A Unifying Model of Strategic Network Formation*

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February 17, 2015

Abstract

We provide a model that merges two basic models of strategic network formation and incorporates them as extreme cases: Jackson and Wolinsky's connections model based on bilateral formation of links, and Bala and Goyal's two-way flow model, where links can be unilaterally formed. In our model a link can be created unilaterally, but when it is only supported by one of the two players the flow through it suffers some friction or decay, but more than when it is supported by both players. When the friction in singly-supported links is maximal (i.e. there is no flow) we have Jackson and Wolinsky's connections model, while when flow in singly-supported links is as good as in doubly-supported links we have Bala and Goyal's two-way flow model. In this setting, a joint generalization of the results relative to efficiency and stability in both seminal papers is achieved, and the robustness in both models is tested with positive results.

JEL Classification Numbers: A14, C72, D20, J00

Key words: Network formation, Unilateral link-formation, Bilateral link-formation, Efficiency, Stability.

*We thank Noemí Navarro for helpful comments. This research is supported by the Spanish Ministerio de Economía y Competitividad under projects ECO2012-31626 and ECO2012-31346. Both authors also benefit from the Basque Government Departamento de Educación, Política Lingüística y Cultura funding for Grupos Consolidados IT869-13 and IT568-13.

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1 Introduction

The importance of the role played by the network structures underlying social and economic phenomena is now widely recognized[1]. From a theoretical point of view, perhaps the most challenging issue is the formation of network structures. There are two main models of strategic network formation in economic literature: that of Jackson and Wolinsky (1996), where a link between two "players" (individuals, firms, towns, etc.) needs the support of both and forms only if both agree, and that of Bala and Goyal (2000a), where players can form links unilaterally. Jackson and Wolinsky's model has two variants: the connections model and the coauthors model. Bala and Goyal's model also has two versions: the one-way flow model, in which flow through a link runs toward a player only if he/she supports it, and the two-way flow model, in which flow runs in both directions through all links.

These seminal models have had a great impact on the literature[2], and are at the root of several extensions resulting from introducing different variations into one model or the other. This paper addresses a different goal: *the unification of the two models by eliminating the dichotomy of unilateral vs. bilateral formation of links.* This is achieved by a model that bridges the gap between the two basic models of strategic network formation[3]. More precisely, we provide a model which has Jackson and Wolinsky's connections model and Bala and Goyal's two-way flow model as extreme cases. In the model introduced here a link can be created unilaterally and flow occurs in both directions with some degree of decay, the same in both directions. However, when a link is only supported by one of the two players (such a link is referred to as a "weak" link) the flow through it suffers a greater decay than when it is supported by both players (a "strong" link). That is, strong links work better than weak links, which may be a reasonable assumption in some contexts[4]. When the decay in weak links is maximal (i.e. there is no flow) we have Jackson and Wolinsky's connections model, where only strong links work, whereas when flow in weak links is as good as in strong links we have Bala and Goyal's two-way flow model, where strong links are inefficient and unstable. In contrast to these two extreme cases, it seems reasonable to consider intermediate situations where both types of link work, but strong doubly-supported links work better than weak singly-supported ones. This joint generalization of both

[1]Goyal (2007), Jackson (2008) and Vega-Redondo (2007) are excellent monographs on social and economic networks.

[2]The number of citations of Jackson and Wolinsky's model exceeds 2000, and citation Bala and Goyal's model exceeds 1300.

[3]In a previous paper (Olaizola and Valenciano, 2015) we provide a transitional model that integrates a variation *without decay* of Jackson and Wolinsky's connections model and Bala and Goyal's two-way flow model, also without decay, as extreme cases.

[4]If links are interpreted as not fully reliable attempts to initiate communication, as in Bala and Goyal (2000b), the lower friction through a strong link could then be interpreted as a higher probability that at least one of the communication attempts will be successful, so communication through weak links might be more likely to fail than that through strong links. Haller and Sarangi (2005) consider a situation where doubly-supported links are more reliable than singly-supported ones.

seminal models *allows for a study of the transition from one to the other,* thus providing a "neighborhood" of each model which offers a point of view for testing the robustness of the results for each of the extreme cases.

