Consistency and its Converse for Roommate Markets

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Abstract

For marriage markets with equal numbers of men and women and where all men find all women acceptable and all women find all men acceptable, Sasaki and Toda (1992) characterize the core by anonymity, Pareto optimality, consistency, and converse consistency. In a recent paper, Nizamogullari and Özkal-Sanver (2012) generalize this result to the full domain of marriage markets by adding individual rationality and by replacing anonymity with gender fairness. We generalize both results by characterizing the core on the domain of no odd rings roommate markets by individual rationality, anonymity, Pareto optimality, consistency, and converse consistency. We also prove that extending this characterization to the domain of solvable roommate markets is not possible.

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1 Introduction

We consider one-to-one matching markets in which agents can either be matched as pairs or remain single. These markets are known as roommate markets and they include, as special cases, the well-known marriage markets (Gale and Shapley, 1962; Roth and Sotomayor, 1990). Furthermore, a roommate market is a simple example of hedonic coalition formation as well as network formation: in a "roommate coalition" situation, only coalitions of size one or two can be formed and in a "roommate network" situation, each agent is allowed or able to form only one link (for surveys of coalition and network formation see Demange and Wooders, 2004; Jackson, 2008).

Various characterizations of the core have been established for marriage markets (Sasaki and Toda, 1992; Toda, 2006). For these well-known "benchmark matching markets," the set of core matchings forms a distributive lattice and reflects polarization between the two sides of the market (Knuth, 1997, attributed this result to John Conway). For one-sided matching problems such as roommate markets and coalition/network formation, the core does not exhibit such strong structural properties and hence it is an interesting question to ask if the structure of the core drives the normative characteristics of the core or if the characterizing properties are strong enough independent of the particular (lattice) structure of the core.

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Toda (2006) provided two characterizations of the core using the well known solidarity property of population monotonicity, which he adapted to the two-sided setup of marriage markets. Klaus (2011) and Can and Klaus (2012) introduced the new properties of competition and resource sensitivity for roommate markets to account for the loss of two-sidedness and established associated core characterizations for various roommate market preference domains (including the domain of marriage markets). However, whether or not the first characterization of the core for marriage markets by Sasaki and Toda (1992) could also be extended to the one-sided model of roommate markets has been an open question for twenty years. We provide the answer to this question in this article.

For marriage markets with equal numbers of men and women and where all men find all women acceptable and all women find all men acceptable, Sasaki and Toda (1992) characterized the core by anonymity,[1] Pareto optimality, consistency,[2] and converse consistency[3]. In a recent paper, Nizamogullari and Özkal-Sanver (2012) generalized this result to the full domain of marriage markets by adding individual rationality and by strengthening anonymity to gender fairness.[4] Furthermore, Özkal-Sanver (2010, Proposition 4.2) showed that on the domain of all roommate markets, no solution satisfies Pareto optimality, anonymity, and converse consistency. However, the proof of this impossibility result used an unsolvable roommate market. Hence, the question whether or not the characterization of Sasaki and Toda (1992) can be extended to the domain of solvable roommate markets or to any of its subdomains (apart from the domain of marriage markets) has not been answered until now.

In this article, we will first extend the characterizations of Sasaki and Toda (1992) and Nizamogullari and Özkal-Sanver (2012) to the domain of no odd rings roommate markets (Theorem 3). Second, we will show that the corresponding properties are compatible on the domain of solvable roommate markets, but that they do not characterize the core (Example 1).

Our paper is organized as follows. In Section 2 we present the roommate model, basic properties of solutions, and the core. In Section 3, we introduce the variable population properties consistency and converse consistency. Section 4 first reviews the above mentioned characterizations of the core (Theorems 1 and 2) and an impossibility result (Özkal-Sanver, 2010, Proposition 4.2). We then establish a new characterization of the core for no odd rings roommate markets (Theorem 3) and establish a new possibility result on the domain of solvable roommate markets (Example 1).

2 Roommate Markets

The following Subsections 2.1 and 2.2 mostly follow Klaus (2011) and Can and Klaus (2012).

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[1] *Anonymity*: matchings assigned by the solution do not depend on agents' names.

[2] *Consistency*: if a set of matched agents leaves, then the solution should still match the remaining agents as before.

[3] *Converse consistency*: matchings assigned by the solution are (conversely) related to the matchings the solution assigns to certain restricted roommate markets (with at most four agents).

[4] *Gender fairness* (Özkal-Sanver, 2004) requires that renaming men as women, and renaming women as men, does not change the outcomes chosen by the solution when taking the renaming of agents into account.

2.1 The Model

We consider Gale and Shapley's (1962, Example 3) roommate markets with variable sets of agents, e.g., because the allocation of dormitory rooms at a university occurs every year for different sets of students.

Let $\mathbb{N}$ be the set of potential agents[5] and $\mathcal{N}$ be the set of all non-empty finite subsets of $\mathbb{N}$, i.e., $\mathcal{N} = \{N \subset \mathbb{N} \mid \infty > |N| > 0\}$. For $N \in \mathcal{N}$, $L(N)$ denotes the set of all linear orders over N.[6] For $i \in N$, we interpret $R_i \in L(N)$ as agent i's strict preferences over sharing a room (being matched) with any of the agents in $N \setminus \{i\}$ and being single (consuming an outside option); e.g., $R_i : j, k, i, l$ means that i would first like to share a room with j, then with k, and then i would prefer to stay alone rather than sharing the room with l. If $j \; P_i \; i$, then agent i finds agent j *acceptable* and if $i \; P_i \; j$, then agent i finds agent j *unacceptable*. $\mathcal{R}^N = \prod_N L(N)$ denotes the set of all preference profiles of agents in N (over agents in N). A *roommate market* consists of a set of agents $N \in \mathcal{N}$ and their preferences $R \in \mathcal{R}^N$ and is denoted by (N, R). A *marriage market* (Gale and Shapley, 1962) is a roommate market (N, R) such that N is the union of two disjoint sets M and W and each agent in M (respectively W) prefers being single to being matched with any other agent in M (respectively W).

A *matching* μ for roommate market (N, R) is a function $\mu : N \to N$ of order two, i.e., for all $i \in N$, $\mu(\mu(i)) = i$. Thus, at any matching μ, the set of agents is partitioned into pairs of agents who share a room and singletons (agents who do not share a room). Agent $\mu(i)$ is agent i's *match* and if $\mu(i) = i$ then i is matched to himself or *single*. For notational convenience, we often denote a matching in terms of the induced partition, e.g., for $N = \{1, 2, 3, 4, 5\}$ and matching μ such that $\mu(1) = 2$, $\mu(3) = 3$ and $\mu(4) = 5$ we write $\mu = \{(1, 2), 3, (4, 5)\}$. For $S \subseteq N$, we denote by $\mu(S)$ the set of agents that are matched to agents in S, i.e., $\mu(S) = \{i \in N \mid \mu^{-1}(i) \in S\}$. We denote the set of matchings for roommate market (N, R) by $\mathcal{M}(N)$ (note that the set of matchings does not depend on preferences R). If it is clear which roommate market (N, R) we refer to, matchings are assumed to be elements of $\mathcal{M}(N)$. Since agents only care about their own matches, we use the same notation for preferences over agents and matchings: for all agents $i \in N$ and matchings μ, μ', $\mu \; R_i \; \mu'$ if and only if $\mu(i) \; R_i \; \mu'(i)$.

Given a roommate market (N, R) and $N' \subseteq N$, we define the *reduced preferences* $R' \in \mathcal{R}^{N'}$ *of* R *to* N' as follows:

(i) for all $i \in N'$, $R'_i \in L(N')$ and

(ii) for all $j, k, l \in N'$, $j \; R'_l \; k$ if and only if $j \; R_l \; k$.

We denote the reduced preferences of R to N' by $R_{N'}$.

Given a roommate market (N, R), a matching $\mu \in \mathcal{M}(N)$, and $N' \subseteq N$ such that $\mu(N') = N'$, the *reduced (roommate) market of* (N, R) *at* μ *to* N' equals $(N', R_{N'})$.

Given a roommate market (N, R), a matching $\mu \in \mathcal{M}(N)$, and $N' \subseteq N$ such that $\mu(N') = N'$, we define the *reduced matching* μ' *of* μ *to* N' as follows:

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[5] All results remain valid for a finite set of potential agents that contains at least 6 agents (the proof of our main result, Theorem 3, as well as the independence of properties in Theorem 3 can be shown with a population of 6 agents).

