Crafting Consensus ⋆

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Abstract

The paper analyses the problem of a committee chair using *favours* at her disposal to maximize the likelihood that her proposal gains committee support. The favours increase the probability of a given member approving the chair's proposal via a smooth *voting function*. The decision-making protocol is any quota voting rule. The paper characterizes the optimal *allocation* of any given level of favours and the optimal *expenditure* minimizing level of favours. The optimal allocation divides favours uniformly among a coalition of the committee members. At a low level of favours, the coalition comprises *all* committee members. At a high level, it is the *minimum winning coalition*. The optimal expenditure level guarantees the chair certain support of the minimum winning coalition if favours are abundant and uncertain support of all committee members if favours are scarce; elitist or egalitarian committees are compatible with a strategic chair. The results are robust to changing the chair's objectives and to alternative voting functions.

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1 Introduction

Committees are responsible for numerous economic and political decisions. Examples abound and include boards of directors, legislative committees, faculty meeting attendees, monetary policy committees or party conferences. While not always codified, the typical code of conduct of most committees is to reach a decision by voting over an agenda set by its chair. The importance of agenda setting power in shaping the final committee decision has long been acknowledged in the political economy literature.

The chair of the committee typically controls other resources in her institution in addition to holding agenda setting power. She may have at her disposal future promotions, bonuses, teaching reductions, electoral campaign fund allocations or patronage. Moreover, she can use these *favours* as a means of influencing the final committee decision to her liking.

In this paper, we analyse the chair's problem of crafting consensus among the committee members by allocating the favours at her disposal. Our model focuses on the moment of an imminent committee vote on the acceptance or rejection of her proposal. We are agnostic about the nature of the proposal; it can represent a particular corporate or political strategy, interest rate level, legislative bill or the hiring of specific job market candidate.

Favours enter our model via their influence on the probability of each of the committee members casting a vote favourable to the chair's proposal, a *yes* vote. In most of our analysis, we capture the relationship between the favours and the probability by simple linear *voting functions*. The decision-making protocol of the committee is a standard quota voting rule requiring a certain fraction of approving votes for the chair's proposal to pass. We allow for any quota voting rule, including unanimity.

We analyse two nested decision problems faced by the chair. The first concerns the optimal allocation of a fixed amount of favours to maximize the probability that her proposal is approved. We call this the *favour allocation* problem. The second problem concerns the optimal level of favours to distribute. Endowed with an *overall budget* of favours, the chair is the residual claimant, conditional on the approval of her proposal, of any favours not allocated among the committee members. Naturally, she attempts to minimize the cost of approval, and we call her doing so the *consensus expenditure* problem. The two problems are nested. For any level of favours distributed in the second problem, she allocates them optimally using the solution from the first.

We characterize the solution to the favour allocation problem and show that it depends crucially on the fixed amount of favours the chair is allocating, on her *budget*. When the budget suffices to buy the certain support of at least a minimum winning coalition of the committee members, she buys its support and ensures that her proposal is approved.

However, when the chair's budget is smaller than the size of the minimum winning coalition in the committee *less one*, she allocates the budget uniformly among *all* committee members. Uniform allocation among committee members receiving positive favours is a general feature of the solution to the favour allocation problem. It holds even when the chair's budget is close to adequate to purchase the support of the minimum winning coalition. However, in this case, the solution focuses on a subset of the committee members that is still larger than the minimum winning one.

We interpret these results as showing the dual role of the *yes* votes. The votes can be both substitutes and complements. Receiving one additional vote beyond the number sufficient for approval has no additional benefit for the chair. The votes are substitutes in this case. However, receiving an additional vote that shifts the committee decision from rejection to acceptance is extremely valuable to the chair. The votes are complements in this case. The size of the budget that the chair controls determines which of the cases applies. For small budgets, the votes are scarce irrespective of the budget allocation and are thus complements. For large budgets, the votes are abundant and are thus substitutes.

Next, we turn attention to the consensus expenditure problem. We derive several cut-offs characterizing the solution to this problem for any committee size and any quota voting rule. The first, the *minimal consensus* cut-off, presents the lower bound on the overall budget ensuring the solution involves the chair purchasing the certain support of the minimum winning coalition of committee members. This implies that her proposal passes and she obtains any remaining favours.

The remaining cut-offs represent increasingly tighter upper bounds on the overall budget that transform the solution first to the chair not purchasing the certain support of the committee, then to the chair purchasing the support of larger than minimum winning coalition and finally to the chair purchasing the support of all committee members. We term the last bound the *maximal consensus* cut-off. In general, all of the bounds are concentrated around the minimum number of votes required for approval, that is, around the size of the minimum winning coalition.

Our results thus imply that committees with scarce resources are more likely to be egalitarian, even in the presence of a strategic chair. Conversely, committees with abundant resources will be elitist in that the chair will allocate favours to a subset of the members. Somewhat surprisingly, the amount of resources and the number of committee numbers receiving these resources are inversely related. Furthermore, for a fixed overall budget and more demanding quota rule, the model implies more egalitarian committees, as the scarcity of resources is measured relative to the voting rule, not relative to committee size.

After solving the two optimization problems, we show that our results, namely the solution to the consensus expenditure problem above the minimal consensus cut-off and the cut-off itself, do not change after altering the benchmark model in several ways. First, we consider alternative utility functions for the chair, incorporating risk aversion, the upfront costs of crafting consensus or the outside options she receives when her proposal is rejected. Second, we show that our results extend to a model with (some) non-homogeneous nonlinear voting functions.

Our work is related to several strands of the literature. The first strand is the study of multilateral bargaining with agenda setting power (see Romer and Rosenthal, 1978, 1979; Baron and Ferejohn, 1989; Banks and Duggan, 2000, 2006; Eraslan, 2002; Cardona and Ponsati, 2007, 2011, among others). Papers in this literature assume, and we depart from, that the agenda setter faces a committee of members voting deterministically. In the language of our model, the literature assumes step voting functions. Below some level of favours, a member votes *yes* with zero probability and with unit probability above that level. The standard result in this literature is that agenda setter purchases the certain support of the minimum winning coalition provided that her budget is large enough to do so. The small budget case is generally discarded on the grounds of not being interesting. Our model smoothes the voting functions, delivers similar results for the large budget case and shows that nontrivial results, robust supermajorities, obtain for the small budget case.

The ability of our model to generate supermajorities that receive positive favours also associates our work with another strand of the literature, that of coalition and government formation (Gamson, 1961; Riker, 1962; Austen-Smith and Banks, 1988; Baron, 1991, 1993; Diermeier and Merlo, 2000; Bassi, 2013, see Laver, 1998, for a survey). Within this literature, minimum winning coalitions arise as a key prediction of many formal models, although such coalitions find mixed empirical support (see Schofield, 1995; Sened, 1996, and references cited therein). Our work both complements and differs from this literature, similarly to the way it both complements and differs from the literature on agenda setting.

Finally, our work relates to the literature on vote buying, which shows, among other interesting findings, that minimum winning coalitions need not be optimal. Groseclose and Snyder (1996) and Banks (2000) present a vote buying model with supermajorities as an equilibrium prediction. The mechanism behind their result is strategic. The first moving vote buyer purchases a supermajority of votes to make vote buying prohibitively expensive for the second moving vote buyer.[1] The key difference between our model and those presented in these two papers as well as in most of the vote buying literature (Myerson, 1993; Diermeier and Myerson, 1999; Dekel, Jackson, and Wolinsky, 2008, 2009; Le Breton and Zaporozhets, 2010; Morgan and Vardy, 2011, 2012; Seidmann, 2011; Le Breton, Sudholter, and Zaporozhets, 2012) is that, while other models examine a setting in which two vote buyers receive the votes they purchase with certainty, in our model, a single vote buyer, the committee chair, does not receive votes against the favours she allocates with certainty. The first difference makes our analysis simpler, enabling us to abstract from strategic interaction between vote buyers and focus on the effect of the second difference, uncertainty in voting, on the chair's behaviour.

Several contributions to the vote buying literature differ from our model in only one of these respects. These include Ferejohn (1986), Snyder (1991) and Dal Bo (2007), with a single vote buyer but no uncertainty in voting, and Le Breton and Zaporozhets (2007), with uncertainty in voting but two

[1] There may be other ways to generate larger than minimum winning coalitions, some of which are mentioned in Groseclose and Snyder (1996). These include norms of universalism, the tendency for ideologically connected coalitions to form, uncertainty about the size of the voting body created by abstention (Koehler, 1975) or the agenda setter's aspiration for political stability and greatness (Doron and Sherman, 1995).

vote buyers. The only paper that combines a single vote buyer and uncertainty in voting is Felgenhauer and Gruner (2008). They study information aggregation under open and closed voting with three players and majoritarian voting rather than the optimal process for building consensus in a committee with arbitrary size and voting rule, as we do.[2]

Finally, in a singular contribution, Mandler (2013) asks how electoral campaigns should be evaluated using a model with a non-degenerate distribution over voters' actions and uncertainty regarding the shape of the distribution itself. He focuses on limit results as the size of the electorate increases and on campaign evaluation, not on characterization of the optimal one.

2 Model

The model posits chair of a committee trying to craft consensus regarding the course of actions the chair has proposed. The committee comprises $n \in \mathbb{N}_{>0}$ members, with the set of the committee members denoted by $N = \{1, \ldots, n\}$. The decision-making protocol within the committee is a standard quota voting rule requiring $q \geq 1$ or more approving votes for the chair's proposal to pass. The most demanding voting rule we allow for is unanimity, so that $q \leq n$.[3]

The chair has overall budget $B > 0$ of *favours* available to her. Choosing to redistribute $b \leq B$ amount of favours among N means dividing b among the committee members, with each member $i \in N$ receiving x_i amount of favours. Upon receiving x_i, $i \in N$ approves the chair's proposal (votes *yes*) with probability $p_i(x_i) = x_i$ and disapproves of it (votes *no*) with the remaining probability. Throughout, we call the functions translating favours into approving votes, $(p_i : [0,1] \to [0,1])_{i=1}^n$, *voting functions*. When

[2] Our model and the rest of the vote buying literature also differ from Dal Bo (2007) and (in part of their analysis) Felgenhauer and Gruner (2008); Morgan and Vardy (2011) in one further respect. The standard assumption is that payments can be conditional either on individual behaviour (voting yes or no) or on the aggregate outcome (acceptance or rejection). They assume that payment to each voter can be conditional on that voter being pivotal, i.e. on the margin of acceptance. This induces competition among voters and the vote buyer is able to ensure acceptance at a significantly lower cost. Similar Bertrand competition among voters arises, as a result of their ability to commit to vote for certain action, in Ferejohn (1986).