We first provide *a characterization of efficient architectures which smoothly extends the results relative to efficiency in the seminal papers* (Proposition 1 in Jackson and Wolinsky (1996), and Proposition 5.5 in Bala and Goyal (2000a)). As it turns out, in spite of the richer variety of feasible structures in this model, possibly combining weak and strong links (which complicates considerably the proofs), only the efficient structures in either model, i.e. the complete network of strong links, the complete network of weak links, the all-encompassing star of strong links or that of weak links, and the empty network, are efficient in this more general setting. No mixed structure is efficient for any value of the parameters.

As both a strictly noncooperative point of view and one allowing for pairwise agreements make sense in this joint generalization, we study the model in the crossfire of both approaches. Thus we study Nash, strict Nash and pairwise stability. The notion of pairwise stability needs to be adapted for this more general model, where an individual's potential actions include creating weak links or even making a preexisting weak link strong by making it double. *A natural adaptation of this notion consistent with this situation is provided.* A study of stability from the two points of view of the efficient structures yields an incomplete characterization, which includes as particular cases the results obtained separately in either model (Proposition 2 in Jackson and Wolinsky (1996), and Proposition 5.3 in Bala and Goyal (2000a)).

Thus, in both respects, i.e. efficiency and stability, transition from one model to the other turns out to be perfectly smooth, so that both models are robust and compatible from the point of view provided by this more general model.

The rest of the paper is organized as follows. Section 2 introduces notation and terminology. Section 3 reviews the strategic models of network formation of Jackson and Wolinsky (1996) and Bala and Goyal (2000a). A model that bridges the gap between these two is presented in Section 4.1, and the pairwise stability notion is adapted to the new setting in 4.2. Section 5.1 addresses the question of efficiency for the intermediate model. In Section 5.2 Nash stable, Nash strictly stable and pairwise stable structures are studied, and Section 6 summarizes the main conclusions and points out some lines of further research.

2 Preliminaries[5]

A *directed N-graph* is a pair (N, Γ), where $N = \{1, 2, ..., n\}$ is a finite set with $n \geq 3$ whose elements are called *nodes*, and Γ is a subset of $N \times N$, whose elements are called *links*. When both (i, j), and (j, i) are in Γ, we say that i and j are connected by a

[5] This brief section is similar to Section 2 in Olaizola and Valenciano (2015), as the notation relative to graphs is the same in both papers.

strong link, while if only one of them is there we say that they are connected by a *weak link*. If $M \subseteq N$, the *M-subgraph* of (N, Γ) is the M-graph $(M, \Gamma\mid_M)$ with

$$\Gamma\mid_M := \{(i,j) \in M \times M : (i,j) \in \Gamma\}.$$

Alternatively, a graph Γ can be specified by a map $g_\Gamma : N \times N \to \{0,1\}$,

$$g_\Gamma(i,j) := \begin{cases} 1, & \text{if } (i,j) \in \Gamma \\ 0, & \text{if } (i,j) \notin \Gamma. \end{cases}$$

When we specify a graph Γ by a map g, we denote $g_{ij} := g(i,j)$, and if $g_{ij} = 1$ link (i,j) is referred to as "link ij in g", and we write $ij \in g$. Note that for $M \subseteq N$, subgraph $\Gamma\mid_M$ is specified by $g\mid_{M\times M}$ but, abusing the notation, this subgraph is denoted by $g\mid_M$. The *empty graph* is denoted by g^e (i.e. $g^e(i,j) = 0$, for all i,j).

If $g_{ij} = 1$ in a graph g, $g - ij$ denotes the graph that results from replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g; and if $g_{ij} = 0$, $g + ij$ denotes the graph that results from replacing $g_{ij} = 0$ by $g_{ij} = 1$. Similarly, if $g_{ij} = g_{ji} = 1$, $g - \overline{ij} = (g - ij) - ji$, and if $g_{ij} = g_{ji} = 0$, $g + \overline{ij} = (g + ij) + ji$. An *isolated* node in a graph g is a node that is not involved in any link, that is, a node i s.t. for all $j \neq i$, $g_{ij} = g_{ji} = 0$. A node is *peripheral* in a graph g if it is involved in a single link (weak or strong).