[6] A linear order over N is a binary relation $\bar{R}$ that satisfies *antisymmetry* (for all $i, j \in N$, if $i \, \bar{R} \, j$ and $j \, \bar{R} \, i$, then $i = j$), *transitivity* (for all $i, j, k \in N$, if $i \, \bar{R} \, j$ and $j \, \bar{R} \, k$, then $i \, \bar{R} \, k$), and *comparability* (for all $i, j \in N$, $i \, \bar{R} \, j$ or $j \, \bar{R} \, i$). By $\bar{P}$ we denote the asymmetric part of $\bar{R}$. Hence, given $i, j \in N$, $i \, \bar{P} \, j$ means that i is strictly preferred to j; $i \, \bar{R} \, j$ means that $i \, \bar{P} \, j$ or $i = j$ and that i is weakly preferred to j.

(i) $\mu' : N' \to N'$ and

(ii) for all $i \in N'$, $\mu'(i) = \mu(i)$.

We denote the reduced matching of μ to N' by $\mu_{N'}$. Note that $\mu_{N'} \in \mathcal{M}(N')$.

In the sequel, we consider various domains of roommate problems: the domain of all roommate markets $\mathfrak{D}$, the domain of marriage markets $\mathfrak{D}_M$, and later the domains of solvable and of no odd rings roommate markets. To avoid notational complexity when introducing solutions and their properties, we use the domain of all roommate markets $\mathfrak{D}$ with the understanding that any other domain could be used as well.

A *solution* φ on $\mathfrak{D}$ is a correspondence that associates with each roommate market $(N,R) \in \mathfrak{D}$ a nonempty subset of matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq \mathcal{M}(N)$ and $\varphi(N,R) \neq \emptyset$. A *subsolution* ψ of φ on $\mathfrak{D}$ is a correspondence that associates with each roommate market $(N,R) \in \mathfrak{D}$ a nonempty subset of matchings in $\varphi(N,R)$, i.e., for all roommate markets $(N,R) \in \mathfrak{D}$, $\psi(N,R) \subseteq \varphi(N,R)$ and $\psi(N,R) \neq \emptyset$. A *proper subsolution* ψ of φ on $\mathfrak{D}$ is a subsolution of φ on $\mathfrak{D}$ such that $\psi \neq \varphi$.

2.2 Basic Properties and the Core

We first introduce a voluntary participation condition based on the idea that no agent can be forced to share a room.

Individual Rationality: Let $(N,R) \in \mathfrak{D}$ and $\mu \in \mathcal{M}(N)$. Then, μ is *individually rational* if for all $i \in N$, $\mu(i)\ R_i\ i$. $IR(N,R)$ denotes the set of all these matchings. A solution φ on $\mathfrak{D}$ is *individually rational* if it only assigns individually rational matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq IR(N,R)$.

An individually rational matching for a marriage market $(N,R) \in \mathfrak{D}_M$ respects the partition of agents into two types and never matches two men or two women. Hence, we embed marriage markets into our roommate market framework by an assumption on preferences (same gender agents are unacceptable) and individual rationality to ensure that no two agents of the same gender are matched. We refer to a marriage market for which matching agents of the same gender is not feasible as a *classical marriage market* (Gale and Shapley, 1962).

Anonymity requires that the agents' matches do not depend on their names. A permutation π of $\mathbb{N}$ is a bijective function $\pi\colon \mathbb{N} \to \mathbb{N}$. By $\Pi^{\mathbb{N}}$ we denote the *set of all permutations of* $\mathbb{N}$.

Let $N \in \mathcal{N}$. Given $\pi \in \Pi^{\mathbb{N}}$ and a roommate market (N,R), let $N^\pi = \{i \in \mathbb{N} \mid \pi^{-1}(i) \in N\}$ and $R^\pi \in \mathcal{R}^{N^\pi}$ be such that for all $i,j,k \in N$, $i\ P^\pi_j\ k$ if and only if $\pi^{-1}(i)\ P_{\pi^{-1}(j)}\ \pi^{-1}(k)$. Furthermore, for $\mu \in \mathcal{M}(N)$ let $\mu^\pi \in \mathcal{M}(N^\pi)$ be such that for all $i \in N^\pi$, $\mu^\pi(i) = \pi\left(\mu(\pi^{-1}(i))\right)$.

Anonymity: A solution φ on $\mathfrak{D}$ is *anonymous* if for all $\pi \in \Pi^{\mathbb{N}}$ and all $(N,R) \in \mathfrak{D}$, $\mu \in \varphi(N,R)$ and $(N^\pi, R^\pi) \in \mathfrak{D}$ imply $\mu^\pi \in \varphi(N^\pi, R^\pi)$.

Next, we introduce the well-known condition of Pareto optimality.

Pareto Optimality: Let $(N,R) \in \mathfrak{D}$ and $\mu \in \mathcal{M}(N)$. Then, μ is *Pareto optimal* if there is no other matching $\mu' \in \mathcal{M}(N)$ such that for all $i \in N$, $\mu'\ R_i\ \mu$ and for some $j \in N$, $\mu'\ P_j\ \mu$. $PO(N,R)$ denotes the set of all these matchings. A solution φ on $\mathfrak{D}$ is *Pareto optimal* if it only assigns Pareto optimal matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq PO(N,R)$.

A matching μ for roommate market $(N, R) \in \mathfrak{D}$ is blocked by a pair $\{i, j\} \subseteq N$ [possibly $i = j$] if $j \, P_i \, \mu(i)$ and $i \, P_j \, \mu(j)$. If $\{i, j\}$ blocks μ, then $\{i, j\}$ is called a *blocking pair for* μ. A matching is individually rational if there is no blocking pair $\{i, j\}$ with $i = j$.

Let $(N, R) \in \mathfrak{D}$ and $\mu \in \mathcal{M}(N)$. Then, μ is *stable* if there is no blocking pair for μ. $S(N, R)$ denotes the set of all these matchings. A roommate market is *solvable* if stable matchings exist, i.e., (N, R) is solvable if and only if $S(N, R) \neq \emptyset$. The domain of solvable roommate markets is denoted by $\mathfrak{D}_S$. Furthermore, on the domain of solvable roommate markets $\mathfrak{D}_S$, a solution φ is *stable* if it only assigns stable matchings, i.e., for all (N, R) such that $S(N, R) \neq \emptyset$, $\varphi(N, R) \subseteq S(N, R)$.

Gale and Shapley (1962) showed that all marriage markets are solvable, i.e., $\mathfrak{D} \supsetneq \mathfrak{D}_S \supsetneq \mathfrak{D}_M$, and they gave an example of an unsolvable roommate market (Gale and Shapley, 1962, Example 3).

For our main result we need the solvability of roommate markets and their reduced markets; e.g., the domain of marriage markets is such a domain of roommate markets because it is closed with respect to the reduction operator, i.e., starting from a marriage market $(N, R) \in \mathfrak{D}_M$, any reduced market $(N', R_{N'})$ of (N, R) is a marriage market.

Chung (2000) introduced a sufficient condition for solvability that also applies to the larger domain of weak preferences. We formulate his well-known *no odd rings* condition for our strict preference setup and refer to it as the *no odd rings* condition.

Let $(N, R) \in \mathfrak{D}$. Then, a *ring* for roommate market (N, R) is an ordered subset of agents $\{i_1, i_2, \ldots, i_k\} \subseteq N$, $k \geq 3$, such that (subscript modulo k) for all $t \in \{1, 2, .., k\}$, $i_{t+1} \, P_{i_t} \, i_{t-1} \, P_{i_t} \, i_t$. If k is odd, then $\{i_1, i_2, \ldots, i_k\}$ is an *odd ring* for roommate market (N, R). A roommate market $(N, R) \in \mathfrak{D}$ is a *no odd rings roommate market* if there exists no odd ring in (N, R). The domain of all such roommate markets is called *the domain of no odd rings roommate markets* and denoted by $\mathfrak{D}_{NOR}$. The domain of no odd rings roommate markets is closed with respect to the reduction operator and $\mathfrak{D} \supsetneq \mathfrak{D}_S \supsetneq \mathfrak{D}_{NOR} \supsetneq \mathfrak{D}_M$.

Another well-known concept for matching problems is the core.