[3] We assume that the chair is not a voting member of the committee. This is not in any way significant for our analysis. If the chair were to vote, we could relabel n and q, if necessary, and proceed without further alterations.

the likelihood of confusion is minimal, we work directly with $(p_i)_{i=1}^n$ instead of using the voting functions in their full specification.[4]

We denote by $\mathbf{p} = (p_i)_{i=1}^n \in \mathbb{R}^n$ the vector of probabilities of the individual committee members approving the chair's proposal. Boldface denotes vectors in general, such that $\mathbf{x} = (x_i)_{i=1}^n$ represents vector of favours allocated to N. Dropping p_i and p_j from $\mathbf{p}$ generates the new vector $\mathbf{p}^{\{ij\}} \in \mathbb{R}^{n-2}$. By $\mathbf{p}(r,p,s)$, we denote the vector of probabilities when r committee members approve with zero probability, s members approve with unit probability and the remaining $n-r-s$ members approve with probability equal to $p = (b-s)/(n-r-s)$. That is,

$$\mathbf{p}(r,p,s) = (\underbrace{0,\dots,0}_{r}, \underbrace{p,\dots,p}_{n-r-s}, \underbrace{1,\dots,1}_{s}). \tag{1}$$

Note that amount of favours required to generate $\mathbf{p}(r,p,s)$ is b, r members are receiving zero favours, s members are receiving unit favours and $n-r-s$ members are receiving $(b-s)/(n-r-s)$ favours.

Because the individual entries in $\mathbf{p}$ represent probabilities, when allocating favours, the chair is not only limited by the total amount of favours she has decided to redistribute, b, but also by the individual entries of $\mathbf{p}$ remaining in the $[0,1]$ interval. For fixed b, we denote the set of possible probability allocations by

$$X(b) = \{\mathbf{p} \in \mathbb{R}^n | \sum_{i=1}^n p_i \le b \ \wedge \ 0 \le p_i \le 1 \ \ \forall\, i \in N\}. \tag{2}$$

Given $\mathbf{p} \in X(b)$, the probability of the chair's proposal being approved is

$$\mathbb{P}_{q|n}[\mathbf{p}] = \sum_{s=q}^n \mathbb{P}^*_{s|n}[\mathbf{p}] \tag{3}$$

where $\mathbb{P}^*_{s|n}[\mathbf{p}]$ is the probability that exactly s out of n committee members approve the chair's proposal when the individual probabilities of approval are equal to $\mathbf{p}$.

We analyse the pair of nested optimization problems the chair faces when

[4] The full specification is required for the discussion of alternative voting functions in section 5. Note that $p_i(x_i) = x_i$ can represent information asymmetry between the chair and $i \in N$. Assuming i is choosing between x_i and $\theta_i \sim \mathcal{U}[0,1]$ outside option, i votes *yes* if and only if $x_i \ge \theta_i$, which occurs with probability $\mathbb{P}[\theta_i \le x_i] = x_i$.

crafting consensus among the committee members. The first optimization problem concerns, for a fixed b, the optimal allocation of favours among the committee members to maximize the probability of the chair's proposal begin approved. We call this optimization problem the *favour allocation* ($\mathbb{FA}$) problem and define it formally as

$$\begin{aligned} \max_{\mathbf{p}\in\mathbb{R}^n} \quad & \mathbb{P}_{q|n}[\mathbf{p}] \\ \text{s.t.} \quad & \sum_{i=1}^{n} p_i \le b \\ \text{s.t.} \quad & 0 \le p_i \le 1 \quad \forall\, i \in N. \end{aligned} \tag{$\mathbb{FA}$}$$

It is easy to see that any solution to $\mathbb{FA}$ must satisfy the Kuhn and Tucker (1951) conditions, with the associated Lagrangian being

$$L(\mathbf{p}, \lambda, \mathbf{m}^+, \mathbf{m}^-) = \mathbb{P}_{q|n}[\mathbf{p}] - \lambda \left[\sum_{i=1}^{n} p_i - b\right] - \mathbf{m}^+ * (\mathbf{p} - \mathbf{1}) + \mathbf{m}^- * \mathbf{p} \tag{4}$$

where λ is the Lagrange multiplier associated with the constraint represented by b, $\mathbf{m}^+ = (m_i^+)_{i=1}^n$ is the vector of multipliers associated with the constraint on the probabilities being less than unity, $\mathbf{1}$ is the unit vector in $\mathbb{R}^n$, $\mathbf{m}^- = (m_i^-)_{i=1}^n$ is the vector of multipliers associated with the constraint on the probabilities being non-negative and $*$ is standard inner product of two vectors in $\mathbb{R}^n$.

For a given b, we denote by $\mathbb{R}_{q|n}[b]$ value of the maximized objective function in $\mathbb{FA}$. That is, if $\mathbf{p}^*$ solves $\mathbb{FA}$ for b, then $\mathbb{R}_{q|n}[b] = \mathbb{P}_{q|n}[\mathbf{p}^*]$. A simple argument shows that $\mathbb{FA}$ has a solution and $\mathbb{P}_{q|n}[\mathbf{p}'^*] = \mathbb{P}_{q|n}[\mathbf{p}''^*]$ if both $\mathbf{p}'^*$ and $\mathbf{p}''^*$ solve $\mathbb{FA}$, and hence $\mathbb{R}_{q|n}[b]$ is well defined as a function of b.

The second optimization problem the chair faces concerns the least expensive means of crafting consensus among the committee members. The incentive to minimize the cost of consensus arises due to her being able to retain the share of the overall budget not redistributed among the committee members, conditional on her proposal being approved. We call this optimization problem the *consensus expenditure* ($\mathbb{CE}$) problem and define it formally by

$$\begin{aligned} \max_b \;\; & (B-b)\mathbb{R}_{q|n}[b] \\ \text{s.t.} \;\; & 0 \le b \le B. \end{aligned} \tag{$\mathbb{CE}$}$$

The two optimization problems are nested, as $\mathbb{CE}$ assumes an optimal allocation of favours for any level of favours redistributed, it nests the solution to $\mathbb{FA}$.[5]

3 Optimal favour allocation

Proposition 1 (Optimal favour allocation)**.** *Let* $\mathbf{p}^*$ *with the associated* λ^**,* $\mathbf{m}^{+,*}$ *and* $\mathbf{m}^{-,*}$ *be a solution to* $\mathbb{FA}$*. Then*

1. $\lambda^* > 0$ *if and only if* $b < q$
2. *if* $\lambda^* = 0$, $\mathbb{R}_{q|n}[b] = 1$ *and* $\mathbf{p}^*$ *solves* $\mathbb{FA}$ *if* $\mathbf{p}^* = \mathbf{p}(r^*, p^*, s^*)$ *where* $s^* \ge q$
3. *if* $\lambda^* > 0$, $\mathbb{R}_{q|n}[b] < 1$, $\sum_{i=1}^n p_i^* = b$ *and* $\mathbf{p}^*$ *solves* $\mathbb{FA}$ *if and only if* $\mathbf{p}^* = \mathbf{p}(r^*, p^*, s^*)$ *for (not necessarily unique)* r^* *and* s^* *satisfying*
   (a) $s^* = 0$
   (b) $r^* \le n - q$
   (c) $r^* = n - q$ *if and only if* $b \ge \hat{b}_q \in \left[q + 2 + \frac{1}{q} - \left(1 + \frac{1}{q}\right)^{q+1}, q\right)$
   (d) $r^* = 0$ *if* $b < q - 1$
   (e) r^* *and* s^* *in parts 3.(a) and 3.(d) are unique,* r^* *in part 3.(c) is unique if* $b > \hat{b}_q$

Proof. See Appendix A1.

Proposition 1 characterizes the solution to the $\mathbb{FA}$ problem. Part two applies to cases in which the chair controls favours sufficient to purchase the certain support of the minimum winning coalition, that is, when $b \ge q$. In

[5] The objective function in $\mathbb{CE}$ embodies the assumption of the chair's favour payments being contingent on the acceptance of her proposal. In the terminology of the vote buying literature (Dekel et al., 2008), this makes the favours equivalent to *campaign promises*. We analyse the other possibility addressed in this literature, *upfront payments* contingent on votes, in section 5. Naturally, in our model, the upfront payments will not guarantee that the chair will receive the votes that she has bought.

this case, she allocates unit favours to at least q of the committee members, thus ensuring the acceptance of her proposal.

In the opposite case where $b < q$, part three applies and shows that any solution to $\mathbb{FA}$ has to have a structure of $\mathbf{p}(r^*, p^*, s^*)$. That is, r^* of the committee members receive zero favours, s^* of the committee members receive unit favours and all remaining members receive the same amount of favours $p^* = \frac{n-s^*}{n-r^*-s^*}$.[6]

The remainder of part three characterizes r^* and s^*. Setting aside the 3.(c) part for now, there are no committee members receiving unit favours, $s^* = 0$, at least the minimum winning number of committee members receives positive favours, $r^* \leq n - q$, and when $b < q - 1$, all committee members receive positive favours. Taken together, these results imply that the solution to $\mathbb{FA}$ has the following simple structure $\mathbf{p}^* = \left(\frac{b}{n}\right)_{i=1}^{n}$ when $b < q - 1$.

Part 3.(c) partially characterizes the remaining interval $b \in [q - 1, q)$. It shows that there exists $\hat{b}_q$ such that for $b > \hat{b}_q$, the solution focuses on the minimum winning coalition, $r^* = n - q$. For $b \leq \hat{b}_q$, all we know based on proposition 1 is that any solution to $\mathbb{FA}$ satisfies $\mathbf{p}(r^*, p^*, 0)$ with some $r^* < n - q$. A subset of committee members larger than the minimum winning coalition receives positive favours, and a potentially empty subset of committee members receives zero favours. Additionally, there might be multiple solutions. $\mathbb{P}_{q|n}[\mathbf{p}(r, p, s)]$ is continuous in b for any r, and we know that as b increases from $q - 1$ to $\hat{b}_q$, r^* increases from 0 to $n - q$. Therefore, there must be a b for which more than one value of r^* solves $\mathbb{FA}$.[7]

The change in the solution from focusing on all committee to focusing on the minimum winning subset of committee members is independent of committee size. It occurs within an interval of unit length. The intuition behind this result is the fact that votes can be both complements and substitutes. They are complements when their number is smaller than $q-1$ and

[6] This observation greatly simplifies the subsequent analysis, as it implies that the uncertainty remaining in the number of votes the chair receives has a Binomial distribution with $n-r^*-s^*$ trials, not a Poisson Binomial distribution. The Poisson Binomial random variable represents the number of successes in n independent Bernoulli trials with success probabilities of $p_1, \ldots, p_n$. The Binomial distribution is the special case when all of the probabilities are equal.

[7] Our conjecture is that r^*, with increasing b, visits any integer in $\{0, \ldots, n - q\}$ in a monotonic fashion. We have (numerically) confirmed the conjecture for $n \leq 13$ and any $q \leq n$. We do not require the conjectured result in what follows, providing us with weak incentives to develop its proof.

substitutes when their number is larger than q. Additionally, irrespective of how many of the committee members receive positive favours from the chair's budget b, the most likely number of *yes* votes she receives approximates the integer part of b. Combining these two observations, increasing b from $q-1$ to q transforms the votes from complements to substitutes. As a result, the chair is, optimally, attempting to acquire the largest possible number of votes when $b < q-1$ and only the minimum number necessary when $b \geq q$, independent of how large the committee is.

To our knowledge, the observation that votes can be complements is novel. The literature related to agenda setting power and minimum winning coalitions discussed in the introduction typically has an agenda setter disposing of a budget sufficient to purchase the votes of at least the minimum winning coalition. Under deterministic voting, assuming a small budget makes the agenda setter's problem trivial; her proposal is never accepted. Our voting functions smooth out the deterministic voting and allow us to uncover arguably interesting results in a region not previously considered worthy of investigation.