Given a graph g, a *path* of *length k from j to i* in g is a sequence of $k+1$ distinct nodes $j_0, j_1, ..., j_k$, s.t. $j = j_0$, $i = j_k$, and for all $l = 1, ..., k$, $g_{j_{l-1}j_l} = 1$ or $g_{j_l j_{l-1}} = 1$. A graph g is *acyclic* or *contains no cycles* if there is no sequence of k $(k > 2)$ distinct nodes, $i_1, ..., i_k$, s.t. for all $l = 1, ..., k-1$, $g_{i_l i_{l+1}} = 1$ or $g_{i_{l+1} i_l} = 1$, and $g_{i_1 i_k} = 1$ or $g_{i_k i_1} = 1$.

Definition 1 *Given a graph g, and $K \subseteq N$, the subgraph $g\mid_K$ is said to be:*
(i) A weak component of g if for any two nodes $i,j \in K$ $(i \neq j)$ there is a path from j to i in g, and no subset of N strictly containing K meets this condition.
(ii) A strong component if for any two nodes $i,j \in K$ $(i \neq j)$ there is a path of strong links from j to i in g, and no subset of N strictly containing K meets this condition.

When a component in either sense consists of a single node we say that it is a *trivial component*. In both senses, an *isolated* node, i.e. a node that is not involved in any link, is a trivial component. The *size* of a component is the number of nodes from which it is formed. Based on these definitions we have two different notions of *connectedness*. We say that a graph g is *weakly (strongly) connected*[6] if g is the unique weak (strong) component of g. Note that strong connectedness implies weak connectedness. A weak (strong) component $g\mid_K$ of a graph g is *minimal* if for all $i,j \in K$ s.t. $g_{ij} = 1$, the

[6]Note that the sense in which the term "strongly connected" is used here differs from its usual meaning in the literature, where a directed network is said to be strongly connected when for any two distinct nodes there is an oriented path from one to the other. In our context, a clear distinction between weak and strong links invites the use of the term in the sense in which we use it here.

number of weak (strong) components of g is smaller than the number of weak (strong) components of $g - ij$. When g is clear from the context, we refer to a component $g\mid_K$ as component K.

A graph is *minimally* weakly (strongly) connected if it is weakly (strongly) connected and minimal. In both cases, a minimally connected graph is a *tree* (of weak links in one case, of strong links in the other), but, in principle, any node in such trees can be seen as the *root*, i.e. a reference node from which there is a unique path connecting it with any other. Note that a weakly connected graph *with no cycles* is a tree in general formed by weak and strong links, and in general neither minimally weakly nor strongly connected.

Given a graph g, the following notation is also used:

$N^d(i;g) := \{j \in N : g_{ij} = 1\}$ (i.e. set of nodes with which i supports a link),
$N^e(i;g) := \{j \in N : g_{ji} = 1\}$ (i.e. set of nodes which support a link with i),
$N^o(i;g) := N^d(i;g) \cup N^e(i;g)$ (i.e. set of nodes involved in a link with i).

The set of nodes connected with i by a path is denoted by $N(i;g)$. Note that none of these sets contains i. Their cardinalities are denoted by $\mu_i^d(g) := \#N^d(i;g)$, $\mu_i^e(g) := \#N^e(i;g)$, $\mu_i^o(g) := \#N^o(i;g)$, and $\mu_i(g) := \#N(i;g)$.

The *distance* between two nodes i, j $(i \neq j)$, denoted by $d(i, j; g)$, is the length of the shortest path connecting them. When there is no path connecting two nodes the distance between them is said to be ∞.

The graph architectures explained hereafter play a role in what follows. A *line* is a graph consisting of a sequence of distinct nodes connected by links where no other links exist. A *star* (*all-encompassing star*) is a graph where one node is involved in links with some (all) other players, and no other links exist. A *mixed* star is a star formed by weak links and strong links. A mixed star consisting of k_s strong links and k_w weak links is denoted by S_{k_s,k_w}. A *wheel* consists of a sequence of nodes connected by links in which the first and the last in that sequence are also linked, and no other links exist. A *complete* (*weak-complete*, *strong-complete*) graph is one where any two nodes are involved in a link (weak link, strong link).