A matching is in the *core* if no coalition of agents can improve their welfare by rematching among themselves. For roommate market $(N, R) \in \mathfrak{D}$, $core(N, R) = \{\mu \in \mathcal{M}(N) \mid$ there exists no $S \subseteq N$ and no $\mu' \in \mathcal{M}(N)$ such that $\mu'(S) = S$, for all $i \in S$, $\mu'(i) R_i \mu(i)$, and for some $j \in S$, $\mu'(j) P_j \mu(j)\}$.

Similarly as in other matching models (e.g., classical marriage markets and college admissions markets with responsive preferences), the core equals the set of stable matchings, i.e., for all $(N, R) \in \mathfrak{D}$, $core(N, R) = S(N, R)$. Hence, the core is a solution on the domain of solvable roommate markets $\mathfrak{D}_S$ and all its subdomains (particularly $\mathfrak{D}_{NOR}$ and $\mathfrak{D}_M$) but not on the domain of all roommate markets $\mathfrak{D}$.

3 Consistency and Converse Consistency

Consistency and converse consistency are key properties in many frameworks with variable sets of agents. Thomson (2013) provides an extensive survey of consistency and its converse for various economic models, including marriage markets. For roommate markets, consistency essentially requires that when a set of matched agents leaves, then the solution should still match the remaining agents as before.

Consistency: A solution φ on $\mathfrak{D}$ is *consistent* if the following holds for each $(N, R) \in \mathfrak{D}$ and each $\mu \in \varphi(N, R)$. If $(N', R_{N'}) \in \mathfrak{D}$ is a reduced market of (N, R) at μ to N' (i.e., $\mu(N') = N'$), then, $\mu_{N'} \in \varphi(N', R_{N'})$.

Lemma 1 (Can and Klaus (2012), Lemma 1)**.** *On either of the roommate market domains* $\mathfrak{D}_S$*,* $\mathfrak{D}_{NOR}$*, or* $\mathfrak{D}_M$*, no proper subsolution of the core satisfies consistency.*

Lemma 1 was first established by Toda (2006, Lemma 3.6) for classical marriage markets. On the domain of all roommate markets, no solution is a subsolution of the core for solvable problems and satisfies consistency (Özkal-Sanver, 2010, Proposition 4.3).

Since stable matchings need not exist for the general domain of all roommate markets, we have to restrict attention to subdomains of solvable roommate markets when studying the core. Considering the whole domain of solvable roommate markets when studying consistency is difficult because a solvable roommate market might well have unsolvable reduced markets. Requiring that a solution only selects matchings that guarantee the solvability of all restricted markets, would already steer results forcefully towards the core. However, two domains of roommate markets we consider, $\mathfrak{D}_M$ and $\mathfrak{D}_{NOR}$, satisfy "closedness" and "solvability under the restriction operation", i.e., for any roommate market in $\mathfrak{D}' \in \{\mathfrak{D}_M, \mathfrak{D}_{NOR}\}$, all possible reduced markets are (i) elements of the domain $\mathfrak{D}'$ and (ii) solvable.

The last property we introduce is converse consistency, a property that determines the desirability of a matching for a roommate market on the basis of the desirability of its restrictions to reduced roommate markets that are obtained by taking two agents and their matches. So, given a matching μ for a roommate market (N, R) and a set of agents $N' \subseteq N$, $|N'| = 2$, if the restriction of μ to the set of agents $M' = N' \cup \mu(N')$ equals the matching chosen by the solution for this type of roommate market, then μ must be a matching assigned by the solution.

Converse Consistency: A solution φ on $\mathfrak{D}$ is *conversely consistent* if the following holds for each $(N, R) \in \mathfrak{D}$ and each $\mu \in \mathcal{M}(N)$. If for all $N' \subseteq N$, $|N'| = 2$, $M' = N' \cup \mu(N')$, and all reduced markets $(M', R_{M'}) \in \mathfrak{D}$, $\mu_{M'} \in \varphi(M', R_{M'})$, then $\mu \in \varphi(N, R)$.

Proposition 1. *On either of the roommate market domains* $\mathfrak{D}_S$*,* $\mathfrak{D}_{NOR}$*, or* $\mathfrak{D}_M$*, the core satisfies individual rationality, anonymity, Pareto optimality, consistency, and converse consistency.*

Proof. It is easy to see that, on any roommate market domain $\mathfrak{D}$, the core satisfies individual rationality, anonymity, and Pareto optimality. Can and Klaus (2012, Proposition 2) proved the consistency of the core for either of the roommate market domains $\mathfrak{D}_S$, $\mathfrak{D}_{NOR}$, or $\mathfrak{D}_M$.

Next, assume that the core is not conversely consistent on domain $\mathfrak{D} \in \{\mathfrak{D}_S, \mathfrak{D}_{NOR}, \mathfrak{D}_M\}$. Then, there exist $(N, R) \in \mathfrak{D}$ and $\mu \in \mathcal{M}(N)$ such that for all $N' \subseteq N$, $|N'| = 2$, $M' = N' \cup \mu(N')$, and all reduced markets $(M', R_{M'})$, $\mu_{M'} \in core(M', R_{M'})$ and $\mu \notin core(N, R)$. Hence, there exists a blocking pair $\{i, j\} \subseteq N$ for μ. Let $\bar{N}' = \{i, j\}$ and $M' = (\bar{N}' \cup \mu(\bar{N}'))$. Then, $\{i, j\}$ is also a blocking pair for $\mu_{M'}$; a contradiction. □

4 Characterizing the Core

4.1 Previous Results

For a subdomain of the *domain of classical marriage markets* for which matching agents of the same gender is not feasible, Sasaki and Toda (1992) characterized the core by anonymity, Pareto optimality, consistency, and converse consistency.

Theorem 1 (Sasaki and Toda (1992), Main Theorem)**.** *On the domain of classical marriage markets with equal numbers of men and women and where all men find all women acceptable and all women find all men acceptable, a solution satisfies anonymity, Pareto optimality, consistency, and converse consistency if and only if it is the core.*

In a recent paper, Nizamogullari and Özkal-Sanver (2012) generalized this result to the full *domain of classical marriage markets* by adding individual rationality and replacing anonymity with a stronger property called gender fairness (introduced in Özkal-Sanver, 2004).[7] We will discuss their characterization result in view of ours in Section 4 (after the proof of Theorem 3).

Theorem 2 (Nizamogullari and Özkal-Sanver (2012), Theorem 3.1)**.** *On the domain of classical marriage markets, a solution satisfies individual rationality, Pareto optimality, consistency, converse consistency, and gender fairness if and only if it is the core.*

Özkal-Sanver (2010, Proposition 4.2) showed that on the domain of all roommate markets, no solution satisfies Pareto optimality, anonymity, and converse consistency. However, the proof of this impossibility result used an unsolvable roommate market. Hence, the question whether or not the characterization of Sasaki and Toda (1992) can be extended to the domain of solvable roommate markets or to any of its subdomains has not been answered until now.

In the next section, we will first extend the characterizations of Sasaki and Toda (1992) and Nizamogullari and Özkal-Sanver (2012) to the domain of no odd rings roommate markets (Theorem 3). Second, we will show that the corresponding properties are compatible on the domain of solvable roommate markets, but that they do not characterize the core (Example 1).

4.2 Core Characterizations for Roommate and Marriage Markets

First, we prove that the characterizations of Sasaki and Toda (1992) and Nizamogullari and Özkal-Sanver (2012) can be extended to the domain of no odd rings roommate markets.

Theorem 3. *On the domain of no odd rings roommate markets, a solution satisfies individual rationality, anonymity, Pareto optimality, consistency, and converse consistency if and only if it is the core.*

We obtain an alternative characterization of the core by replacing individual rationality with mutually best (see Corollary 2 in Appendix A).

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[7] Loosely speaking, gender fairness requires that renaming men as women, and renaming women as men, does not change the outcomes chosen by the solution when taking the renaming of agents into account.

Proof. By Proposition 1, the core satisfies all the properties in the theorem on $\mathfrak{D}_{NOR}$. To prove the uniqueness part, let φ be a solution on the domain of no odd rings roommate markets $\mathfrak{D}_{NOR}$ that satisfies individual rationality, anonymity, Pareto optimality, consistency, and converse consistency. We will first show that $\varphi \subseteq core$.