4 Optimal consensus expenditure

We now analyse solution to the chair's problem of crafting consensus in the least expensive manner. We present three results characterizing the solution to the $\mathbb{CE}$ problem, b^*. The first, proposition 2, derives upper bound on the overall budget B such that $b^* < q$ and pair of closely related bounds. The second result, theorem 1, presents the opposite result, that is, the lower bound on B ensuring that $b^* = q$. Notice that the solution to $\mathbb{CE}$ can never satisfy $b^* > q$. For any b above q, the chair's proposal is accepted with certainty, and hence the only effect of increasing b is giving away favours she values. The last result, theorem 2, is similar to theorem 1 but applies to large committees.

Proposition 2 (Optimal consensus expenditure with small B)**.** *Let b^* be a solution to $\mathbb{CE}$. Then*

1. *if $B < q+1$, $b^* < q$*
2. *if $B < q - \frac{1}{q}$, b^* implies $r^* < n-q$ in the associated $\mathbb{FA}$*

3. *if* $B < q - 1$, b^* *implies* $r^* = 0$ *in the associated* $\mathbb{FA}$

Proof. See Appendix A1.

Theorem 1 (Optimal consensus expenditure with large B)**.** *Let* b^* *be a solution to* $\mathbb{CE}$*. Then,* $b^* = q$ *if*

1. $B \geq n + 1$
2. $B \geq (q+1)(2 - \frac{q}{n})$
3. $B \geq q + (1 - \frac{q}{n})\,{}_2F_1(1, 1+n, 1+q, \frac{q}{n})$
4. $B \geq n$ *and* $q \leq n - 1$

where the first three bounds are in descending order. b^* *is unique.*

Proof. See Appendix A1.

Proposition 2 presents series of upper bounds on the overall budget. The first bound, $q+1$, is the largest overall budget, implying that the solution to the $\mathbb{CE}$ problem involves the chair *not* purchasing the certain support of the committee members. Doing so would be either too costly or impossible given the overall budget at her disposal. We call this the *uncertain consensus* cut-off. The second bound, $q - \frac{1}{q}$, is the largest overall budget such that solution to $\mathbb{CE}$ in the associated $\mathbb{FA}$ has the chair allocating strictly positive favours to more than the minimum winning subset of committee members. For this reason, we call it the *non-minimal consensus* cut-off. Notice that for large committees and q expressed as a fraction of n, the non-minimal consensus cut-off approaches q. Finally, the third bound, $q-1$, is the largest overall budget, implying that the solution to $\mathbb{CE}$ is such that the associated $\mathbb{FA}$ involves the chair allocating strictly positive favours to all committee members. Correspondingly, we call it the *maximal consensus* cut-off.

Theorem 2 present series of lower bounds on the overall budget. Each of these bounds implies that the solution to $\mathbb{CE}$ involves the chair purchasing the certain support of the committee by allocating unit favours to the minimum winning coalition. For this reason, we call it the *minimal consensus* cut-off. The three minimal consensus cut-offs in theorem 2 increase both in their strength and complexity. The first, $n+1$, is extremely simple but relatively weak. It requires that the overall budget be more than sufficient to purchase the certain support of the entire committee. The second,

$(q+1)(2-\frac{q}{n})$, depends on the voting rule in addition to committee size. The third, expressed in terms of a hypergeometric function, is the strongest of the three but does not lend itself to straightforward evaluation.[8]

Taking the results in proposition 2 and theorem 1 in their entirety, we have shown that given a small overall budget, the chair allocates part of it uniformly among the committee members and is able to retain the rest, conditional on her proposal passing. Conversely, given a large overall budget, she purchases the certain support of the minimum winning coalition of committee members.

We stress that all of the bounds are sufficient but not necessary, except for special values of q. In fact, the uncertain and minimal consensus cut-offs would be equal if they were necessary. The gap between the cut-offs reflects our approach to proving theorem 1. We ensure, and the minimal consensus cut-off guarantees, that the chair's expected utility is strictly increasing in b, for any number of the committee members receiving zero favours $r \in \{0, \ldots, n-q\}$. With respect to the gap between the cut-offs, there is none if $q = n$ and it is generally small for large or small q. For a majority voting rule such that $q = \frac{n}{2}$, the strongest bound in theorem 1 is proportional to $\frac{n}{2} + \frac{\sqrt{\pi n}}{2\sqrt{2}}$, and hence the gap between the cut-offs approximately equals $\frac{\sqrt{\pi n}}{2\sqrt{2}}$.[9]

The final component of theorem 1 partially closes the gap between the cut-offs. Taking $q = n-1$ and $B = n$ as an example of a case when it applies not covered by the remaining parts, the chair's expected utility is decreasing in b close to q when allocating favours to the whole committee. To ensure that $b^* = q$, we had to confirm that when this happens, it is indeed better for her to allocate positive favours to $n-1$ of the committee members. Such a comparison of expected utilities is technically complex, preventing us from using it to derive further results.

The final result we present uses a large n argument to close the gap between the cut-offs. It shows that when the overall budget B is larger than the quota voting rule q, it will not be optimal for the chair to set $b < q$.

[8] A Gaussian or ordinary hypergeometric function ${}_2F_1$ is a special function that includes many other (special) functions as special cases. See the preliminary section to the proofs in appendix A1 or Olver, Lozier, Boisvert, and Clark (2010) for details.

[9] $\frac{1}{2}\,{}_2F_1(1, 1+n, 1+\frac{n}{2}, \frac{1}{2})$ equals $z(n) = \frac{1}{2} + \frac{\sqrt{\pi}\,\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(\frac{1+n}{2}\right)}$ where Γ is the Euler gamma function with $\lim_{n\to\infty} z(n)/\sqrt{n} = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

When increasing n, we fix all the variables as fractions of n.

Theorem 2 (Optimal consensus expenditure in large committees)**.** *Fix* $b' = \frac{b}{n}$*,* $q' = \frac{q}{n}$ *and* $B' = \frac{B}{n}$ *such that* $b' < q' < B'$*. Then, for any* $\epsilon = q' - b' > 0$*, there exists* $\underline{n}_\epsilon$*, such that* b' *cannot be a solution to* $\mathbb{CE}$ *for any* $n \geq \underline{n}_\epsilon$*.*

Proof. See Appendix A1.

5 Extensions and discussion

We present a series of alternatives to our benchmark model and investigate how these changes affect the results presented thus far. The first group of extensions relates to the chair's objective function, while the second concerns the voting functions. We omit formal proofs, as these are straightforward.

Chair

First, the objective function in $\mathbb{CE}$ assumes that the chair is risk neutral. The rejection of her proposal results in zero payoff and zero utility. Assuming that she is risk averse and maximizes the concave utility function, u would change the objective function in $\mathbb{CE}$ to

$$u(B-b)\mathbb{R}_{q|n}[b] + u(0)(1-\mathbb{R}_{q|n}[b]). \tag{5}$$

Theorem 1 still applies. Any degree of risk aversion only reinforces the optimality of $b^* = q$, as it involves the certain acceptance of the chair's proposal. However, proposition 1 no longer applies.[10] Consider a logarithmic u as an example. The chair would never set $b^* < q$ for $B > q$, as doing so would result in the possibility of zero payoff with infinite disutility.[11]

Next, assume that rejection of the chair's proposal does not result in zero payoff as she receives some outside option o. This changes her objective to

$$(B-b)\mathbb{R}_{q|n}[b] + o\,(1-\mathbb{R}_{q|n}[b]) = (B-o-b)\mathbb{R}_{q|n}[b] + o \tag{6}$$

[10] That is, parts one and two of proposition 1 no longer apply. Part three still holds, not only here but for any alternative objective function considered in this subsection. What is important for part three of the proposition is that the solution to $\mathbb{FA}$ remains unchanged.

[11] Changing the chair's objective function leaves the $\mathbb{FA}$ problem unchanged, as she still attempts to allocate favours in an optimal way.

and with o being constant, proposition 1 and theorem 1 continue to hold after simple relabelling. In this light, B should not be interpreted in absolute terms but in terms of its size relative to the chair's outside option o.

Finally, assume that the chair has to pay the committee members upfront, not conditional on the acceptance of her proposal as in the benchmark model. In the spirit of the vote buying literature, we discuss *campaign promises* when referring to the benchmark model. With *upfront payments*, the chair's objective function changes to

$$B\,\mathbb{R}_{q|n}[b] - b \tag{7}$$

generating two effects.[12] The first effects shifts the solution to $\mathbb{CE}$ towards $b^* = q$. If $b^* = q$ solves $\mathbb{CE}$ under campaign promises, it also has to solve it, *modulo* the second effect, under upfront payments. b^* implies certain acceptance, making the distinction between conditional and unconditional payments irrelevant and for any $b < q$, the objective function under upfront payments is lower than under campaign promises. Theorem 1 still applies, while proposition 1 no longer holds. It is easy to construct examples when $b^* < q$ under campaign promises changes to $b^* = q$ under upfront payments. The second effect shifts the solution to $\mathbb{CE}$ towards $b^* = 0$. If the chair's overall budget is small such that $B\,\mathbb{R}_{q|n}[b] - b < 0$ for any $b \leq B$, her optimal strategy will be to set $b^* = 0$. If this occurs, theorem 1 no longer holds while proposition 1 still applies.

Note that upfront payments imply a larger or smaller level of favours distributed, relative to campaign promises, conditional on the size of the overall budget. For a small overall budget, the level will be lower and *vice versa*. The lower level of favours redistributed under upfront payments is reminiscent of the same result in Dekel et al. (2008). In their model, the difference between campaign promises and upfront payments can be likened

[12] Assuming that the chair only pays for the votes she actually receives makes the analysis considerably more involved. In the $\mathbb{CE}$ problem, the chair now maximizes, by choosing $\mathbf{p} \in \mathbb{R}^n$ subject to the ex-post budget constraint $\mathbf{p} * \mathbf{1} \leq B$ and $p_i \in [0,1]$ for $\forall i \in N$, either $(B - \mathbf{p} * \mathbf{p})\mathbb{P}_{q|n}[\mathbf{p}]$ or $B\mathbb{P}_{q|n}[\mathbf{p}] - \mathbf{p} * \mathbf{p}$ depending on whether she pays for the votes only when her proposal passes or not. The complication arises because $\mathbb{P}_{q|n}[\mathbf{p}]$ and $-\mathbf{p} * \mathbf{p}$ must be maximized jointly so that we lose the nested nature of the $\mathbb{FA}$ and $\mathbb{CE}$ problems. The only observation we make that does not require extended additional analysis is that, for $B < q - 1$, any solution to the $\mathbb{CE}$ problem must involve a uniform allocation of favours among committee members. This is because, for any $b' < B$, $p_i = \frac{b'}{n}$ for $\forall i \in N$ maximizes both $\mathbb{P}_{q|n}[\mathbf{p}]$ and $-\mathbf{p} * \mathbf{p}$, the former by proposition 1 and the latter by simple argument.

to the difference between English and all-pay auctions. The same intuition applies in our model. On the other hand, the larger level of favours redistributed under upfront payments is specific to our model. The upfront payments can be considered an investment, and the chair attempts to ensure that her investment yields positive profits by increasing the investment itself.