3 Unilateral vs. bilateral link-formation

We consider situations where individuals may initiate or support *links* with other individuals under certain assumptions, thus creating a network formalized as a graph. We assume that at each node $i \in N$ there is an agent identified by label i and referred to as *player*[7] i. Each player i may invest in links with other players[8]. A map $g_i : N\backslash\{i\} \to \{0, 1\}$ describes the links in which player i invests. We write $g_{ij} := g_i(j)$,

[7] In order to avoid biased language, we often refer to players by the more neutral term "nodes".

[8] This is similar to Myerson's (1977) model, where all players simultaneously announce the set of players with whom they wish form links. But while in Myerson's model links are formed if and only if they were proposed by both, we consider a different scenario here.

and $g_{ij} = 1$ ($g_{ij} = 0$) means that i invests (does not invest) in a link with j. Thus, vector $g_i = (g_{ij})_{j \in N \backslash \{i\}} \in \{0,1\}^{N \backslash \{i\}}$ specifies the links in which i invests and is referred to as a *strategy* of player i. $G_i := \{0,1\}^{N \backslash \{i\}}$ denotes the set of i's strategies and $G_N = G_1 \times G_2 \times ... \times G_n$ the set of *strategy profiles*. A strategy profile $g \in G_N$ univocally determines a *graph* (N, Γ_g) of links invested in, where $\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\}$. Given a strategy profile $g \in G_N$ and $i \in N$, g_{-i} denotes the $N \backslash \{i\}$ strategy profile that results by eliminating g_i in g, i.e. all links in which player i invests [9], and (g_{-i}, g_i'), where $g_i' \in G_i$, denotes the strategy profile that results from replacing g_i by g_i' in g.

Let g be a strategy profile representing the links invested in by each player. The following is generally assumed:

1. Investment by player i in a link with player j entails a *cost* $c_{ij} > 0$ for all $j \neq i$.

2. The player at node j has a particular type of information or other good[10] of *value* v_{ij} for player i.

3. If $\mathbf{v} = (v_{ij})_{i,j \in N}$ is the matrix of values, $\mathbf{c} = (c_{ij})_{i,j \in N}$ is the matrix of cost (assuming[11] $c_{ii} = v_{ii} = 0$), and g is the strategy profile, the payoff of a player is given by a function

$$\Pi_i(g) = I_i(g^*, \mathbf{v}) - c_i(g, \mathbf{c}), \tag{1}$$

where $I_i(g^*, \mathbf{v})$ is the *information* received by i through the actual network g^* under strategy profile g, and $c_i(g, \mathbf{c}) = \sum_{j \in N^d(i;g)} c_{ij}$ the *cost* incurred by i.

Under different assumptions, different models specify g^* and I_i differently. In all cases a game in strategic form is specified: $(G_N, \{\Pi_i\}_{i \in N})$. In Jackson and Wolinsky (1996) only doubly supported links actually form, i.e.

$$g^*_{ij} = \min\{g_{ij}, g_{ji}\}. \tag{2}$$

In Bala and Goyal (2000a) links are created unilaterally, i.e.

$$g^*_{ij} = \max\{g_{ij}, g_{ji}\}. \tag{3}$$

In both models the information flow through a link suffers some degree of decay, with δ ($0 < \delta < 1$) being the fraction of the value of information at one node that reaches another node through a link[12]. Thus, for the right instantiation of g^*, i.e. (2) or (3), in both cases we have:

$$I_i(g^*, \mathbf{v}) = \sum_{j \in N(i;g)} v_{ij} \; \delta^{d(i,j;g^*)}. \tag{4}$$

The point of this work is to provide and study a model that bridges the gap between these two.

[9] Note that if $ji \in g$, then $ji \in g_{-i}$.

[10] Although other interpretations are possible, in general, we give preference to the interpretation in terms of information.

[11] Only to make it possible to call $\mathbf{c}$ and $\mathbf{v}$ matrices. Nevertheless, in practice c_{ii} and v_{ii} play no role. Note also that by definition g_{ii} remains undefined for any strategy.

[12] Bala and Goyal (2000a) also consider the case of no decay, i.e. $\delta = 1$.