Assume, by contradiction, that there exists a no odd rings roommate market $(\bar{N}, \bar{R}) \in \mathfrak{D}_{NOR}$ such that $\varphi(\bar{N}, \bar{R}) \nsubseteq core(\bar{N}, \bar{R})$. Then, there exists a matching $\bar{\mu} \in \varphi(\bar{N}, \bar{R})$ with a blocking pair $\{i, j\}$ for $\bar{\mu}$. By individual rationality, $i \neq j$. Let $N' = \{i, j\}$. Set $N = N' \cup \bar{\mu}(N') = \{i, j, \bar{\mu}(i), \bar{\mu}(j)\}$, $R = \bar{R}_N$, and $\mu = \bar{\mu}_N$. Consider the reduced market $(N, R) \in \mathfrak{D}_{NOR}$. Since $\bar{\mu} \in \varphi(N, R) \setminus core(N, R)$, by consistency, $\mu \in \varphi(N, R) \setminus core(N, R)$. We consider three cases depending on the cardinality of N.

In the sequel, we indicate the matches of matching μ (or corresponding other matchings being discussed at the time) in preference tables by a cycle around an agent's match.

Case 1 ($|N|$=2): Then, $N = \{i, j\}$ and $\{i, j\}$ being a blocking pair for μ implies that agents' preferences are as follows:

$$\begin{array}{c|cc} R_i & j & \textcircled{i} \\ R_j & i & \textcircled{j} \end{array}$$

However, note that agents i and j would both prefer being matched with each other than being single at μ; a contradiction to Pareto optimality.

Case 2 ($|N|$=3): Without loss of generality, let $k = \mu(i)$ and j is single at μ. Then, $N = \{i, j, k\}$. By individual rationality and $\{i, j\}$ being a blocking pair for μ, agents' partial preferences are as follows:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & & i & \textcircled{j} \\ R_k & \textcircled{i} & k & \end{array}$$

Among all no odd rings roommate markets, there is exactly one that complies with the above partial preferences:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & k & i & \textcircled{j} \\ R_k & \textcircled{i} & j & k \end{array}$$

It is noteworthy to mention that the above preference profile illustrates why we need to impose the no odd rings domain restriction in our theorem. The above roommate market is not solvable and solution φ is not defined for this subproblem. Therefore, for this specific preference profile, we cannot obtain a contradiction. We use this insight to construct a counterexample (Example 1) of a solution that is defined on the domain of solvable roommate markets, that does not equal the core, and that satisfies all properties in the theorem. We now proceed with the proof knowing that none of the subproblems we consider contains an odd ring and hence φ is defined and all properties apply.

Preferences R_j and R_k above are incomplete and we can distinguish 8 cases of complete and no odd ring preferences R_j and R_k. Relying heavily on anonymity, we will deal with all possible cases at the same time, but the interested reader can find a case by case proof in Appendix C.

Let $\pi_a \in \Pi^{\mathbb{N}}$ be such that

$$\pi_a(i) = 1,\ \pi_a(j) = 2,\ \text{and } \pi_a(k) = 3$$

and let $\pi_b \in \Pi^{\mathbb{N}}$ be such that

$$\pi_b(i) = 3,\ \pi_b(j) = 4,\ \text{and } \pi_b(k) = 1.$$

For $\tilde{N} = \{1, 2, 3, 4\}$ we define matching $\tilde{\mu} = \{(1, 3), 2, 4\}$ and $\tilde{R} \in \mathcal{R}^{\tilde{N}}$ such that

$$\tilde{R}_1: \quad 2 \quad \textcircled{3} \quad 1 \quad \| \quad [4]$$

and notation $\|$ [4] here means that the position of agent 4 relative to agents 3 and 1 is determined by the position of agent $j = \pi_b^{-1}(4)$ relative to agents $i = \pi_b^{-1}(3)$ and $k = \pi_b^{-1}(1)$ in $R_k = R_{\pi_b^{-1}(1)}$,

$$\tilde{R}_2: \quad 1 \quad \textcircled{2} \quad 4 \quad \| \quad [3]$$

and notation $\|$ [3] here means that the position of agent 3 relative to agents 1 and 2 is determined by the position of agent $k = \pi_a^{-1}(3)$ relative to agents $i = \pi_a^{-1}(1)$ and $j = \pi_a^{-1}(2)$ in $R_j = R_{\pi_a^{-1}(2)}$,[8]

$$\tilde{R}_3: \quad 4 \quad \textcircled{1} \quad 3 \quad \| \quad [2]$$

and notation $\|$ [2] here means that the position of agent 2 relative to agents 1 and 3 is determined by the position of agent $j = \pi_a^{-1}(2)$ relative to agents $i = \pi_a^{-1}(1)$ and $k = \pi_a^{-1}(3)$ in $R_k = R_{\pi_a^{-1}(3)}$, and

$$\tilde{R}_4: \quad 3 \quad \textcircled{4} \quad 2 \quad \| \quad [1]$$

and notation $\|$ [1] here means that the position of agent 1 relative to agents 3 and 4 is determined by the position of agent $k = \pi_b^{-1}(1)$ relative to agents $i = \pi_b^{-1}(3)$ and $j = \pi_b^{-1}(4)$ in $R_j = R_{\pi_b^{-1}(4)}$.[9]

We show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(1, 2), (3, 4)\}$ is preferred by everybody; a contradiction to Pareto optimality.

- $N' = M' = \{1, 3\}$:

$$\begin{array}{c|cc} \tilde{R}_1 & \textcircled{3} & 1 \\ \tilde{R}_3 & \textcircled{1} & 3 \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{2, 4\}$:

$$\begin{array}{c|cc} \tilde{R}_2 & \textcircled{2} & 4 \\ \tilde{R}_4 & \textcircled{4} & 2 \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

---

[8] We could choose the position of agent 4 in $\tilde{R}_2$ freely.

[9] We could choose the position of agent 2 in $\tilde{R}_4$ freely.

- $N' = \{1,2\}$ or $N' = \{2,3\}$ and $M' = \{1,2,3\}$:

$$\begin{array}{c|ccc}
\tilde{R}_1 & 2 & \textcircled{3} & 1 \\
\tilde{R}_2 & 1 & \textcircled{2} & \parallel [3] \\
\tilde{R}_3 & \textcircled{1} & 3 & \parallel [2]
\end{array}
\quad
\begin{array}{c}
\xleftarrow{\pi_a} \\
\xleftarrow{\pi_a} \\
\xleftarrow{\pi_a}
\end{array}
\quad
\begin{array}{c|ccc}
R_i & j & \textcircled{k} & i \\
R_j & i & \textcircled{j} & \parallel [k] \\
R_k & \textcircled{i} & k & \parallel [j]
\end{array}$$

Note that $N^{\pi_a} = M'$. In the above preference table, we have already indicated that $\tilde{R}_{M'} = R^{\pi_a}$. Then, by anonymity, $\tilde{\mu}_{M'} = \mu^{\pi_a} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{1,4\}$ or $N' = \{3,4\}$ and $M' = \{1,3,4\}$:

$$\begin{array}{c|ccc}
\tilde{R}_1 & \textcircled{3} & 1 & \parallel [4] \\
\tilde{R}_3 & 4 & \textcircled{1} & 3 \\
\tilde{R}_4 & 3 & \textcircled{4} & \parallel [1]
\end{array}
\quad
\begin{array}{c}
\xleftarrow{\pi_b} \\
\xleftarrow{\pi_b} \\
\xleftarrow{\pi_b}
\end{array}
\quad
\begin{array}{c|ccc}
R_k & \textcircled{i} & k & \parallel [j] \\
R_i & j & \textcircled{k} & i \\
R_j & i & \textcircled{j} & \parallel [k]
\end{array}$$

Note that $N^{\pi_b} = M'$. In the above preference table, we have already indicated that $\tilde{R}_{M'} = R^{\pi_b}$. Then, by anonymity, $\tilde{\mu}_{M'} = \mu^{\pi_b} \in \varphi(M', \tilde{R}_{M'})$.