Voting functions

The voting functions are undoubtedly central to our analysis. Their homogeneity and linearity allowed us to carry the analysis very far. However, their exact shape is not crucial for many of our results. First, note that changing the slope of the voting functions does not alter any of the results presented thus far. Changing the benchmark voting functions to $(p_i(x_i) = tx_i)_{i=1}^n$ for some strictly positive t and adjusting the budget in $\mathbb{FA}$ to $\frac{b}{t}$ and the overall budget in $\mathbb{CE}$ to $\frac{B}{t}$ represents a simple change of units. The benchmark voting functions measure favours in probability units. The altered voting functions, with an appropriate choice of t, can measure favours in monetary or any other units.

Next, consider changing the benchmark voting functions $(p_i(x_i) = x_i)_{i=1}^n$ to $(p'_i)_{i=1}^n$ such that

$$\begin{gathered} p'_i(x_i) \le p_i(x_i) \quad \text{for } x_i \in [0,1] \quad \text{and } i \in N \\ \#\{i \in N \,|\, p'_i(1) = 1\} \ge q. \end{gathered} \tag{8}$$

$(p'_i)_{i=1}^n$ can be arbitrary, possibly failing monotonicity, differentiability or even continuity. Naturally, the solution to $\mathbb{FA}$ will depend on the exact shape of the voting functions, and hence proposition 1, and proposition 2 that heavily depends on it, no longer holds. However, theorem 1 still applies. With $b^* = q$ all that matters is that $p'_i(1) = 1$ for the minimum winning coalition of the committee members and $(p'_i)_{i=1}^n$ do not increase the chair's expected utility relative to the benchmark model for any $b < q$. Condition (8) ensures precisely that.

Notice that these two ways of shaping the voting functions without altering the results of theorem 1 allow the model to incorporate a large number of environments with asymmetric information between the chair and the committee members. Assume that each $i \in N$, when voting, compares the favours he has been offered x_i to disutility from the chairman's proposal θ_i, which is his private information, and votes *yes* if and only if $\theta_i \leq x_i$.[13] The chair only knows the distribution of $\theta_i \sim \mathcal{D}$; hence from her perspective, i votes for her proposal with probability $p_i(x_i) = \mathbb{P}_{\mathcal{D}}[\theta_i \leq x_i] = F_{\mathcal{D}}(x_i)$, assuming that $\mathcal{D}$ has a well defined cumulative distribution function $F_{\mathcal{D}}$.

Naturally, there is no guarantee that $F_{\mathcal{D}}(x_i)$ will fit (8). However many distributions can be made to do so if defined on finite support $[u, v]$ and $F_{\mathcal{D}}(v)' = f_{\mathcal{D}}(v) > 0$. Let us illustrate the procedure with an example. Take θ_i distributed according to the Beta distribution on $[0, 1]$ with its two parameters $\alpha = \beta = \frac{1}{2}$, that is, $\theta_i \sim \mathcal{B}_{\frac{1}{2},\frac{1}{2}}[0, 1]$. Then $F_{\mathcal{B}}(x_i) = \frac{2\sin^{-1}\sqrt{x_i}}{\pi}$, which alone does not fit (8). However, we can shift the entire support of θ_i to $[s, 1 + s]$. Naturally, $F_{\mathcal{B}}(1 + s) = 1$ due to the finite support of $\mathcal{B}$ and the (satisfied) requirement $F_{\mathcal{B}}(v)' = f_{\mathcal{B}}(v) > 0$ ensures that there will exist s such that $F_{\mathcal{B}}(x_i) \leq \frac{x_i}{s}$ for any $x_i \in [0, s]$.[14] Numerous other statistical distributions can be used in a similar manner, including truncated versions of those supported on the real line.

The altered voting functions $(p_i')_{i=1}^n$ still have to satisfy $p_i'(0) = 0$. That is, both the benchmark and altered specifications of the voting functions requires zero probability of acceptance in return for zero favours. Assuming that the chair faces such a disapproving committee might not be the best way to model certain situations. Assume instead that i votes for the chair's proposal with non-zero probability, even when allocated zero favours. That is, change each of the voting functions to

$$p_i(x_i) = \begin{cases} \epsilon & \text{for } x_i \leq \epsilon \\ x_i & \text{for } x_i \geq \epsilon \end{cases} \tag{9}$$

for small ϵ and $\forall\, i \in N$. We call these voting functions with *minimal support*.

The impact of the minimal support functions is most pronounced for the favour allocations where zero favours are awarded to a non-empty sub-

[13] In the benchmark model, the game we have in mind is the chair committing to pay $\mathbf{x}$ conditional on the acceptance of her proposal. Alternatively, the chair can announce $\mathbf{x}$ and only pay for the votes actually received, thereby changing her objective to that discussed in the previous subsection. In any case, we have to assume the *stage undominated voting* of Baron and Kalai (1993) to make the comparison between x_i and θ_i the one determining the voting decision.

[14] For $F_{\mathcal{D}}(v)' = f_{\mathcal{D}}(v) = 0$, the linear function $\frac{x_i}{s}$ can never 'squeeze' in between unity and the cumulative distribution function $F_{\mathcal{D}}$ for values of x_i close to the upper bound of the support of $\mathcal{D}$.

set of committee members. In the $\mathbb{FA}$ problem, this generally moves the solution in the direction of the minimum winning coalition. Nevertheless, we can readily generate examples where allocating favours to all committee members remains optimal.

For the $\mathbb{CE}$ problem, the chair's objective function generally increases for $b \leq q$. However, the impact is rather small. For small values of b, the chair generally finds it optimal to allocate favours to supermajorities, and for large b, the effect is of the second order. To see this, assume that $b = q - \kappa$ for a sufficiently small κ such that the solution to $\mathbb{FA}$ involves the allocation of positive favours to q of the committee members. Call a committee member with zero or near unit favours z and u, respectively. The minimal support only matters when u rejects and z accepts the chair's proposal. This occurs with probability $\frac{\kappa}{q}\epsilon$, which for small values of κ and ϵ is of second order. Nevertheless, the minimal support voting functions can induce $b^* < q$, where the benchmark voting functions would have $b^* = q$. Intuitively, the chair attempts to save the cost of crafting consensus by relying on minimal support.

These basic insights suggest that our results hold beyond the singular case of $\epsilon = 0$. We have not investigated the minimal support further and do not claim that any of the results above remain unchanged, as the impact will obviously depend on the magnitude of ϵ. In fact, for $\epsilon = 1$, the chair sets (uniquely) $b^* = 0$ in $\mathbb{CE}$ and her problem becomes trivial.

Finally, we comment on the relationship between the shape, concavity or convexity of the voting functions and the complement substitute quality of votes.[15] First, consider a convex voting function and initial situation of two committee members, i and j, receiving the same amount of favours $x_i = x_j$ and voting *yes* with the same probability $p_i = p_j$. Increasing x_i and decreasing x_j by the same amount increases p_i to a greater extent than it decreases p_j. This creates a clear incentive to set $x_i \gg x_j$. However, when the probability of the chair's proposal passing, $\mathbb{P}_{q|n}$, is small, votes are still complements, creating incentives to set $x_i \approx x_j$, with the optimal favour allocation determined by which of the two effects dominates. Conversely,

[15] As an example, consider binary voting over alternatives yielding utility x_i and y in a quantal response model such as the Quantal Response Equilibrium of McKelvey and Palfrey (1995). Then, the probability that i votes for x_i is $p_i(x_i) = \frac{\exp \lambda x_i}{\exp \lambda x_i + \exp \lambda y}$, where $\lambda \geq 0$ measures the precision in i's best response. p_i is increasing as a function of x_i, convex for $x_i \leq y$ and concave otherwise.

when $\mathbb{P}_{q|n}$ is large, votes are substitutes and the convexity of the voting function reinforces the tendency to set $x_i \gg x_j$.

The opposite holds for a concave voting function. Increasing x_i and decreasing x_j by the same amount increases p_i to a lesser extent than it decreases p_j, creating incentives to set $x_i \approx x_j$. When $\mathbb{P}_{q|n}$ is large, the substitutability of votes and the concavity of the voting function cancel out. When $\mathbb{P}_{q|n}$ is small, the complementarity of votes and the concavity of the voting function reinforce each other, creating the tendency to set $x_i \approx x_j$.[16]

The standard deterministic voting functions used in the agenda setting literature discussed in the introduction, equal to zero for x_i below some threshold and increasing to unity above the threshold, can be considered a limiting case of a convex voting function. The interval where x_i is small is relevant for any agenda setter. Combined with a typically large budget at the agenda setter's disposal and hence votes being substitutes, it is not surprising that the literature has minimum winning coalitions as one of its key predictions.

6 Conclusion

Crafting consensus within a committee of economic or political agents is a non-trivial task. Idiosyncracies of the individual committee members make the acceptance of any proposal before the committee an uncertain event. Its chair, seeking support for any proposal she put in front of the committee, might use the favours at her disposal to overcome these idiosyncracies and increase the likelihood of her proposal gaining committee support.

We have investigated the optimal means of using the favours, both in terms of their allocation and the amount to use. The best way to allocate the favours is to redistribute them evenly among a coalition of committee members. If the amount of the favours at the chair's disposal is small, the coalition comprises all committee members. If the amount of the favours is large, the coalition comprises the minimum winning coalition. This result

[16] For the Quantal Response Equilibrium mentioned in footnote 15, we have constructed an example with $n = 3$ and $q = 2$ to illustrate how concavity creates incentives for $x_i \approx x_j$ and convexity for $x_i \gg x_j$. As y increases from 0 to 1, the voting functions change in a smooth manner from concave to convex, assuming $b = 1$. Correspondingly, the optimal favour allocation focuses first on all 3 players and then switches to focusing on 2, minimum winning, players.

is driven by votes of the individual committee members being complements when favours are scarce and substitutes when favours are abundant.

The optimal amount of favours to use when the chair can claim any unspent favours is to purchase the certain support of the minimum winning coalition of committee members, provided the favours at the chair's disposal are sufficient. In the opposite case, the chair optimally retains some favours for herself and typically divides the remainder among all committee members. Having a strategic chair leads to egalitarian committees when favours are scarce and elitist committees when favours are abundant.

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A1 Proofs

A1.1 Preliminaries

We present a series of auxiliary results that facilitate the proofs below. First, consider a random variable X_j representing success or failure in a single Bernoulli trial with a probability of success $p \in [0, 1]$. The number of successes in n independent identical Bernoulli trials follows a Binomial distribution $B(n, p)$ with the probability mass function $f(k, n, p)$, giving a probability of exactly $k \in \{0, \ldots, n\}$ successes.

Lemma A1. *For $p \in (0, 1)$, $f(k+1, n, p) \geq f(k, n, p)$ if and only if $k + 1 \leq (n + 1)p$.*

Proof. From $f(k, n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$ we have for $p \in (0, 1)$

$$\frac{f(k+1, n, p)}{f(k, n, p)} = \frac{(n-k)p}{(k+1)(1-p)}. \tag{A1}$$

The lemma follows using simple algebra. □

We denote the probability of exactly k or more successes by $F(k, n, p) = \sum_{s=k}^{n} f(s, n, p)$. Below, we will need to differentiate $F(k, n, p)$ with respect to p. To this end, it is helpful to express $F(k, n, p)$ in terms of a regularized incomplete beta function (see Olver et al., 2010, for details)

$$F(k, n, p) = I_p(k, n - (k - 1)) \tag{A2}$$

where

$$\begin{aligned} I_x(a, b) &= \frac{B(x, a, b)}{B(a, b)} \\ B(x, a, b) &= \int_0^x t^{a-1}(1-t)^{b-1} dt \\ B(a, b) &= \frac{(a-1)!(b-1)!}{(a+b-1)!} \end{aligned} \tag{A3}$$

are a regularized incomplete beta function, an incomplete beta function and a complete beta function, respectively. The derivative of $F(k, n, p)$ for $p \in (0, 1)$ and $k \in \{1, \ldots, n\}$ is thus equal to

$$\frac{\partial F(k, n, p)}{\partial p} = \frac{k}{p} f(k, n, p). \tag{A4}$$

Note that $f(k,n,p) > 0$ when $p \in (0,1)$ for any $k \in \{0,\ldots,n\}$ and any $n \in \mathbb{N}_{>0}$.