4 Bridging the gap

4.1 A merging of two models

In both Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000a) two-way flow model with decay, a level of friction in the flow through a link is assumed, so that only a fraction of the information at one node reaches the other through that link. In both models the flow is assumed to be *homogeneous* (i.e. the same through all actual links). In order to bridge the gap between these two models, making a transition from one to the other possible, we introduce a very simple form of *endogenous* heterogeneity[13] relative to the level of decay. We consider a model where *information flows through all links with some degree of decay, the same in both directions, but friction is smaller through strong links.*

More precisely, we consider the following model. Let δ $(0 \leq \delta \leq 1)$ be the fraction of the value of information at one node that reaches another node through a *strong* link, and let α $(0 \leq \alpha \leq \delta \leq 1)$ be the fraction of the value of information at one node that reaches another node through a *weak* link. For a graph g representing a strategy profile and a pair of nodes $i \neq j$, let $\mathcal{P}_{ij}(g)$ denote the set of paths in g from i to j. For $p \in \mathcal{P}_{ij}(g)$, let $\ell(p)$ denote the length of p and $\omega(p)$ the number of weak links in p. Then i values information originating from j that arrives via p by

$$I_i(p, \mathbf{v}) = v_{ij}\delta^{\ell(p)-\omega(p)}\alpha^{\omega(p)}.$$

If information is routed via the best possible route from j to i, then i's valuation of information originating from j is

$$I_{ij}(g, \mathbf{v}) = \max_{p \in \mathcal{P}_{ij}(g)} I_i(p, \mathbf{v})$$

and i's overall benefit from g (ignoring costs) is

$$I_i(g, \mathbf{v}) = \sum_{j \in N(i;g)} I_{ij}(g, \mathbf{v}).$$

Thus (1) becomes (note that now the actual network is the strategy profile, i.e. $g^* = g$):

$$\Pi_i(g) = I_i(g, \mathbf{v}) - c_i(g, \mathbf{c}) = \sum_{j \in N(i;g)} v_{ij} \max_{p \in \mathcal{P}_{ij}(g)} \delta^{\ell(p)-\omega(p)}\alpha^{\omega(p)} - c_i(g, \mathbf{c}). \qquad (5)$$

Observe that:

• $0 = \alpha < \delta < 1$ yields Jackson and Wolinsky's original connections model: information flows only through links in which both players invest.

[13] See Bloch and Dutta (2009) for a model with endogenous heterogeneity where players may invest their endowments across links.

• $0 < \alpha = \delta < 1$ yields Bala and Goyal's two-way flow model with decay: information flows in both directions with the same decay through weak and strong links.

Thus, the intermediate situations, i.e. $0 \leq \alpha \leq \delta < 1$ *yield a bridge-model between Jackson and Wolinsky's original connections model* ($\alpha = 0$) *and Bala and Goyal's two-way flow connections model with decay* ($\alpha = \delta$).

4.2 Stability notions

From a conceptual point of view, the first interesting issue raised by this "intermediate" model is how to adapt the different notions of stability used in each of the two benchmark models to this "mixed" situation. In Bala and Goyal's purely noncooperative model Nash and strict Nash equilibrium are the natural stability notions. In Jackson and Wolinsky's model, stability analysis is based on the notion of "pairwise" stability. In this transitional model a noncooperative approach based on Nash and strict Nash equilibrium makes sense, but adapting the pairwise stability notion (Jackson and Wolinsky, 1996) is more delicate. The concept introduced by Jackson and Wolinsky, in a context where only strong links make sense and actually form, consists of two requirements: (i) no player gains by severing a link ("link deletion proofness"); and (ii) no two players who are not linked have an incentive to create a strong link ("link addition proofness"). Part (i) is the stability requirement for the noncooperative dimension of Jackson and Wolinsky's model, but in the current transitional model individual players have other options, given that weak links can be created unilaterally, and so can strong links by making double an existing weak link. Thus we add a requirement of "link creation proofness" to that of "link deletion proofness", and the simplest way of "strategy-proofness" w.r.t. the modification which results from combining the two: no player gains by changing his/her investment from one link into a new one.

Thus we consider the following three forms of stability.