Case 3 ($|N|$=4): Let $k = \mu(i)$, $l = \mu(j)$, and $N = \{i,j,k,l\}$. By individual rationality and $\{i,j\}$ being a blocking pair for μ, agents' partial preferences are as follows:

$$\begin{array}{c|l}
R_i & j \ \textcircled{k} \ i \\
R_j & i \ \textcircled{l} \ j \\
R_k & \textcircled{i} \ k \\
R_l & \textcircled{j} \ l
\end{array}$$

There are $4 \cdot 4 \cdot 12 \cdot 12 = 2304$ preference profiles that comply with the above partial preferences. Even though some of these profiles are not part of the no odd rings domain, a case by case proof as for Case 2, Appendix C, is a bit too much work. Luckily, as in Case 2, we can offer a compact proof using anonymity. Note that even though we will not mention it explicitly, the following proof excludes preference profiles that contain odd rings (as in Case 2, for odd rings roommate markets solution φ is not defined).

Let $\pi_a \in \Pi^{\mathbb{N}}$ be such that

$$\pi_a(i) = 3,\ \pi_a(j) = 5,\ \pi_a(k) = 1, \text{ and } \pi_a(l) = 6,$$

let $\pi_b \in \Pi^{\mathbb{N}}$ be such that

$$\pi_b(i) = 6,\ \pi_b(j) = 4,\ \pi_b(k) = 5 \text{ and } \pi_b(l) = 2,$$

and let $\pi_c \in \Pi^{\mathbb{N}}$ be such that

$$\pi_c(i) = 2,\ \pi_c(j) = 1,\ \pi_c(k) = 4 \text{ and } \pi_c(l) = 3.$$

For $\tilde{N} = \{1,2,3,4,5,6\}$ we define matching $\tilde{\mu} = \{(1,3),(2,4),(5,6)\}$ and $\tilde{R} \in \mathcal{R}^{\tilde{N}}$ such that

$$\tilde{R}_1 : \quad 2 \quad \textcircled{3} \quad 1 \quad \parallel \quad [4],[5],[6]$$

and notation $\parallel [4]], [5], [6]$ here means that the position of agents 5 and 6 relative to each other and to agents 3 and 1 is determined by the position of agents $j = \pi_a^{-1}(5)$ and $l = \pi_a^{-1}(6)$ relative to each other and to agents $i = \pi_a^{-1}(3)$ and $k = \pi_a^{-1}(1)$ in $R_k = R_{\pi_a^{-1}(1)}$ and that the position of agent 4 relative to agents 2, 3, and 1 is determined by the position of agent $k = \pi_c^{-1}(4)$ relative to agents $i = \pi_c^{-1}(2)$, $l = \pi_c^{-1}(3)$, and $j = \pi_c^{-1}(1)$ in $R_j = R_{\pi_c^{-1}(1)}$,

$$\tilde{R}_2 : \quad 1 \quad \textcircled{4} \quad 2 \quad \parallel \quad [3], [5], [6]$$

and notation $\parallel [3], [5], [6]$ here means that the position of agents 5 and 6 relative to each other and to agents 4 and 2 is determined by the position of agents $k = \pi_b^{-1}(5)$ and $i = \pi_b^{-1}(6)$ relative to each other and to agents $j = \pi_b^{-1}(4)$ and $l = \pi_b^{-1}(2)$ in $R_l = R_{\pi_b^{-1}(2)}$ and that the position of agent 3 relative to agents 1, 4, and 2 is determined by the position of agent $l = \pi_c^{-1}(3)$ relative to agents $j = \pi_c^{-1}(1)$, $k = \pi_c^{-1}(4)$, and $i = \pi_c^{-1}(2)$ in $R_i = R_{\pi_c^{-1}(2)}$,

$$\tilde{R}_3 : \quad 5 \quad \textcircled{1} \quad 3 \quad \parallel \quad [2], [4], [6]$$

and notation $\parallel [2], [4], [6]$ here means that the position of agent 6 relative to agents 5, 1, and 3 is determined by the position of agent $l = \pi_a^{-1}(6)$ relative to agents $j = \pi_a^{-1}(5)$, $k = \pi_a^{-1}(1)$, and $i = \pi_a^{-1}(3)$ in $R_i = R_{\pi_a^{-1}(3)}$ and that the position of agents 2 and 4 relative to each other and to agents 1 and 3 is determined by the position of agents $i = \pi_c^{-1}(2)$ and $k = \pi_c^{-1}(4)$ relative to each other and to agents $j = \pi_c^{-1}(1)$ and $l = \pi_c^{-1}(3)$ in $R_l = R_{\pi_c^{-1}(3)}$,

$$\tilde{R}_4 : \quad 6 \quad \textcircled{2} \quad 4 \quad \parallel \quad [1], [3], [5]$$

and notation $\parallel [1], [3], [5]$ here means that the position of agent 5 relative to agents 6, 2, and 4 is determined by the position of agent $k = \pi_b^{-1}(5)$ relative to agents $i = \pi_b^{-1}(6)$, $l = \pi_b^{-1}(2)$, and $j = \pi_b^{-1}(4)$ in $R_j = R_{\pi_b^{-1}(4)}$ and that the position of agents 1 and 3 relative to each other and to agents 2 and 4 is determined by the position of agents $j = \pi_c^{-1}(1)$ and $l = \pi_c^{-1}(3)$ relative to each other and to agents $i = \pi_c^{-1}(2)$ and $R_k = \pi_c^{-1}(4)$ in $R_k = R_{\pi_c^{-1}(4)}$,

$$\tilde{R}_5 : \quad 3 \quad \textcircled{6} \quad 5 \quad \parallel \quad [1], [2], [4]$$

and notation $\parallel [1], [2], [4]$ here means that the position of agent 1 relative to agents 3, 6, and 5 is determined by the position of agent $k = \pi_a^{-1}(1)$ relative to agents $i = \pi_a^{-1}(3)$, $l = \pi_a^{-1}(6)$, and $j = \pi_a^{-1}(5)$ in $R_j = R_{\pi_a^{-1}(5)}$ and that the position of agents 2 and 4 relative to each other and to agents 6 and 5 is determined by the position of agents $l = \pi_b^{-1}(2)$ and $j = \pi_b^{-1}(4)$ relative to each other and to agents $i = \pi_b^{-1}(6)$ and $k = \pi_b^{-1}(5)$ in $R_k =_{\pi_b^{-1}(5)}$, and

$$\tilde{R}_6 : \quad 4 \quad \textcircled{5} \quad 6 \quad \parallel \quad [1], [2], [3]$$

and notation $\parallel [1], [2], [3]$ here means that the position of agents 1 and 3 relative to each other and to agents 5 and 6 is determined by the position of agents $k = \pi_a^{-1}(1)$ and $i = \pi_a^{-1}(3)$ relative to each other and to agents $j = \pi_a^{-1}(5)$ and $l = \pi_a^{-1}(6)$ in $R_l = R_{\pi_a^{-1}(6)}$ and that the position of agent 2 relative to agents 4, 5, and 6 is determined by the position of agent $l = \pi_b^{-1}(2)$ relative to agents $j = \pi_b^{-1}(4)$, $k = \pi_b^{-1}(5)$, and $i = \pi_b^{-1}(6)$ in $R_i = R_{\pi_b^{-1}(6)}$.

We show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(1,2),(3,5),(4,6)\}$ is preferred by everybody; a contradiction to Pareto optimality.

- $N' = M' = \{1,3\}$:

$$\begin{array}{c|cc} \tilde{R}_1 & \textcircled{3} & 1 \\ \tilde{R}_3 & \textcircled{1} & 3 \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{2,4\}$:

$$\begin{array}{c|cc} \tilde{R}_2 & \textcircled{4} & 2 \\ \tilde{R}_4 & \textcircled{2} & 4 \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{5,6\}$:

$$\begin{array}{c|cc} \tilde{R}_5 & \textcircled{6} & 5 \\ \tilde{R}_6 & \textcircled{5} & 6 \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{1,5\}$, $N' = \{1,6\}$, $N' = \{3,5\}$, or $N' = \{3,6\}$ and $M' = \{1,3,5,6\}$:

$$\begin{array}{c|cccc c c|cccc}
\tilde{R}_1 & \textcircled{3} & 1 & \| \, [5], & [6] & \overset{\pi_a}{\longleftarrow} & R_k & \textcircled{i} & k & \| \, [j], & [l] \\
\tilde{R}_3 & 5 & \textcircled{1} & 3 & \| \, [6] & \overset{\pi_a}{\longleftarrow} & R_i & j & \textcircled{k} & i & \| \, [l] \\
\tilde{R}_5 & 3 & \textcircled{6} & 5 & \| \, [1] & \overset{\pi_a}{\longleftarrow} & R_j & i & \textcircled{l} & j & \| \, [k] \\
\tilde{R}_6 & \textcircled{5} & 6 & \| \, [1], & [3] & \overset{\pi_a}{\longleftarrow} & R_l & \textcircled{j} & l & \| \, [k], & [i]
\end{array}$$

