numerical methods for the shallow water system:
  - A third order Roe method based on the PHM reconstruction operator.
  - A well-balanced third order method based on the generalized hydrostatic reconstruction, the Roe flux for the homogeneous problem, and the well-balanced modification of the PHM reconstruction operator
- We consider first a transcritical (transonic) stationary solution corresponding corresponding to:

$$q = 2.5, \quad g(h - H) + \frac{(q)^2}{2h^2} = 17.56957396120237, \quad g = 9.812.$$

for the depth function:

$$H(x) = \begin{cases} -0.25(1 + \cos(5\pi(x + 0.5))) & \text{if } 1.3 \leq x \leq 1.7, \\ 0 & \text{otherwise.} \end{cases}$$

The critical (sonic) point is located at $x = 1.5$.

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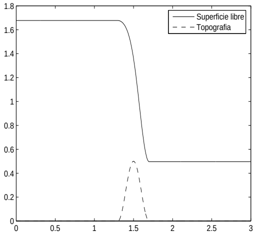

Figure: Transcritical stationary solution.

n. cells	error h	error q
50	9.99e-17	5.32e-17
100	1.04e-16	1.27e-15
200	1.03e-15	7.95e-15
400	3.36e-15	2.91e-14

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100	5.17e-4	3.02e-3	-	-
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400	1.03e-5	6.56e-5	2.82	2.76
800	1.43e-6	9.30e-5	2.84	2.81

Table: Left: third-order well-balanced numerical scheme; right: third order-Roe scheme. CFL=0,9.

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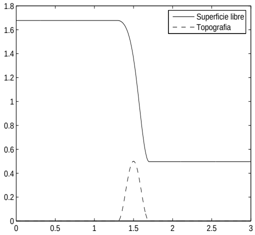

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GHR: Applications to the shallow water systems

- A small perturbation of the order of Δx is applied to h in the interval $[1.1, 1.2]$. The evolution of the perturbation is simulated with the two numerical schemes. The differences between the numerical solution and the stationary solution are depicted in the Figures.

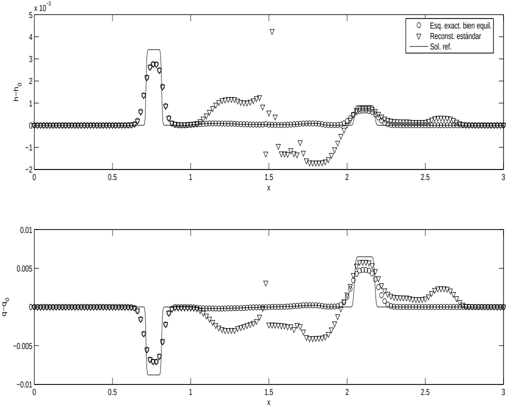

Figure: Evolution of the perturbation in a mesh with 150 cells at the instant $t = 0.15$.

GHR: Applications to the shallow water systems

- We consider finally a stationary solution with a stationary shock (a hydraulic jump). The depth function is:

$$H(x) = \begin{cases} -0.2 + 0.05(x-10)^2 & \text{if } 8 \leq x \leq 12, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

- the initial conditions

$$q(x,0) = 0, \quad h(x,0) = 0.33 + H(x);$$

- and the boundary conditions $h = 0.33$ at the left extreme of the interval, and $q = 0.18$ at the right one.

GHR: Applications to the shallow water systems

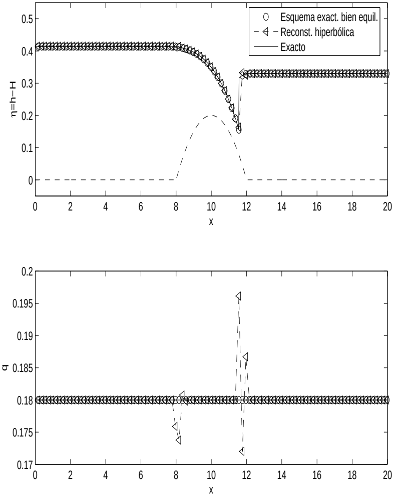

Figure: Free surface and discharge corresponding to the numerical stationary solution.