Definition 2 *A strategy profile g is*:
(*i*) *A Nash equilibrium if* $\Pi_i(g_{-i}, g'_i) \leq \Pi_i(g)$, *for all i and all* $g'_i \in G_i$.
(*ii*) *A strict Nash equilibrium if* $\Pi_i(g_{-i}, g'_i) < \Pi_i(g)$, *for all i and all* $g'_i \in G_i$ ($g'_i \neq g_i$).
(*iii*) *Pairwise stable if*:
- *for all* $ij \in g$, $\Pi_i(g - ij) \leq \Pi_i(g)$,
- *for all* $ij \notin g$, $\Pi_i(g + ij) \leq \Pi_i(g)$,
- *for all* $ij \in g$, $ij' \notin g$, $\Pi_i((g - ij) + ij') \leq \Pi_i(g)$, *and*
- *for all* i, j ($i \neq j$) *s.t.* $g_{ij} = g_{ji} = 0$, *if* $\Pi_i(g + \overline{\overline{ij}}) > \Pi_i(g)$ *then* $\Pi_j(g + \overline{\overline{ij}}) < \Pi_j(g)$.

5 The transition

In what follows we assume homogeneity in costs and values across players, i.e. we assume $v_{ij} = 1$ and $c_{ij} = c > 0$, for all i, j. Consequently, we drop $\mathbf{v}$ and $\mathbf{c}$ in (5), and write i's payoff as:

$$\Pi_i(g) = I_i(g) - c_i(g) = \sum_{j \in N(i;g)} \max_{p \in \mathcal{P}_{ij}(g)} \delta^{\ell(p)-\omega(p)} \alpha^{\omega(p)} - c\mu_i^d(g) \tag{6}$$

We first address the question of efficiency and then that of stability.

5.1 Efficiency

A network is said to be *efficient* for a particular configuration of values of the parameters if it maximizes the aggregate payoff, referred to as the *value* of the network. When the value of network g, denoted by $v(g)$, is greater than or equal to that of network g' we say that g *dominates* g'. Both Jackson and Wolinsky (1996) and Bala and Goyal (2000a) provide a characterization of efficient networks in their settings. Propositions 1 and 2 present their results. The statements are adapted to the terminology used here.

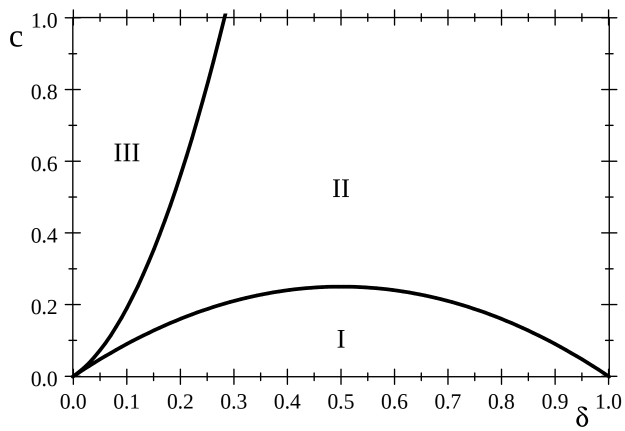

Figure 1: Efficiency: $\alpha = 0$ (Jackson and Wolinsky), $n = 20$

Proposition 1 (*Proposition* 1*, Jackson and Wolinsky,* 1996) *In Jackson and Wolinsky's connections model, the unique efficient network is*:
(*i*) *The strong-complete graph if* $c < \delta - \delta^2$ (*Region I in Figure* 1).
(*ii*) *All-encompassing stars of strong links if* $\delta - \delta^2 < c < \delta + (n-2)\,\delta^2/2$ (*Region II in Figure* 1).
(*iii*) *The empty network if* $\delta + (n-2)\,\delta^2/2 < c$ (*Region III in Figure* 1).

Figure 1 shows the regions where these architectures are efficient. The cost, c, is represented on the vertical axis, and the fraction of the unit of information at one node that reaches another one through a link, δ, on the horizontal axis. In order to keep the different regions of values of the parameters bounded, only the part of the picture for $c \leq 1$ is represented in the figures, although *no upper bound is imposed on c*.