Note that $N^{\pi_a} = M'$. In the above preference table, we have already indicated that $\tilde{R}_{M'} = R^{\pi_a}$. Then, by anonymity, $\tilde{\mu}_{M'} = \mu^{\pi_a} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{2,5\}$, $N' = \{2,6\}$, $N' = \{4,5\}$, or $N' = \{4,6\}$ and $M' = \{2,4,5,6\}$:

$$\begin{array}{c|cccc c c|cccc}
\tilde{R}_2 & \textcircled{4} & 2 & \| \, [5], & [6] & \overset{\pi_b}{\longleftarrow} & R_l & \textcircled{j} & l & \| \, [k], & [i] \\
\tilde{R}_4 & 6 & \textcircled{2} & 4 & \| \, [5] & \overset{\pi_b}{\longleftarrow} & R_j & i & \textcircled{l} & j & \| \, [k] \\
\tilde{R}_5 & \textcircled{6} & 5 & \| \, [2], & [4] & \overset{\pi_b}{\longleftarrow} & R_k & \textcircled{i} & k & \| \, [l], & [j] \\
\tilde{R}_6 & 4 & \textcircled{5} & 6 & \| \, [2] & \overset{\pi_b}{\longleftarrow} & R_i & j & \textcircled{k} & i & \| \, [l]
\end{array}$$

Note that $N^{\pi_b} = M'$. In the above preference table, we have already indicated that $\tilde{R}_{M'} = R^{\pi_b}$. Then, by anonymity, $\tilde{\mu}_{M'} = \mu^{\pi_b} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{1,2\}$, $N' = \{1,4\}$, $N' = \{3,2\}$, or $N' = \{3,4\}$ and $M' = \{1,2,3,4\}$:

$$\begin{array}{c|cccccc|cccc}
\tilde{R}_1 & 2 & \textcircled{3} & 1 & \| \; [4] & \overset{\pi_c}{\longleftarrow} & R_j & i & \textcircled{l} & j & \| \; [k] \\
\tilde{R}_2 & 1 & \textcircled{4} & 2 & \| \; [3] & \overset{\pi_c}{\longleftarrow} & R_i & j & \textcircled{k} & i & \| \; [l] \\
\tilde{R}_3 & \textcircled{1} & 3 & \| \; [2], & [4] & \overset{\pi_c}{\longleftarrow} & R_l & \textcircled{j} & l & \| \; [i], & [k] \\
\tilde{R}_4 & \textcircled{2} & 4 & \| \; [1], & [3] & \overset{\pi_c}{\longleftarrow} & R_k & \textcircled{i} & k & \| \; [j], & [l]
\end{array}$$

  Note that $N^{\pi_c} = M'$. In the above preference table, we have already indicated that $\tilde{R}_{M'} = R^{\pi_c}$. Then, by anonymity, $\tilde{\mu}_{M'} = \mu^{\pi_c} \in \varphi(M', \tilde{R}_{M'})$.

Cases 1, 2, and 3 have now all resulted in contradictions. Hence, our assumption that $\varphi \nsubseteq core$ was incorrect and we have now shown that $\varphi \subseteq core$. However, by Lemma 1, no proper subsolution of the core satisfies consistency. Hence, $\varphi = core$. □

We prove the independence of properties in Theorem 3 in Appendix B.

We next consider the marriage market domain $\mathfrak{D}_M \subsetneq \mathfrak{D}_{NOR}$. Recall that we model the classical marriage market restriction that no two men and no two women are matched together via the agents' preferences together with individual rationality: no woman finds another woman acceptable and no man finds another man acceptable. Note that our anonymity property, for any marriage market, will respect the division into men and women that is ingrained in the agents' preferences and that the proof of our characterization in Theorem 3 can easily be adapted to show the following corollary.[10]

Corollary 1. *On the domain of marriage markets, a solution satisfies individual rationality, anonymity, Pareto optimality, consistency, and converse consistency if and only if it is the core.*

Nizamogullari and Özkal-Sanver (2012, Example 3.2) showed for *classical marriage markets* that the above characterization *does not seem to apply.* The reason why, on the domain of classical marriage markets, a solution different from the core satisfies the properties individual rationality, anonymity, Pareto optimality, consistency, and converse consistency is that for classical marriage markets anonymity is a much weaker property: it only allows to rename men within the group of men and women within the group of women. Hence, "classical marriage market anonymity" as introduced by Sasaki and Toda (1992) and adapted by Nizamogullari and Özkal-Sanver (2012) still allows a solution to discriminate based on gender and hence does not capture the full spirit of anonymity. It suffices to strengthen anonymity by adding the requirement that whenever there is an equal number of men and women present, then renaming men as women and women as men does not essentially change the matchings a solution assigns (up to the corresponding name changes): this stronger property is referred to as gender fairness (Özkal-Sanver, 2004; Nizamogullari and Özkal-Sanver, 2012).

Özkal-Sanver (2010, Proposition 4.2) showed that on the domain of all roommate markets, no solution satisfies Pareto optimality, anonymity, and converse consistency. By Proposition 1, the core satisfies all these properties on the domain of solvable roommate markets. However, Theorem 3 does not hold on $\mathfrak{D}_S$ as we will demonstrate with the following counter example.

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[10] Proposition 1 and Lemma 1 hold on the marriage market domain $\mathfrak{D}_M$. Furthermore, since all marriage markets are no odd rings roommate markets but not all no odd rings roommate markets are marriage markets, the proof in principle is just shorter.

Example 1. We define solution $\hat{\varphi} \supsetneq \mathit{core}$ on $\mathfrak{D}_S$ using the following roommate markets and matching. Let roommate market $(\hat{N}, \hat{R})$ be such that $\hat{N} = \{1, 2, 3, 4, 5, 6\}$ and partial preferences $\hat{R}$ are given as follows (we indicate the matches of matching $\hat{\mu}$ below by a cycle around an agent's match.):

$$\begin{array}{c|lcl}
\hat{R}_1 & 2\ 3\ \textcircled{1}\ \ldots & \qquad & \mu = \{(1,3),(2,6),(4,5)\} \\
\hat{R}_2 & \textcircled{3}\ 6\ 1\ 2\ \ldots & & \hat{\mu} = \{(2,3),(4,6),1,5\} \\
\hat{R}_3 & 1\ \textcircled{2}\ 3\ \ldots & & \\
\hat{R}_4 & 5\ \textcircled{6}\ 4\ \ldots & & \mathit{core}(\hat{N},\hat{R}) = \{\mu\} \\
\hat{R}_5 & 6\ 4\ \textcircled{5}\ \ldots & & \\
\hat{R}_6 & \textcircled{4}\ 2\ 5\ 6\ \ldots & &
\end{array}$$

A roommate market of the form $(\hat{N}, \hat{R})$ contains two odd rings $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Now consider matching $\hat{\mu} = \{(2, 3), (4, 6), 1, 5\} \in PO(\hat{N}, \hat{R}) \cap IR(\hat{N}, \hat{R})$ and its reduced markets $M' = \{i, j, \hat{\mu}(i), \hat{\mu}(j)\}$ with $i, j \in \hat{N}$. Note that the reduced markets $(M', \hat{R}_{M'})$ with $M' \in \{\{1, 2, 3\}, \{4, 5, 6\}\}$ are not solvable and that $\hat{\mu}_{M'}$ is a stable matching for the reduced market $(M', \hat{R}_{M'})$ with $M' = \{2, 3, 4, 6\}$ (these facts will guarantee that by converse consistency no non-stable matchings will be added to larger roommate markets that include $(\hat{N}, \hat{R})$ as reduced market). For all roommate markets of the form $(\hat{N}, \hat{R})$, solution $\hat{\varphi}$ adds the unstable matching $\hat{\mu}$ to the core, i.e., $\hat{\varphi}(\hat{N}, \hat{R}) = \mathit{core}(\hat{N}, \hat{R}) \cup \{\hat{\mu}\}$. Furthermore, for all $\pi \in \Pi^{\mathbb{N}}$, $\hat{\varphi}(\hat{N}^{\pi}, \hat{R}^{\pi}) = \mathit{core}(\hat{N}^{\pi}, \hat{R}^{\pi}) \cup \{\hat{\mu}^{\pi}\}$. For all other roommate markets $(N, R) \in \mathfrak{D}_S$, $\hat{\varphi}(N, R) = \mathit{core}(N, R)$. Solution $\hat{\varphi}$, by construction, satisfies individual rationality, anonymity, Pareto optimality, consistency, and converse consistency. ◇

Appendix

A Mutually Best

Mutually best requires that two agents who are "mutually best agents" are always matched with each other.