Working with $F(k,n,p)$ is in general problematic. However it can be bounded for large n. The following is direct application of theorem 1 in Hoeffding (1963) to the random variables X_j representing the success or failure in the Bernoulli trials.

Theorem A1 (Hoeffding (1963))**.** *Let random variable $X_j \in \{0,1\}$ represent success ($X_j = 1$) or failure ($X_j = 0$) with $\mathbb{P}[X_j = 1] = p \in (0,1)$ for $\forall j \in \{1,\ldots,n\}$. Then*

$$\mathbb{P}\left[\sum_{j=1}^{n} X_j \geq n(p+t)\right] \leq \exp\left[-2nt^2\right] \tag{A5}$$

for $0 < t \leq 1-p$ and $\forall n \in \mathbb{N}_{>0}$.

The final result represents the bound on the upper tail of the Binomial distribution expressed as a fraction of its probability mass function.

Theorem A2 (Diaconis and Zabell (1991))**.** *For any integer k satisfying $k > np$ where $n \in \mathbb{N}_{>0}$ and $p \in (0,1)$*

$$\frac{F(k,n,p)}{f(k,n,p)} \leq \frac{k(1-p)}{k-np}. \tag{A6}$$

We use bound on the upper tail as an exact expression in terms of a hypergeometric function (for details see Olver et al., 2010, section 8.17)

$$\frac{F(k,n,p)}{f(k,n,p)} = (1-p)\,{}_2F_1(1, 1+n, 1+k, p) \tag{A7}$$

is difficult to work with. The hypergeometric function itself is defined as

$${}_2F_1(a,b,c,z) = \sum_{s=0}^{\infty} \frac{a)_s b)_s}{c)_s} \frac{z^s}{s!} \tag{A8}$$

where $a)_n$ is Pochhammer's symbol defined as $a(a+1)\ldots(a+n-1)$ if $n > 1$ and 1 for $n = 0$.

A1.2 Proof of proposition 1

First, note that the $\mathbb{FA}$ problem has a solution, as it involves the maximization of the continuous objective function $\mathbb{P}_{q|n}$ over a compact region $X(b)$. $\mathbb{P}_{q|n}$ is differentiable with respect to every element of $\mathbf{p}$; hence any solution to $\mathbb{FA}$ necessarily satisfies the standard Kuhn and Tucker (1951) conditions. No further constraint qualification is required, as the constraints of $\mathbb{FA}$ are cut-out by affine functions (Pardalos, 2009). The Lagrangian for $\mathbb{FA}$ is

$$L(\mathbf{p}, \lambda, \mathbf{m}^+, \mathbf{m}^-) = \mathbb{P}_{q|n}[\mathbf{p}] - \lambda \left[\sum_{i=1}^{n} p_i - b \right] - \mathbf{m}^+ * (\mathbf{p} - \mathbf{1}) + \mathbf{m}^- * \mathbf{p} \quad \text{(A9)}$$

where λ, $\mathbf{m}^+ = (m_i^+)_{i=1}^n$ and $\mathbf{m}^- = (m_i^-)_{i=1}^n$ are Lagrange multipliers and $\mathbf{1}$ is the unit vector in $\mathbb{R}^n$. Throughout, we denote the solution to $\mathbb{FA}$ by $\mathbf{p}^*$, λ^*, $\mathbf{m}^{+,*}$ and $\mathbf{m}^{-,*}$. The typical element of $\mathbf{p}^*$ is p_i^* and similarly for $\mathbf{m}^{+,*}$ and $\mathbf{m}^{-,*}$.

Part one of the proposition is immediate. Favours satisfying $b < q$, irrespective of how they are allocated, do not suffice for the certain acceptance of the chair's proposal; $\mathbb{P}_{q|n}[\mathbf{p}] < 1$ for any $\mathbf{p} \in X(b)$. Increasing the amount of favours from b to some $b + \epsilon < q$ allows the chair to increase the probability of her proposal passing. Hence $\lambda^* > 0$. Conversely, if $b \geq q$, the chair can allocate $p_i = 1$ to q committee members and increasing b has no effect on the maximized objective function in $\mathbb{FA}$. Hence, $\lambda^* = 0$.

Part two of the proposition now follows. $\lambda^* = 0$ is equivalent to $b \geq q$, which implies that any solution to $\mathbb{FA}$ allocates unit favours to at least q committee members. One such solution is $\mathbf{p}(r^*, p^*, s^*)$ with $s^* \geq q$, so that $\mathbb{R}_{q|n}[b] = \mathbb{P}_{q|n}[\mathbf{p}(r^*, p^*, s^*)] = 1$.

Part three of the proposition is the crux of the proof. $\lambda^* > 0$ is equivalent to $b < q$, and hence trivially $\mathbb{R}_{q|n}[b] < 1$ and simple argument shows that the constraint presented by b has to be binding in any solution to $\mathbb{FA}$, so that $\sum_{i=1}^n p_i^* = b$. The following lemma helps in proving the remaining claims.[17]

Lemma A2. *Assume $b < q$. $\mathbf{p}^*$ solves $\mathbb{FA}$ if and only if any two elements of $\mathbf{p}^*$ with $0 < p_i^* < 1$ and $0 < p_j^* < 1$ (if such elements exist) satisfy $p_i^* = p_j^*$.*

[17] The *if* part of lemma A2 and the *if* parts of parts 3.(a) and 3.(d) of proposition 1 we will prove with the aid of lemma A2 follow from Hoeffding (1956). We have decided to provide the full proof here, as the results are difficult to extract from Hoeffding's proof. Additionally, our proof relies on constrained optimization techniques standard in the economic literature.

Proof. Using (A9), the first order necessary condition for the optimality of $\mathbf{p}^*$ for any p_i^* satisfying $0 < p_i^* < 1$ is

$$\frac{\partial L(\mathbf{p}^*, \lambda^*, \mathbf{m}^{+,*}, \mathbf{m}^{-,*})}{\partial p_i} = \frac{\partial \mathbb{P}_{q|n}[\mathbf{p}^*]}{\partial p_i} - \lambda^* = 0. \tag{A10}$$

Recall that $\mathbf{p}^{\{ij\}}$ is obtained from $\mathbf{p}$ by dropping p_i and p_j, and $\mathbb{P}^*_{s|n}[\mathbf{p}]$ denotes the probability of exactly s out of n committee members accepting when allocated favours $\mathbf{p}$. Now

$$\begin{aligned}\mathbb{P}_{q|n}[\mathbf{p}] =& \mathbb{P}_{q|n-2}[\mathbf{p}^{\{ij\}}](1-p_i)(1-p_j)+\\ & \mathbb{P}_{q-1|n-2}[\mathbf{p}^{\{ij\}}](p_i(1-p_j)+(1-p_i)p_j)+\\ & \mathbb{P}_{q-2|n-2}[\mathbf{p}^{\{ij\}}]p_ip_j\\ =& \mathbb{P}_{q|n-2}[\mathbf{p}^{\{ij\}}] + (p_i+p_j)\mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\}}]+\\ & p_ip_j\left[\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\}}] - \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\}}]\right].\end{aligned} \tag{A11}$$

$\mathbb{P}^*_{q|n}$ in the above expression is well defined if $n \geq 1$ and $n \geq q \geq 0$. For $n \geq 1$ and $q > n$ or $q < 0$, we use, by convention, $\mathbb{P}^*_{q|n} = 0$. Similarly, for $n = q = 0$, we use $\mathbb{P}^*_{q|n} = 1$ and for $n = 0$ and $q \neq 0$ we use $\mathbb{P}^*_{q|n} = 0$. We do not need to consider $n < 0$, as the lemma clearly holds for $n = 1$.

If there exist $0 < p_i^* < 1$ and $0 < p_j^* < 1$ for $i \neq j$, then from (A10)

$$\left(p_i^* - p_j^*\right)\left[\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] - \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}]\right] = 0 \tag{A12}$$

showing that if $p_i^* = p_j^*$ then $\mathbf{p}^*$ solves $\mathbb{FA}$.

To show the converse, assume, towards the contradiction, that $\mathbf{p}^*$ with $0 < p_i^* < 1$ and $0 < p_j^* < 1$ solves $\mathbb{FA}$ and $p_i^* \neq p_j^*$. Then, from (A12), $\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] = \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}]$. This is impossible with $n = 2$; hence for the remainder of the proof of lemma A2, assume that $n \geq 3$.

First, note that $\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] = \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}] > 0$. If not then we would have $\mathbb{P}^*_{s|n-2}[\mathbf{p}^{\{ij\},*}] = 0$ for all $s \geq q-2$ and consequently $\mathbb{P}_{q|n}[\mathbf{p}^*] = 0$, contradicting the optimality of $\mathbf{p}^*$ in $\mathbb{FA}$ as $\mathbb{P}_{q|n}[\mathbf{p}(0, p, 0)] > \mathbb{P}_{q|n}[\mathbf{p}^*] = 0$. Now $\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] = \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}] > 0$ implies that there exists at least one element of $\mathbf{p}^{\{ij\},*}$, p_k^* satisfying $0 < p_k^* < 1$. If all of the entries of $\mathbf{p}^{\{ij\},*}$ were equal either to zero or unity, then we would have

$\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] \neq \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}]$. Rewriting

$$\begin{aligned}\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}]-&\mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}] = \\ &= p^*_k \left[\mathbb{P}^*_{q-3|n-3}[\mathbf{p}^{\{ijk\},*}] - \mathbb{P}^*_{q-2|n-3}[\mathbf{p}^{\{ijk\},*}]\right] + \\ &\quad (1-p^*_k)\left[\mathbb{P}^*_{q-2|n-3}[\mathbf{p}^{\{ijk\},*}] - \mathbb{P}^*_{q-1|n-3}[\mathbf{p}^{\{ijk\},*}]\right].\end{aligned} \tag{A13}$$

With $0 < p^*_k < 1$, $\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] - \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}] = 0$ holds in either of the three cases.