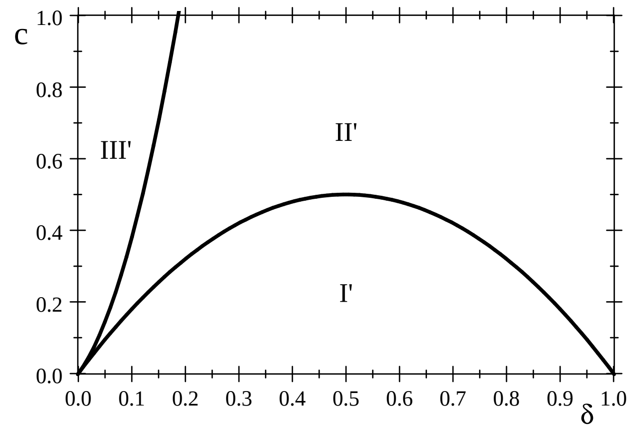

Figure 2: Efficiency: $\alpha = \delta$ (Bala and Goyal), $n = 20$

Proposition 2 (*Proposition* 5.5*, Bala and Goyal,* 2000*a*) *In Bala and Goyal's two-way flow model with decay, the unique efficient network is*:
(*i*) *The weak-complete graph if* $c < 2\left(\delta - \delta^2\right)$ (*Region I' in Figure* 2).
(*ii*) *All-encompassing stars of weak links if* $2\left(\delta - \delta^2\right) < c < 2\delta + (n-2)\,\delta^2$ (*Region II' in Figure* 2).
(*iii*) *The empty network if* $2\delta + (n-2)\,\delta^2 < c$ (*Region III' in Figure* 2).

As we presently show, the only efficient architectures in our setting are, depending on the values of the parameters (α, δ, c and n): the strong-complete, the weak-complete, the star of strong links, the star of weak links and the empty network. In order to have a complete characterization, the region where each of them is efficient must be determined. This is established in Proposition 3, where only the region where the strong-complete network is efficient, and part of the region where the weak-complete network is efficient are directly established. The rest is the result of several lemmas, which in a patchwork-like way cover the whole region where the parameters vary. In spite of the complexity of this piecewise study, the strategy of the proof is easy to understand. The basic idea is, as in the seminal papers, to compare the value of an arbitrary component with that of certain "dominant" structures. Nevertheless, the possibility of weak and strong links makes this comparison more complicated. In different regions of values of the parameters, it is proved that a component of a network is dominated by a star with the same number of nodes (Lemmas 1, 2, 3 and 4). Then it is shown that a mixed star is dominated by a star with the same number of links (either all strong or all weak) (Lemma 5). To conclude the proof, a region containing the boundary between the regions where the star of strong links and the weak-complete graph are (later proved to be) efficient remains to be studied. This requires the use of different dominant structures, namely, two interesting sorts of "hybrid" structure (Figure 3) between a star of strong links and a weak-complete network (Lemma 6).

Finally, such hybrid structures are proved to be dominated in that region either by a weak-complete network or by a star of strong links with the same number of nodes (Lemma 7).

Lemma 1 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$ *and* $c > \max\{\delta - \delta^2, 2\left(\alpha - \delta^2\right)\}$*, then the maximal value of a weak component containing* m *nodes and* $m-1$ *or more strong links is only reached by a star with* $m-1$ *strong links.*

Proof. Let K be a weak component containing m nodes and $k_s \geq m-1$ strong links and $k_w \geq 0$ weak links. Without loss of generality, it can be assumed that no link is superfluous. Then,

$$v(K) = k_s\left(2\delta - 2c\right) + k_w\left(2\alpha - c\right) + p(\alpha, \delta),$$

where $p(\alpha, \delta)$ is a polynomial on α and δ with integer positive coefficients (summing up to $\max\{m\left(m-1\right) - 2\left(k_s + k_w\right), 0\}$, i.e. twice the number of pairs of nodes non-directly connected) multiplying monomials of the form $\alpha^q \delta^r$ with $q + r \geq 2$. As $\alpha \leq \delta$, the maximal value of this polynomial is obtained when $p(\alpha, \delta) = (m\left(m-1\right) - 2\left(k_s + k_w\right))\delta^2$, that is

$$v(K) \leq k_s\left(2\delta - 2c\right) + k_w\left(2\alpha - c\right) + \left(m\left(m-1\right) - 2\left(k_s + k_w\right)\right)\delta^2,$$

while the value of a star of $m-1$ strong links with m nodes is

$$v(S_{m-1,0}) = (m-1)\left(2\delta - 2c\right) + (m-1)\left(m-2\right)\delta^2.$$