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- The first order numerical schemes provided by the original Hydrostatic Reconstruction Technique are positive and entropy-preserving if the chosen numerical flux has these properties, but these properties can be lost for the high order extensions presented here.
- The first order well-balanced for the shallow water obtained by the GHR is not in general positive nor entropy-preserving even if the chosen numerical flux has these properties.
- In Bouchut & Morales, SINUM 2010 a first order numerical scheme which is positive, entropy preserving and well-balanced for subcritical stationary solutions has been presented.

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References

Abgrall R., Karni S.
A comment on the computation of non-conservative products. J. Comput. Phys., 45 (2010), 382–403.

Audusse E., Bouchut F., Bristeau M.O., Klein R., Perthame B.
A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows. SIAM J. Sci. Comp., 25 (2004), 2050–2065.

Bermúdez Al, Vázquez M.E.
Upwind methods for hyperbolic conservation laws with source terms. Comput. & Fluids, 23 (1994), 1049–1071.

Bouchut F.
Nonlinear Stability of Finite Volume Methods for Hyperbolic Conservation Laws and Well-Balanced Schemes for Sources. Birkhäuser, Basel, Switzerland, 2004.

Bouchut F., Morales T.
A Subsonic well-balanced reconstruction scheme for shallow water flows. SIAM J. Num. Anal. 48, 1733-1758.

References

Castro M.J., Fernández-Nieto E.D., Ferreiro A.M., Parés C.
Two-dimensional sediment transport models in shallow water equations. A second order finite volume approach on unstructured meshes. Comput. Methods Appl. Mech. Engrg. 198 (2009), 2520–2538.

Castro M.J., Fernández-Nieto E.D., Ferreiro A.M., García J.A., Parés C.
High order extensions of Roe schemes for two dimensional nonconservative hyperbolic systems. J. Sci. Comput. 114 (2009), 67–114.

Castro M.J., Macías J., Parés C.
A Q-scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system. M2AN Math. Mod. Numer. Anal., 35 (2001), 107–127.

Castro M.J., Gallardo J.M., López J.A., Parés C.
Well-balanced high order extensions of Godunov's method for semilinear balance laws. SIAM J. Num. Anal., 46 (2008), 1012–1039.

References

Castro M.J., Gallardo J.M., Parés C.
Finite volume schemes based on WENO reconstruction of states for solving nonconservative hyperbolic systems. Applications to shallow water systems. Math. Comp. 75 (2006), 1103–1134.

Castro M.J., LeFloch P.G., Muñoz M.L., Parés C.
Why many theories of shock waves are necessary: convergence error in formally path-consistent schemes. J. Comput. Phys 227 (2008), 8107–8129.

Castro M.J., Pardo A., Parés C.
Well-balanced numerical schemes based on a generalized hydrostatic reconstruction technique. Math. Mod. Meth. App. Sci., 17 (2007), 2055-2113.

Castro M.J., Pardo A., Parés C., Toro E.F.
On some fast well-balanced first order solvers for nonconservative systems. Math. Comp. 79 (2010), 1427–1472.

Dal Maso G., LeFloch P.G., Murat F.
Definition and weak stability of nonconservative products. J. Math. Pures Appl. 74 (1995), 483–548.

References

Dumbser M., Castro M.J., Pars C., Toro E.F.
ADER schemes on unstructured meshes for nonconservative hyperbolic systems: applications to geophysical flows. Comp. & Fluids, 38 (2009), 1731-1748.

Dumbser M., Hidalgo A., Castro M.J., Pars C., Toro E.F.
FORCE schemes on unstructured meshes II: Non-conservative hyperbolic systems. Comp. Meth. Appl. Mech. Eng. 199 (2010), 625–647.

Gallardo J.M., Parés C., Castro M.J.
On a well-balanced high-order finite volume scheme for shallow water equations with topography and dry areas. J. Comp. Phys. 227 (2007), 574–601.

Gosse L.
A well-balanced flux-vector splitting scheme designed for hyperbolic systems of conservation laws with source terms. Comput. Math. Appl., 39 (2000), 135–159.

Gosse L.
A well-balanced scheme using nonconservative products designed for hyperbolic systems of conservation laws with source terms.
Math. Models Methods Appl. Sci., 11 (2001), 339–365.

References

Gosse L.
Localization effects and measure source terms in numerical schemes for balance laws. Math. Comp., 71 (2002), 553–582.

Gottlieb S., Shu C.-W.
Total variation diminishing Runge-Kutta schemes. Math. Comp. 67 (1998) , 73–85.