Mutually Best: Let $(N, R) \in \mathfrak{D}$ and $i, j \in N$ [possibly $i = j$] such that for all $k \in N$, $i\, R_j\, k$ and $j\, R_i\, k$. Then, i and j are *mutually best agents for* (N, R). A matching is a *mutually best matching* if all mutually best agents are mutually matched. $MB(N, R)$ denotes the set of all these matchings. A solution φ on $\mathfrak{D}$ is *mutually best* if it only assigns matchings at which all mutually best agents are matched, i.e., for all roommate markets $(N, R) \in \mathfrak{D}$, $\varphi(N, R) \subseteq MB(N, R)$.

Our notion of mutually best is slightly stronger than that used in Toda (2006) for marriage markets (because he considers mutually best man-woman pairs, he does not allow for a single mutually best agent $i = j$).

Lemma 2 (Can and Klaus (2012), Lemma 2)**.** *On the domain of no odd rings roommate markets, mutually best and consistency imply individual rationality.*

Corollary 2. *On the domain of no odd rings roommate markets, a solution satisfies anonymity, Pareto optimality, mutually best, consistency, and converse consistency if and only if it is the core.*

B Independence of Properties in Theorem 3

- The solution IR that always assigns the set of individually rational matchings satisfies individual rationality, anonymity, consistency, and converse consistency, but not Pareto optimality.
- The intersection of the Pareto solution PO with solution IR, $PO \cap IR$, satisfies individual rationality, anonymity, Pareto optimality, and consistency, but not converse consistency.

  To see that converse consistency is violated, consider the following roommate market $(N, R) \in \mathfrak{D}_{NOR}$ with $N = \{1, 2, 3, 4, 5, 6\}$ and matching $\mu = \{(1, 2), (3, 4), (5, 6)\}$:

R_1	4	②	1	...
R_2	5	①	2	...
R_3	6	④	3	...
R_4	1	③	4	...
R_5	2	⑥	5	...
R_6	3	⑤	6	...

  It is easy to check that converse consistency would imply that $\mu \in PO \cap IR$ even though all agents prefer matching $\{(1, 4), (2, 5), (3, 6)\}$.
- The solution ζ that for any roommate market $(N, R) \in \mathfrak{D}_{NOR}$ such that $|N| \leq 4$ assigns the core and otherwise equals $PO \cap IR$ satisfies individual rationality, anonymity, Pareto optimality, and converse consistency, but not consistency.
- We define a subsolution of the Pareto solution $\psi \subsetneq PO$ by elimination all matchings that have a blocking pair composed of two different agents, i.e., in difference to the core this solution allows for degenerate blocking pairs of the form $\{i, i\}$ and hence violates individual rationality.

  Solution ψ satisfies anonymity, Pareto optimality, consistency, and converse consistency.
- We define solution ξ as follows. First, consider the following roommate market $(\hat{N}, \hat{R}) \in \mathfrak{D}_{NOR}$ with $\hat{N} = \{1, 2, 3\}$ and matching $\hat{\mu} = \{(1, 3), 2\}$:

$\hat{R}_1$	2	③	1
$\hat{R}_2$	1	②	3
$\hat{R}_3$	①	3	2

  The unique stable matching for this roommate market is $\{(1, 2), 3\}$ and solution ξ adds matching $\hat{\mu}$ to the core, i.e., $\xi(\hat{N}, \hat{R}) = core(\hat{N}, \hat{R}) \cup \{\hat{\mu}\}$.

  Next, consider roommates markets $(N, R) \in \mathfrak{D}_{NOR}$ such that $\hat{N} \subseteq N$, $R_{\hat{N}} = \hat{R}$ and let $\bar{\mu} \in core(N, R)$ such that $\bar{\mu}_{\hat{N}} = \{(1, 2), 3\}$ and at matching $\tilde{\mu} = \bar{\mu}_{N \setminus \hat{N}} \cup \hat{\mu}$, $\{1, 2\}$ is the unique blocking pair. Then, solution ξ adds the unstable matching $\tilde{\mu}$ to $core(N, R)$. Note that solution ξ might add several of these additional unstable matchings that replace the component $\{(1, 2), 3\}$ in a stable matching with component $\hat{\mu}$. For all remaining roommates markets $(N, R) \in \mathfrak{D}_{NOR}$, $\xi(N, R) = core(N, R)$. Note that the addition of matchings with the "component" $\hat{\mu}$ is designed such that ξ satisfies converse consistency.

  Solution ξ satisfies individual rationality, Pareto optimality, consistency, and converse consistency, but not anonymity.

C Case by case proof for Case 2 ($|N|$=3) in the proof of Theorem 3

Case 2.1:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & k & i & \textcircled{j} \\ R_k & j & \textcircled{i} & k \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R} \in \mathcal{R}^{\tilde{N}}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i, k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$$\begin{array}{c|cccc} \tilde{R}_i & l & j & \textcircled{k} & i \\ \tilde{R}_j & k & i & \textcircled{j} & l \\ \tilde{R}_k & j & l & \textcircled{i} & k \\ \tilde{R}_l & i & k & \textcircled{l} & j \end{array}$$

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matchings $\{(i, j), (k, l)\}$ and $\{(i, l), (j, k)\}$ are preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

$$\begin{array}{c|cc} \tilde{R}_i & \textcircled{k} & i \\ \tilde{R}_k & \textcircled{i} & k \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

$$\begin{array}{c|cc} \tilde{R}_j & \textcircled{j} & l \\ \tilde{R}_l & \textcircled{l} & j \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & j & \textcircled{k} & i \\ \tilde{R}_j & k & i & \textcircled{j} \\ \tilde{R}_k & j & \textcircled{i} & k \end{array}$$

  This is the original roommate market (N, R) of Case 2.1 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & l & \textcircled{k} & i \\ \tilde{R}_k & l & \textcircled{i} & k \\ \tilde{R}_l & i & k & \textcircled{l} \end{array}$$

Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^\pi, R^\pi)$ and $\tilde{\mu}_{M'} = \mu^\pi$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

Case 2.2:

$$\begin{array}{c|ccc} \tilde{R}_i & j & \textcircled{k} & i \\ \tilde{R}_j & i & k & \textcircled{j} \\ \tilde{R}_k & j & \textcircled{i} & k \end{array}$$

Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = j$. Note that we can obtain Case 2.2 by applying permutation π to Case 2.1. Hence, we obtain a contradiction in a similar fashion (up to name changes according to π).

Case 2.3:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & k & i & \textcircled{j} \\ R_k & \textcircled{i} & k & j \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i, k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$$\begin{array}{c|cccc} \tilde{R}_i & j & \textcircled{k} & i & l \\ \tilde{R}_j & k & i & \textcircled{j} & l \\ \tilde{R}_k & l & \textcircled{i} & k & j \\ \tilde{R}_l & i & k & \textcircled{l} & j \end{array}$$

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(i, j), (k, l)\}$ is preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

$$\begin{array}{c|cc} \tilde{R}_i & \textcircled{k} & i \\ \tilde{R}_k & \textcircled{i} & k \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

$$\begin{array}{c|cc} \tilde{R}_j & \textcircled{j} & l \\ \tilde{R}_l & \textcircled{l} & j \end{array}$$

Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & j & \textcircled{k} & i \\ \tilde{R}_j & k & i & \textcircled{j} \\ \tilde{R}_k & \textcircled{i} & k & j \end{array}$$

This is the original roommate market (N, R) of Case 2.3 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & \textcircled{k} & i & l \\ \tilde{R}_k & l & \textcircled{i} & k \\ \tilde{R}_l & i & k & \textcircled{l} \end{array}$$

Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^\pi, R^\pi)$ and $\tilde{\mu}_{M'} = \mu^\pi$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