Case 1 has $\mathbb{P}^*_{q-s|n-3}[\mathbf{p}^{\{ijk\},*}] = 0$ for $s \in \{1, 2, 3\}$, which is impossible by the argument similar to that we used to show $\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] > 0$. Case 2 has $\mathbb{P}^*_{q-s|n-3}[\mathbf{p}^{\{ijk\},*}] - \mathbb{P}^*_{q-s+1|n-3}[\mathbf{p}^{\{ijk\},*}] = 0$ for $s \in \{2, 3\}$, which is impossible as $\mathbb{P}^*_{q|n}[\mathbf{p}]$ is, as a function of q, first strictly increasing and then strictly decreasing, except possibly for two equal maxima (Darroch, 1964). The result in Darroch (1964) relies on all of the entries of $\mathbf{p}$ being strictly between zero and unity, but it extends to our setting as well. Denote the number of zero entries in $\mathbf{p}$ by z, the number of unit entries in $\mathbf{p}$ by u and by $\mathbf{p}^{\{zu\}}$ the original $\mathbf{p}$ after dropping all the zero and unit elements. Then $\mathbb{P}^*_{q|n}[\mathbf{p}] = \mathbb{P}^*_{q-u|n-z-u}[\mathbf{p}^{\{zu\}}]$. Notice $n - z - u > 0$, otherwise we cannot be in case 2.

Case 3, the only one possible, has $\mathbb{P}^*_{q-3|n-3}[\mathbf{p}^{\{ijk\},*}] < \mathbb{P}^*_{q-2|n-3}[\mathbf{p}^{\{ijk\},*}]$ and $\mathbb{P}^*_{q-2|n-3}[\mathbf{p}^{\{ijk\},*}] > \mathbb{P}^*_{q-1|n-3}[\mathbf{p}^{\{ijk\},*}]$. Now, as $p^*_i \neq p^*_j$, either $p^*_i \neq p^*_k$ or $p^*_j \neq p^*_k$. Without loss of generality, assume that $p^*_i \neq p^*_k$. Replacing p^*_k with p^*_i in (A13), we have

$$\mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{jk\},*}] - \mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{jk\},*}] \neq 0 \tag{A14}$$

which by (A12) implies that $p^*_j = p^*_k$ as $0 < p^*_k < 1$. Finally, if $p^*_i \neq p^*_j$ are part of $\mathbf{p}^*$, solving $\mathbb{FA}$, then so are, by (A12), $p'^*_i = p^*_i - \epsilon$ and $p'^*_j = p^*_j + \epsilon$ for all sufficiently small $\epsilon \neq 0$. Repeating the same argument that lead to $p^*_j = p^*_k$, using p'^*_i and p'^*_j instead of p^*_i and p^*_j, we find $p'^*_j = p^*_k$, contradicting the obvious $p^*_j \neq p'^*_j$. □

By lemma A2, the $\mathbf{p}^*$ solving $\mathbb{FA}$ satisfies $\mathbf{p}^* = \mathbf{p}(r^*, p^*, s^*)$ where $p^* = \frac{b-s^*}{n-r^*-s^*}$. What remains are parts 3.(a) through 3.(e) characterizing r^* and s^*. Part 3.(b), $r^* \leq n - q$, is obvious. With $r^* > n - q$, the number of committee members receiving non-zero favours is strictly less than q and $\mathbb{P}_{q|n}[\mathbf{p}(r^*, p^*, s^*)] = 0$, which can always be improved upon by $\mathbb{P}_{q|n}[\mathbf{p}(0, p, 0)] > 0$

For parts 3.(a), $s^* = 0$, and 3.(d), $r^* = 0$, if $b < q-1$, first note that both hold for $n = 1$; hence until we prove 3.(a) and 3.(d), assume that $n \geq 2$. To proceed, we analyse the first order necessary conditions for optimality derived from (A9) that read

$$\begin{aligned} &\frac{\partial U(\mathbf{p}^*)}{\partial p_i} - \lambda^* + m^{-,*} = 0 && \text{if } p_i^* = 0 \\ &\frac{\partial U(\mathbf{p}^*)}{\partial p_i} - \lambda^* = 0 && \text{if } 0 < p_i^* < 1 \\ &\frac{\partial U(\mathbf{p}^*)}{\partial p_i} - \lambda^* - m^{+,*} = 0 && \text{if } p_i^* = 1 \end{aligned} \tag{A15}$$

which, using the non-negativity of the Lagrange multipliers, can be rewritten as

$$\left.\frac{\partial U(\mathbf{p}^*)}{\partial p_i}\right|_{p_i^*=1} \geq \left.\frac{\partial U(\mathbf{p}^*)}{\partial p_i}\right|_{0<p_i^*<1} \geq \left.\frac{\partial U(\mathbf{p}^*)}{\partial p_i}\right|_{p_i^*=0} \tag{A16}$$

and implies that

$$\mathbb{P}^*_{q-1|n-2}[\mathbf{p}^{\{ij\},*}] \geq \mathbb{P}^*_{q-2|n-2}[\mathbf{p}^{\{ij\},*}] \tag{A17}$$

for any pair of $p_i^* \neq p_j^*$.

For $q = 1$, 3.(a) holds by the virtue of the fact that $b < 1$ and 3.(d) holds vacuously as $b < q - 1 = 0$. For $q = n$, 3.(a) holds as $s^* > 0$ implies the left hand side of (A17) is equal to zero and hence $\mathbb{P}_{q|n}[\mathbf{p}(r^*, p^*, s^*)] = 0$. Note that we can use argument based on (A17), as it is not possible for all of the elements of p^* to equal unity. Still for $q = n$, 3.(d) also holds because $r^* > 0$ implies that $\mathbb{P}_{q|n}[\mathbf{p}(r^*, p^*, s^*)] = 0$, which can be improved upon. Notice that by covering $q = 1$ and $q = n$ cases, we have shown that 3.(a) and 3.(d) for $n = 2$; hence until we prove 3.(a) and 3.(d) fully, assume that $n \geq 3$.

Now consider a number of entries in $\mathbf{p}^* = \mathbf{p}(r^*, p^*, s^*)$ different from zero and unity, $n - r^* - s^*$. Naturally, $n - r^* - s^* = 0$ only for integer value of b, but then $\mathbb{P}_{q|n}[\mathbf{p}^*] = 0$ as the largest integer b satisfying $b < q$ is $q - 1$. Moreover, $n - r^* - s^* \neq 1$. If this were so, then the only case in which $\mathbb{P}_{q|n}[\mathbf{p}^*] > 0$ would be $s^* = q - 1$. Then, we could obtain $\mathbf{p}^{\{ij\},*}$ in (A17) by dropping the non-zero non-unit entry and one unit entry. This would make the $\mathbf{p}^{\{ij\},*}$ $n - 2$ vector with $q - 2$ unit entries and $n - q$ zero entries. Subsequently, the left hand side of (A17) would equal zero and the right hand side would equal unity. As a result, we have $n - r^* - s^* \geq 2$.

Writing $\mathbf{p}^{\{ij\},*}$ in general form, it will have $r^* - z$ zero entries, $s^* - u$ unit entries and $n - 2 - (r^* - z) - (s^* - u)$ of the p^* entries, where $0 \leq z \leq 2$, $0 \leq u \leq 2$ and $0 \leq 2 - (z + u) \leq 2$, with z and u denoting the number of zero and unit entries dropped from the original p^*, respectively. Using lemma A1, we can rewrite (A17) as

$$\begin{aligned}(n - 2 - (r^* - z) - (s^* - u) + 1)\, p^* &\overset{z+u=1}{=} b - s^* \\ &\geq \quad q - 2 - (s^* - u) + 1.\end{aligned} \tag{A18}$$

Now, assume that $s^* \geq 1$ in $\mathbf{p}^*$ solving $\mathbb{FA}$. Then, we can set $u = 1$ in (A18) to obtain $b \geq q$, a contradiction to $b < q$ proving 3.(a). Similarly, assume that $r^* \geq 1$ in $\mathbf{p}^*$ solving $\mathbb{FA}$ for $b < q - 1$. Then, we can set $z = 1$ in (A18) to obtain $b \geq q - 1$, a contradiction proving that 3.(d). As $r^* = s^* = 0$ are the only remaining possibilities, this also proves the uniqueness claim from 3.(e).

What remains to be shown is 3.(c) and the uniqueness claim from 3.(e). The results thus far imply we can express the value of the maximized objective function in $\mathbb{FA}$ as

$$\mathbb{P}_{q|n}[\mathbf{p}^*] = F(q, n - r, b/(n - r)) \tag{A19}$$

with $r = 0$ when $b < q - 1$ and $r \in \{0, \dots, n - q\}$ when $q - 1 \leq b \leq q$. The argument below and proof of theorem 1 both use arguments based on the derivative of $F(q, n - r, b/(n - r))$ with respect to b. The derivative can be expressed in several alternative ways summarized in the following lemma.

Lemma A3. *Assume* $0 \leq b \leq q$ *and* $r \in \{0, \dots, n - q\}$*. Then,*

$$\begin{aligned}\frac{\partial F(q, n - r, b/(n - r))}{\partial b} &= \frac{\left(\frac{b}{n-r}\right)^{q-1} \left(\frac{n-r-b}{n-r}\right)^{n-r-q}}{B(k, n - r - (q - 1))} \frac{1}{n - r} \\ &= f(q - 1, n - r - 1, b/(n - r)) \\ &= \frac{q}{b} f(q, n - r, b/(n - r)) \\ &= \frac{n - r - (q - 1)}{n - r - b} f(q - 1, n - r, b/(n - r))\end{aligned} \tag{A20}$$

with $\frac{\partial F(q,n-r,b/(n-r))}{\partial b} > 0$ *for all* $0 < b \leq q$.

Proof. The lemma is easily proved using (A4). $\square$

To prove 3.(c), we first show that the interval for b such that $r^* = n - q$ is convex. This will follow from the next lemma showing that $F(q, q, b/q)$ increases faster in b than $F(q, n-r, b/(n-r))$ for $r \in \{0, \ldots, n-q-1\}$ and $b \in [q-1, q]$. Note that we only need to focus on $b \in [q-1, q]$. For $b < q-1$, we know that $r^* = 0$.

Lemma A4. *Assume that* $q - 1 \leq b \leq q$. *Then,*

$$\frac{\partial F(q, q, b/q)}{\partial b} > \frac{\partial F(q, n-r, b/(n-r))}{\partial b} \tag{A21}$$

for $r \in \{0, \ldots, n-q-1\}$.

Proof. Evaluating $\frac{\partial F(q,n-r-1,b/(n-r-1))}{\partial b} > \frac{\partial F(q,n-r,b/(n-r))}{\partial b}$ at $b = q-1$ for $r \in \{0, \ldots, n-q-1\}$ using lemma A3, we have

$$f(q-1, n-r-1, (q-1)/(n-r-1)) > f(q-1, n-r-1, (q-1)/(n-r)). \tag{A22}$$

This holds because $f(q-1, n-r-1, p)$ is strictly increasing in p for $p < \frac{q-1}{n-r-1}$, and we have $\frac{q-1}{n-r} < \frac{q-1}{n-r-1}$. Thus $\left.\frac{\partial F(q,n-r,b/(n-r))}{\partial b}\right|_{b=q-1}$ is strictly increasing in r for $r \in \{0, \ldots, n-q-1\}$.