Greenberg J.M., LeRoux A.Y.
A well-balanced scheme for the numerical processing of source terms in hyperbolic equations. SIAM J. Numer. Anal., 33 (1996), 1–16.

Greenberg J.M., LeRoux A.Y., Baraille R., Noussair A.
Analysis and approximation of conservation laws with source terms. SIAM J. Numer. Anal., 34 (1997), 1980–2007.

Harten A., Engquist B., Osher S., Chakravarthy S.
Uniformly High Order Accurate Essentially Non Oscillatory Schemes III. J. Comput. Phys. 71 (1987), 231–303.

References

Hou T.Y., LeFloch P.G.
Why nonconservative schemes converge to wrong solutions: error analysis.
Math. Comp. 62 (1994), 497–530.

LeFloch P.G., Tzavaras A.E.
Representation of weak limits and definition of nonconservative products.
SIAM J. Math. Anal., 30 (1999), 1309–1342.

LeVeque R.
Balancing source terms and flux gradients in high-resolution Godunov methods: The quasi-steady wave-propagation algorithm. J. Comput. Phys., 146 (1998), 346–365.

López, J.A.
Métodos numéricos bien equilibrados de alto orden para sistemas hiperbólicos no conservativos. Aplicaciones a modelos de aguas someras. Ph.D. Thesis. Universidad de Málaga, 2011.

Marquina A.
Local piecewise hyperbolic reconstructions for nonlinear scalar conservation laws. SIAM J. Sci. Comp. 15 (1994), 892–915.

References

Muñoz M.L., Parés C.
Godunov method for nonconservative hyperbolic systems, Math. Model. Numer. Anal. 41 (2007), 169–185.

Muñoz M.L., Parés C.
On some difficulties of the numerical approximation of Nonconservative Hyperbolic Systems. SEMA Journal 47 (2009), 23–52.

Muñoz M.L., Parés C.
On the convergence and well-balanced property of path-conservative numerical schemes for systems of balance laws, J. Sci. Comp. 48 (2011) 274–295.

Noelle S., Pankratz N., Puppo G., Natvig J.R.
Well-balanced finite volume schemes of arbitrary order of accuracy for shallow water flows. J. Comput. Phys.213 (2006) , 474–499.

Noelle S., Xing Y., Shu C.-W.
High Order Well-balanced Finite Volume WENO Schemes for Shallow Water Equation with Moving Water. J. Comput. Phys. 226 (2007), 29–59.

References

Parés C.,
Numerical methods for nonconservative hyperbolic systems: a theoretical framework. SIAM J. Num. Anal. 44 (2006) no. 1, 300–321.

Parés C. Castro M.J.
On the well-balanced property of Roe's method for nonconservative hyperbolic systems. Applications to shallow-water systems. M2AN Math. Model. Numer. Anal., 38 (2004), 821–852.

Perthame B., Simeoni C.
A kinetic scheme for the Saint–Venant system with a source term. Calcolo, 38 (2001), 201–231.

Perthame B., Simeoni C.
Convergence of the upwind interface source method for hyperbolic conservation laws, in Hyperbolic Problems: Theory, Numerics, Applications, Thou and Tadmor, ed., Springer, Berlin, 2003.

Puppo G., Russo G. eds.
Numerical Methods for Balance Laws. Quaderni di Matematica, vol. 24. Dipartamento di Matematica Seconda Universitá di Napoli, 2009.

References

Roe P.L.
Upwind differencing schemes for hyperbolic conservation laws with source term. in Nonlinear Hyperbolic Problems, C. Carasso, P. A. Raviart, and D. Serre, eds., Lecture Notes in Math. 1270, 41–51. Springer, 1987.

Russo G., Khe A.
High order well balanced schemes for systems of balance laws. In Hyperbolic problems: theory, numerics and applications, 919–928. Amer. Math. Soc. 2009.

Shu C.-W.
Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, ICASE Report 97-65, 1997.

Shu C.-W., Osher S.
Efficient implementation of essentially non-oscillatory shock capturing schemes. J. Comput. Phys. 77 (1988) 439–471.

Shu C.-W., Osher S.
Efficient implementation of essentially non-oscillatory shock capturing schemes II.. J. Comput. Phys. 83 (1989) 32–78.