Case 2.4:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & i & k & \textcircled{j} \\ R_k & \textcircled{i} & j & k \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i, k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$$\begin{array}{c|cccc} \tilde{R}_i & j & \textcircled{k} & l & i \\ \tilde{R}_j & i & k & \textcircled{j} & l \\ \tilde{R}_k & l & \textcircled{i} & j & k \\ \tilde{R}_l & k & i & \textcircled{l} & j \end{array}$$

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(i, j), (k, l)\}$ is preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

$$\begin{array}{c|cc} \tilde{R}_i & \enclose{circle}{k} & i \\ \tilde{R}_k & \enclose{circle}{i} & k \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

$$\begin{array}{c|cc} \tilde{R}_j & \enclose{circle}{j} & l \\ \tilde{R}_l & \enclose{circle}{l} & j \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & j & \enclose{circle}{k} & i \\ \tilde{R}_j & i & k & \enclose{circle}{j} \\ \tilde{R}_k & \enclose{circle}{i} & j & k \end{array}$$

  This is the original roommate market (N, R) of Case 2.4 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & \enclose{circle}{k} & l & i \\ \tilde{R}_k & l & \enclose{circle}{i} & k \\ \tilde{R}_l & k & i & \enclose{circle}{l} \end{array}$$

  Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^\pi, R^\pi)$ and $\tilde{\mu}_{M'} = \mu^\pi$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

Case 2.5:

$$\begin{array}{c|ccc} R_i & j & \enclose{circle}{k} & i \\ R_j & i & k & \enclose{circle}{j} \\ R_k & \enclose{circle}{i} & k & j \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i, k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$$\begin{array}{c|cccc} \tilde{R}_i & j & \enclose{circle}{k} & i & l \\ \tilde{R}_j & i & k & \enclose{circle}{j} & l \\ \tilde{R}_k & l & \enclose{circle}{i} & k & j \\ \tilde{R}_l & k & i & \enclose{circle}{l} & j \end{array}$$

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(i,j),(k,l)\}$ is preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

$$\begin{array}{c|cc} \tilde{R}_i & \textcircled{k} & i \\ \tilde{R}_k & \textcircled{i} & k \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

$$\begin{array}{c|cc} \tilde{R}_j & \textcircled{j} & l \\ \tilde{R}_l & \textcircled{l} & j \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & j & \textcircled{k} & i \\ \tilde{R}_j & i & k & \textcircled{j} \\ \tilde{R}_k & \textcircled{i} & k & j \end{array}$$

  This is the original roommate market (N, R) of Case 2.5 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & \textcircled{k} & i & l \\ \tilde{R}_k & l & \textcircled{i} & k \\ \tilde{R}_l & k & i & \textcircled{l} \end{array}$$

  Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^\pi, R^\pi)$ and $\tilde{\mu}_{M'} = \mu^\pi$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

Case 2.6:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & i & \textcircled{j} & k \\ R_k & j & \textcircled{i} & k \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i,k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$\tilde{R}_i$	j	l	$\textcircled{k}$	i
$\tilde{R}_j$	i	$\textcircled{j}$	k	l
$\tilde{R}_k$	l	j	$\textcircled{i}$	k
$\tilde{R}_l$	k	$\textcircled{l}$	j	i

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(i,j),(k,l)\}$ is preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

$\tilde{R}_i$	$\textcircled{k}$	i
$\tilde{R}_k$	$\textcircled{i}$	k

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

$\tilde{R}_j$	$\textcircled{j}$	l
$\tilde{R}_l$	$\textcircled{l}$	j

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

$\tilde{R}_i$	j	$\textcircled{k}$	i
$\tilde{R}_j$	i	$\textcircled{j}$	k
$\tilde{R}_k$	j	$\textcircled{i}$	k

  This is the original roommate market (N, R) of Case 2.6 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$\tilde{R}_i$	l	$\textcircled{k}$	i
$\tilde{R}_k$	l	$\textcircled{i}$	k
$\tilde{R}_l$	k	$\textcircled{l}$	i

  Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^{\pi}, R^{\pi})$ and $\tilde{\mu}_{M'} = \mu^{\pi}$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

Case 2.7:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & i & \textcircled{j} & k \\ R_k & \textcircled{i} & j & k \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i, k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$$\begin{array}{c|cccc} \tilde{R}_i & j & \textcircled{k} & l & i \\ \tilde{R}_j & i & \textcircled{j} & k & l \\ \tilde{R}_k & l & \textcircled{i} & j & k \\ \tilde{R}_l & k & \textcircled{l} & i & j \end{array}$$

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(i, j), (k, l)\}$ is preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

  $$\begin{array}{c|cc} \tilde{R}_i & \textcircled{k} & i \\ \tilde{R}_k & \textcircled{i} & k \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

  $$\begin{array}{c|cc} \tilde{R}_j & \textcircled{j} & l \\ \tilde{R}_l & \textcircled{l} & j \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

  $$\begin{array}{c|ccc} \tilde{R}_i & j & \textcircled{k} & i \\ \tilde{R}_j & i & \textcircled{j} & k \\ \tilde{R}_k & \textcircled{i} & j & k \end{array}$$

  This is the original roommate market (N, R) of Case 2.7 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & \textcircled{k} & l & i \\ \tilde{R}_k & l & \textcircled{i} & k \\ \tilde{R}_l & k & \textcircled{l} & i \end{array}$$

  Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^\pi, R^\pi)$ and $\tilde{\mu}_{M'} = \mu^\pi$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

Case 2.8:

$$\begin{array}{c|ccc} R_i & j & \textcircled{k} & i \\ R_j & i & \textcircled{j} & k \\ R_k & \textcircled{i} & k & j \end{array}$$

Let $\tilde{N} = \{i, j, k, l\}$ and consider the following preference profile $\tilde{R}$ such that $\tilde{R}_N = R$ as well as matching $\tilde{\mu} = \{(i, k), j, l\}$ such that $\tilde{\mu}_N = \mu$:

$$\begin{array}{c|cccc} \tilde{R}_i & j & \textcircled{k} & i & l \\ \tilde{R}_j & i & \textcircled{j} & k & l \\ \tilde{R}_k & l & \textcircled{i} & k & j \\ \tilde{R}_l & k & \textcircled{l} & i & j \end{array}$$

We will show that for all $N' \subseteq \tilde{N}$, $|N'| = 2$, $M' = N' \cup \tilde{\mu}(N')$, and all reduced markets $(M', \tilde{R}_{M'})$, $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$. Hence, by converse consistency, $\tilde{\mu} \in \varphi(\tilde{N}, \tilde{R})$. However, note that matching $\{(i, j), (k, l)\}$ is preferred by everybody; a contradiction to Pareto optimality.

In the following steps we denote the market $(M', \tilde{R}_{M'})$ with the corresponding preference table and indicate the matches of matching $\tilde{\mu}_{M'}$ by a cycle around an agent's match.

- $N' = M' = \{i, k\}$:

$$\begin{array}{c|cc} \tilde{R}_i & \textcircled{k} & i \\ \tilde{R}_k & \textcircled{i} & k \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = M' = \{j, l\}$:

$$\begin{array}{c|cc} \tilde{R}_j & \textcircled{j} & l \\ \tilde{R}_l & \textcircled{l} & j \end{array}$$

  Pareto optimality implies that $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, j\}$ or $N' = \{j, k\}$ and $M' = \{i, j, k\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & j & \textcircled{k} & i \\ \tilde{R}_j & i & \textcircled{j} & k \\ \tilde{R}_k & \textcircled{i} & k & j \end{array}$$

  This is the original roommate market (N, R) of Case 2.8 and $\tilde{\mu}_{M'} = \mu$. Hence $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

- $N' = \{i, l\}$ or $N' = \{k, l\}$ and $M' = \{i, k, l\}$:

$$\begin{array}{c|ccc} \tilde{R}_i & \enclose{circle}{k} & i & l \\ \tilde{R}_k & l & \enclose{circle}{i} & k \\ \tilde{R}_l & k & \enclose{circle}{l} & i \end{array}$$

  Consider $\pi \in \Pi^{\mathbb{N}}$ such that $\pi(i) = k$, $\pi(k) = i$, and $\pi(j) = l$. Then, $(M', \tilde{R}_{M'}) = (N^\pi, R^\pi)$ and $\tilde{\mu}_{M'} = \mu^\pi$. Since $\mu \in \varphi(N, R)$, by anonymity $\tilde{\mu}_{M'} \in \varphi(M', \tilde{R}_{M'})$.

This case by case proof hence establishes a contradiction whenever $|N| = 3$.

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