Using lemma A3 again $\frac{\partial F(q,q,b/q)}{\partial b} > \frac{\partial F(q,n-r,b/(n-r))}{\partial b}$ rewrites as

$$1 > \binom{n-r}{q} \left(\frac{q}{n-r}\right)^q \left(\frac{n-r-q}{n-r}\right)^{n-r-q} \left(\frac{n-r-b}{n-r-q}\right)^{n-r-q} \tag{A23}$$

which we know holds for $r \in \{0, \ldots, n-q-1\}$ at $b = q-1$. However, it then has to hold for $b \in [q-1, q]$ as the last term, the only one that depends on b, is decreasing in b. $\square$

By lemma A4, we know that if $F(q, q, b/q) \geq F(q, n-r, b/(n-q))$ holds for some $b = b' \in [q-1, q]$ and $r \in \{0, \ldots, n-q-1\}$, then it has to hold strictly for all $b \in (b', q]$. Denote by $\hat{b}_{q|r}$ the value of $b \in [q-1, q]$ solving $F(q, q, b/q) = F(q, n-r, b/(n-q))$ for $r \in \{0, \ldots, n-q-1\}$. By the preceding argument, there is either exactly one such b or it does not exist. In the latter case, set $\hat{b}_{q|r} = q-1$. Define $\hat{b}_q = \max_{r \in \{0,\ldots,n-q-1\}} \hat{b}_{q|r}$. Clearly, $r^* = n - q$

if and only if $b \in [\hat{b}_q, q]$, and by lemma A4 $r^* = n - q$ is unique solution to $\mathbb{FA}$ if $b > \hat{b}_q$.

What remains to be shown is that $q > \hat{b}_q \geq q + 2 + \frac{1}{q} - \left(1 + \frac{1}{q}\right)^{q+1}$. This follows from the definition of $\hat{b}_q$, easily verifiable $\hat{b}_{q|n-q-1} = q + 2 + \frac{1}{q} - \left(1 + \frac{1}{q}\right)^{q+1}$, and the fact that $r^* = n - q$ at $b = q$ combined with the continuity of $F(q, n-r, b/(n-r))$ in b and $F(q, q, 1) = 1 > F(q, n-r, q/(n-r))$ for any $r \in \{0, \ldots, n-q-1\}$. Note that $q > q+2+\frac{1}{q}-\left(1 + \frac{1}{q}\right)^{q+1} \geq q-1$. It holds for $q = 1$ and otherwise is easily observed using the monotonic decreasing convergence of $\left(1 + \frac{1}{q}\right)^{q+1}$ to $\lim_{q\to\infty} \left(1 + \frac{1}{q}\right)^{q+1} = \mathrm{e}$ (Sandor, 2007). □

A1.3 Proof of proposition 2

First, notice that b^* maximizing $(B - b)\mathbb{R}_{q|n}[b]$ has to satisfy $b^* \leq q$ for any B. For $b \geq q$, $\mathbb{R}_{q|n}[b] = 1$; hence the objective function of $\mathbb{CE}$ is strictly decreasing in b for $b > q$. We must therefore show that $b^* \neq q$ for $B < q + 1$.

Assume that $b^* = q$ and $B < q + 1$. Then, we have $\mathbb{R}_{q|n}[b^*] = 1$, and the value of the objective function in $\mathbb{CE}$ is equal to $B - q$. Consider now a deviation consisting of allocating $p_i = 1 - \epsilon$ to exactly one of the committee members receiving unit favours for some small $\epsilon > 0$. The expected utility of this deviation is $(B - q + \epsilon)(1 - \epsilon)$. Because, by conjecture, $b^* = q$ solves $\mathbb{CE}$

$$\begin{aligned} B - q &\geq (B - q + \epsilon)(1 - \epsilon) \\ B &\geq q + 1 - \epsilon. \end{aligned} \tag{A24}$$

The second inequality has to hold for any $\epsilon > 0$ so it rewrites as $B \geq q + 1$, contradicting $B < q + 1$. This proves part one of the theorem.

For part two, assume that b^* is such that r^* in the associated $\mathbb{FA}$ satisfies $r^* = n - q$. With voting rule q, this implies that all committee members allocated positive favours have to vote *yes* for the chair's proposal to pass. Her objective function in $\mathbb{CE}$ under $r^* = n - q$ is thus

$$(B - b)\left(\frac{b}{q}\right)^q \tag{A25}$$

which is maximized at $b = B\frac{q}{q+1}$. From proposition 1, we know that $r^* = n - q$ cannot be optimal in $\mathbb{FA}$ for $b < q - 1$. Condition in part two of the theorem, $B\frac{q}{q+1} < q - 1$, which can be rewritten as $B < q - \frac{1}{q}$, thus implies a contradiction to $r^* = n - q$.

Part three is immediate given the characterization of the solution to $\mathbb{FA}$. With $B < q - 1$, $b^* < q - 1$, which implies that $r^* = 0$. □

A1.4 Proof of theorem 1

As in the proof of proposition 2, any b^* solving $\mathbb{CE}$ has to satisfy $b^* \leq q$. Therefore, we need to show that $b < q$ cannot solve $\mathbb{CE}$. Additionally, notice that if we show that $b^* = q$ for given size of the overall budget B', then $b^* = q$ for any $B \geq B'$. To see this, $b^* = q$ for B' means that

$$B' - q \geq (B' - b)\mathbb{R}_{q|n}[b] \tag{A26}$$

holds for all $b \leq q$. Adding a positive c to both sides

$$\begin{aligned}(B' + c) - q &\geq (B' - b)\mathbb{R}_{q|n}[b] + c \\ &\geq ((B' + c) - b)\mathbb{R}_{q|n}[b]\end{aligned} \tag{A27}$$

where the second inequality follows from $\mathbb{R}_{q|n}[b] \in [0, 1]$.

Because the characterization of the solution to $\mathbb{FA}$ in proposition 1 does not specify the number of the committee members receiving zero favours for $b \in [q - 1, q]$, we will be working with $F(q, n - r, b/(n - r))$ instead of working with $\mathbb{R}_{q|n}[b]$. However, we know that any solution to $\mathbb{FA}$ has to have $r \in \{0, \ldots, n - q\}$. To avoid unnecessary repetitions, from now on statements 'for all r' are taken to mean 'for all $r \in \{0, \ldots, n - q\}$'.

Our approach to proving theorem 1, except for the last part, will be to show that the derivative of the objective function in $\mathbb{CE}$ is increasing as a function of b. The following lemma summarizes behaviour of the derivative.

Lemma A5. *Assume $b \leq q$. Then*

$$\begin{aligned}&\frac{\partial}{\partial b}(B - b)F(q, n - r, b/(n - r)) = \\ &\qquad = (B - b)\frac{q}{b}f(q, n - r, b/(n - r)) - F(q, n - r, b/(n - r)) \\ &\qquad \geq \frac{q}{b}f(q, n - r, b/(n - r))\left[B - n - 1 + r + q - b\right].\end{aligned} \tag{A28}$$

Proof. The equality follows from lemma A3. To demonstrate the inequality, we have $F(q, n - r, b/(n - r)) = \sum_{s=q}^{n-r} f(s, n - r, b/(n - r))$. Furthermore, by applying lemma A1, we have $f(q, n-r, b/(n-r)) \geq f(s, n-r, b/(n-r))$ for $s \in \{q+1, \dots, n-r\}$. This implies that

$$\frac{q}{b} f(q, n-r, b/(n-r)) \, [n-r-q+1] \geq F(q, n-r, b/(n-r)) \tag{A29}$$

providing us the result to be shown. □

Using lemma A5, for $B = n+1$, and hence for any larger B by the opening remark, the derivative of the objective function in $\mathbb{CE}$ with respect to b is strictly positive for $b < q$ and any r. This proves part one of the theorem.

The current setup now lends itself to proving part four. We will return to parts two and three shortly. We begin the proof of part four, which applies for $B \geq n$ and $q \leq n-1$, again using lemma A5. Now it implies that the objective function in $\mathbb{CE}$ is strictly increasing in b for $b < q$ and any $r \geq 1$. Therefore, assume that $r = 0$ until we prove part four. For $r = 0$, we cannot use lemma A5, as there is one term too many in $F(q, n, b/n)$, each of which we are replacing by $f(q, n, b/n)$. However, the next lemma shows that the sum of the *four* terms in $F(n-3, n, b/n)$, as a fraction of $f(n-3, n, b/n)$, is less than *three* for any $b \leq n-3$. This will prove part four of the theorem for any $q \leq n-3$, as the square brackets in (A28) become $[B-n+r+q-b]$. Naturally, we can only use this argument for $n \geq 4$.

Lemma A6. *Assume that $b \leq n-3$ and $n \geq 4$. Then,*

$$\frac{\sum_{s=n-3}^{n} f(s, n, b/n)}{f(n-3, n, b/n)} \leq 3. \tag{A30}$$

Proof. Using simple algebra, the expression can be rewritten as

$$\begin{aligned} 1 &+ \frac{3}{n-2} \frac{b}{n-b} + \frac{6}{(n-1)(n-2)} \left(\frac{b}{n-b} \right)^2 \\ &+ \frac{6}{n(n-1)(n-2)} \left(\frac{b}{n-b} \right)^3 . \end{aligned} \tag{A31}$$

It is clearly continuous and increasing in b. Therefore, it remains to show that it is less than 3 when evaluated at $b = n-3$. Doing so along with some algebra yields

$$\frac{1}{9} \left[26 - \frac{4}{n-2} - \frac{8}{n-1} - \frac{27}{n} \right] \leq \frac{27}{9} \tag{A32}$$

for any $n \geq 4$. $\square$

What remains for part four of the theorem are $q = n - 2$ and $q = n - 1$ when $r = 0$. We cannot use the same argument as above, based on a strictly increasing objective function in $\mathbb{CE}$, as the objective function might have local maxima for $r = 0$. Instead, we will directly confirm that $(B - b)\mathbb{R}_{q|n}[b] \leq B - q$ for any $b \leq q$. We set $B = n$, which suffices to prove part four of the theorem by the opening remark.

With $B = n$ and $r = 0$, the objective function in $\mathbb{CE}$ rewrites as, using $p = \frac{b}{n}$, $n(1-p)F(q, n, p)$. Expanding $F(q, n, p)$, the objective function will be a sum of $n - q + 1$ terms with a typical one being

$$n(1-p)\binom{n}{s}p^s(1-p)^{n-s}. \tag{A33}$$

Maximizing each term individually, its maximum is attained at $p = \frac{s}{n+1}$, with a value of

$$n\left(\frac{n+1-s}{s}\right)^{n+1-s}\binom{n}{s}\left(1 - \frac{n+1-s}{n+1}\right)^{n+1} \tag{A34}$$

yielding

$$\begin{aligned} &\left(1 - \frac{1}{n+1}\right)^{n+1} && \text{for } s = n \\ \frac{4}{1}\frac{n^2}{(n-1)^2}&\left(1 - \frac{2}{n+1}\right)^{n+1} && \text{for } s = n-1 \\ \frac{27}{2}\frac{n^2(n-1)}{(n-2)^3}&\left(1 - \frac{3}{n+1}\right)^{n+1} && \text{for } s = n-2. \end{aligned} \tag{A35}$$

Naturally, maximizing each term individually yields the maximum value weakly larger than joint maximization. Therefore, we need to show that the sum of the first two terms in (A35), for $q = n - 1$, and the sum of the first three terms in (A35), for $q = n - 2$, is less than 1 and 2, respectively, values of the objective function in $\mathbb{CE}$ for $B = n$, $r = n - q$ and $b = q$.

All of the terms in (A35) of the form $\left(1 - \frac{s}{n+1}\right)^{n+1}$ are monotone increasing to their limit $\exp[-s]$. This can be seen from arithmetic geometric mean inequality using a similar technique as in Sandor (2007), the details of which we do not believe need to be provided in full here. Thus, we need

$$\begin{aligned} \frac{4}{1}\frac{n^2}{(n-1)^2}\frac{1}{\mathrm{e}^2}+\frac{1}{\mathrm{e}} &\leq 1 \\ \frac{27}{2}\frac{n^2(n-1)}{(n-2)^3}\frac{1}{\mathrm{e}^3}+\frac{4}{1}\frac{n^2}{(n-1)^2}\frac{1}{\mathrm{e}^2}+\frac{1}{\mathrm{e}} &\leq 2 \end{aligned} \tag{A36}$$

which holds for $n \geq 14$. For the remaining values of n, that is for $n \in \{2, \dots, 13\}$ under $q = n-1$ and $n \in \{3, \dots, 13\}$ under $q = n-2$, we have numerically confirmed that the maximized value of $n(1-p)F(q, n, p)$, maximized jointly with respect to p, is below 1 and 2 for the applicable values of q. The details of the procedure, the *Mathematica* notebook, are available upon request.

We now return to part two, which applies for $B \geq (q+1)(2-\frac{q}{n})$. We are aware that part one of the theorem can be proven from part two by maximizing $(q+1)(2-\frac{q}{n})$ with respect to q. However proving part one separately allowed us to focus on the case of $r = 0$ in the proof of part four. Additionally, it was instrumental in explicitly showing the main approach to the proof, ensuring that the derivative of the expected utility in $\mathbb{CE}$ is strictly increasing in b.

We use the same approach, if somewhat differently, now. The following expression presents a series of inequalities for the derivative of the objective function in $\mathbb{CE}$. We explain the source of each inequality after stating all of them. For any $b \leq q$

$$\begin{aligned}
&\frac{\partial}{\partial b}(B-b)F(q,n-r,b/(n-r)) = \\
&\overset{0}{=}(B-b)\frac{q}{b}f(q,n-r,b/(n-r)) - F(q,n-r,b/(n-r)) \\
&\overset{1}{\geq}\frac{q}{b}f(q,n-r,b/(n-r))\left[B-b-\frac{F(q,n-r,b/(n-r))}{f(q,n-r,b/(n-r))}\right] \\
&\overset{2}{\geq}\frac{q}{b}f(q,n-r,b/(n-r))\left[B-b-\frac{F(q,n-r,q/(n-r))}{f(q,n-r,q/(n-r))}\right] \\
&\overset{3}{=}\frac{q}{b}f(q,n-r,b/(n-r))\left[B-b-\left(1+\frac{F(q+1,n-r,q/(n-r))}{f(q,n-r,q/(n-r))}\right)\right] \\
&\overset{4}{\geq}\frac{q}{b}f(q,n-r,b/(n-r))\left[B-b-\left(1+\frac{F(q+1,n-r,q/(n-r))}{f(q+1,n-r,q/(n-r))}\right)\right] \\
&\overset{5}{\geq}\frac{q}{b}f(q,n-r,b/(n-r))\left[B-b-\left(1+\frac{(q+1)(1-\frac{q}{n-r})}{q+1-q}\right)\right] \\
&\overset{6}{\geq}\frac{q}{b}f(q,n-r,b/(n-r))\left[B-b-\left(1+\frac{(q+1)(n-q)}{n}\right)\right].
\end{aligned} \tag{A37}$$

$\overset{0}{=}$ uses lemma A3 to rewrite the derivative. $\overset{1}{\geq}$ follows from $\frac{q}{b} \geq 1$. $\overset{2}{\geq}$ follows from $\frac{F(q,n-r,b/(n-r))}{f(q,n-r,b/(n-r))} = \frac{\sum_{s=q}^{n-r} f(s,n-r,b/(n-r))}{f(q,n-r,b/(n-r))}$ increasing in b. To see this, the typical term in the sum will be of the form

$$\begin{aligned}
\frac{f(s,n-r,b/(n-r))}{f(q,n-r,b/(n-r))} &= \frac{\binom{n-r}{s}\left(\frac{b}{n-r}\right)^s\left(\frac{n-r-b}{n-r}\right)^{n-r-s}}{\binom{n-r}{q}\left(\frac{b}{n-r}\right)^q\left(\frac{n-r-b}{n-r}\right)^{n-r-q}} \\
&= \frac{\binom{n-r}{s}}{\binom{n-r}{q}}\left(\frac{b}{n-r-b}\right)^{s-q}
\end{aligned} \tag{A38}$$

where $\frac{b}{n-r-b}$ is clearly increasing in b. $\overset{3}{=}$ is an identity cancelling the first term in $\frac{\sum_{s=q}^{n-r} f(s,n-r,b/(n-r))}{f(q,n-r,b/(n-r))}$. We can only use this for $q \leq n-1$ and $r \leq n-q-1$. It is easy to see that for the remaining cases, the bound on B ensuring a strictly increasing objective function in $\mathbb{CE}$ is $q+1$. This is weaker than the bound we will derive below. $\overset{4}{\geq}$ follows from lemma A1, implying that $f(q,n-r,q/(n-r)) \geq f(q+1,n-r,q/(n-r))$. $\overset{5}{\geq}$ follows from Diaconis and Zabell's theorem A2. Finally, $\overset{6}{\geq}$ follows from $-\frac{q}{n-r}$ decreasing in r.

Part two of the theorem is now immediate. The objective function in $\mathbb{CE}$ is strictly increasing in b for any $b < q$ if $B - b > 1 + \frac{(q+1)(n-q)}{n}$. This can be rewritten as $B \geq q + 1 + \frac{(q+1)(n-q)}{n} = (q+1)(2 - \frac{q}{n})$.

Part three remains. As the line of the argument is very similar to part two, we only indicate where it differs regarding (A37). Lines $\overset{3}{=}$ and $\overset{4}{\geq}$ can be omitted, and the remaining lines become

$$\begin{aligned} &\overset{5'}{\geq} \frac{q}{b} f(q, n-r, b/(n-r)) \left[B - b - \frac{F(q,n,q/n)}{f(q,n,q/n)} \right] \\ &\overset{6'}{\geq} \frac{q}{b} f(q, n-r, b/(n-r)) \left[B - b - \left(1 - \tfrac{q}{n}\right) {}_2F_1(1, 1+n, 1+q, \tfrac{q}{n}) \right]. \end{aligned} \tag{A39}$$

where ${}_2F_1$ is an ordinary or Gaussian hypergeometric function. To prove that $\overset{5'}{\geq}$, we need to show that $\frac{F(q,n-r,q/(n-r))}{f(q,n-r,q/(n-r))}$ is decreasing in r. $\overset{6'}{\geq}$ is then a matter of simply rewriting the tail probability to the probability mass function ratio in terms of the hypergeometric function.

Lemma A7. $\frac{F(q,n-r,q/(n-r))}{f(q,n-r,q/(n-r))}$ *is decreasing in* r *for* $r \in \{0, \ldots, n-q-1\}$.

Proof. Rewriting the expression

$$\frac{F(q, n-r, q/(n-r))}{f(q, n-r, q/(n-r))} = \frac{\sum_{s=q}^{n-r} f(s, n-r, q/(n-r))}{f(q, n-r, q/(n-r)} \tag{A40}$$

note that increasing r implies fewer terms in the sum. For $s > k$, the typical term in the sum will be equal to

$$\begin{aligned} \frac{f(s, n-r, q/(n-r))}{f(q, n-r, q/(n-r))} &= \frac{\binom{n-r}{s}}{\binom{n-r}{k}} \frac{\left(\frac{k}{n-r}\right)^s}{\left(\frac{k}{n-r}\right)^k} \frac{\left(\frac{n-r-k}{n-r}\right)^{n-r-s}}{\left(\frac{n-r-k}{n-r}\right)^{n-r-k}} \\ &= \frac{k!}{s!} \left(\frac{k}{n-r-k}\right)^{s-k} \frac{(n-r-k)!}{(n-r-s)!} \\ &= \frac{k! k^{s-k}}{s!} \underbrace{\frac{(n-r-k)}{(n-r-k)} \cdot \ldots \cdot \frac{(n-r-s+1)}{(n-r-k)}}_{s-k} \end{aligned} \tag{A41}$$

with the typical term among the last $s-k$ ones $\frac{n-r-s+i}{n-r-k}$ for $i \in \{1, \ldots, s-k\}$. It is easy to confirm that $\frac{n-r-s+i}{n-r-k}$ is decreasing in r when $i - (s-k) \leq 0$, which clearly holds. The expression in the lemma, for larger r, is thus not only sum of a smaller number of terms, but each of its summands is smaller for larger values of r. The lemma follows. $\square$

Finally, the theorem asserts that the bounds of parts one through three are decreasing. $n+1 \geq (q+1)(2-\frac{q}{n})$ can be observed by maximizing the lower bound over $q \in \{1,\ldots,n\}$. $(q+1)(2-\frac{q}{n}) \geq q + (1-\frac{q}{n})\,{}_2F_1(1,1+n,1+q,\frac{q}{n})$ then follows from the fact that the larger bound is derived using an approximation of the exact value embedded in the lower bound. □

A1.5 Proof of theorem 2

To prove the theorem, we use Hoeffding's theorem A1. As in the statement of the theorem, express the key model parameters as a fraction of n, $b' = \frac{b}{n}$, $q' = \frac{q}{n}$, $B' = \frac{B}{n}$ and $r' = \frac{r}{n} \in [0, 1-q']$ and fix all the primed variables when n changes. We ignore possible non-integer values of the implied q and r.

With $n - r = n(1-r')$ of the committee members receiving positive favours and hence voting *yes* with probability $p = \frac{b}{n-r} = \frac{b'}{1-r'}$, we can set $t = \frac{q-b}{n-r} = \frac{q'-b'}{1-r'}$ in the statement of theorem A1. Notice $t > 0$ can be rewritten as $q' > b'$ and $t \leq 1-p$ as $r' \leq 1-q'$. Now using theorem A1, we thus have

$$\begin{aligned} F(q, n-r, b/(n-r)) &= F((n-r)(p+t), n(1-r'), b'/(1-r')) \\ &\leq \exp\left[-2(n-r)t^2\right] \\ &\leq \exp\left[-2n\frac{(q'-b')^2}{1-r'}\right] \end{aligned} \tag{A42}$$

where $\frac{(q'-b')^2}{1-r'}$ is strictly positive for $q' > b'$ and any $r' \in [0, 1-q']$.

Now, the condition for the given value of b' and the implied value of b to be optimal in $\mathbb{CE}$ is

$$\begin{aligned} (B-b)F(q, n-r, b/(n-r)) &\geq B - q \\ F(q, n(1-r'), b'/(1-r')) &\geq \frac{B'-q'}{B'-b'}. \end{aligned} \tag{A43}$$

Fixing b' such that $\epsilon = q' - b' > 0$ and assuming that $B' > q'$, there exists $\underline{n}_\epsilon$ such that for any $n \geq \underline{n}_\epsilon$

$$F(q, n(1-r'), b'/(1-r')) \leq \exp\left[-2n\frac{(q'-b')^2}{1-r'}\right] < \frac{B'-q'}{B'-b'} \tag{A44}$$

as $\lim_{n\to\infty} \exp\left[-2n\frac{(q'-b')^2}{1-r'}\right] = 0$, which concludes the proof. □