Thus, the difference is

$$v(S_{m-1,0}) - v(K) \geq \left(m - 1 - k_s\right)\left(2\delta - 2c - 2\delta^2\right) + k_w\left(2\delta^2 - 2\alpha + c\right) \geq 0,$$

given that $m - 1 - k_s \leq 0$, $2\delta - 2c - 2\delta^2 < 0$ and $2\delta^2 - 2\alpha + c > 0$. Moreover, it is 0 only for $k_s = m-1$ and $k_w = 0$. Finally, a component with $k_s = m-1$ and $k_w = 0$ is necessarily minimally strongly connected, and the maximal value of a minimally strongly connected component is only reached by stars of strong links. ■

Lemma 2 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$ *and* $c > 2\alpha$*, then a weak component containing* m *nodes and fewer than* $m-1$ *strong links is dominated by a mixed star with the same number of strong links.*

Proof. Let K be a weak component containing m nodes and $k_s < m-1$ strong links and $k_w \geq m - 1 - k_s > 0$ weak links. Without loss of generality, it can be assumed that no link is superfluous. Thus, reasoning as in the preceding lemma, we have:

$$v(K) \leq k_s\left(2\delta - 2c\right) + k_w\left(2\alpha - c\right) + k_s\left(k_s - 1\right)\delta^2$$
$$+ k_s\left(m - 1 - k_s\right)2\alpha\delta + \left(m - 1 - k_s\right)\left(m - 2 - k_s\right)\alpha^2,$$

while the value of a star with k_s strong links and $m-1-k_s$ weak links is

$$v\left(S_{k_s,m-1-k_s}\right) = k_s\left(2\delta - 2c\right) + \left(m-1-k_s\right)\left(2\alpha - c\right)$$
$$+k_s\left(k_s-1\right)\delta^2 + k_s\left(m-1-k_s\right)2\alpha\delta + \left(m-1-k_s\right)\left(m-2-k_s\right)\alpha^2.$$

Thus, the difference is

$$v\left(S_{k_s,m-1-k_s}\right) - v(K) = \left(c - 2\alpha\right)\left(k_s + k_w - (m-1)\right) \geq 0,$$

given that $k_s + k_w \geq m-1$ and $c > 2\alpha$. And it is 0 only for $k_s + k_w = m-1$. Finally, the maximal value of a component with $m-1$ links is only reached by stars. ■

The next lemma establishes the same result for $2\left(\alpha - \alpha^2\right) < c < 2\alpha$.

Lemma 3 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$ *and* $2\left(\alpha - \alpha^2\right) < c < 2\alpha$, *then a weak component containing* m *nodes and fewer than* $m-1$ *strong links is dominated by a mixed star with the same number of strong links.*

Proof. Let K be a weak component containing m nodes, $k_s < m-1$ strong links and $k_w \geq m-1-k_s > 0$ weak links. Without loss of generality, it can be assumed that no link is superfluous. The maximal value of the component here requires a more detailed discussion than in the case $c > 2\alpha$ addressed in the preceding lemma. Thus we have

$$v(K) \leq k_s\left(2\delta - 2c\right) + k_w\left(2\alpha - c\right) + A\delta^2 + B\alpha\delta + C\alpha^2,$$

where

$$A = \min\{k_s(k_s-1), m(m-1) - 2k_s - 2k_w\}.$$

Two cases must be considered depending on which of these numbers is smaller:

1^{st} *case*: $A = m(m-1) - 2k_s - 2k_w$. In this case $B = C = 0$, and we have

$$v(K) \leq k_s\left(2\delta - 2c\right) + k_w\left(2\alpha - c\right) + (m(m-1) - 2k_s - 2k_w)\delta^2,$$

while the value of a star with k_s strong links and $m-1-k_s$ weak links is

$$v\left(S_{k_s,m-1-k_s}\right) = k_s\left(2\delta - 2c\right) + \left(m-1-k_s\right)\left(2\alpha - c\right)$$
$$+k_s\left(k_s-1\right)\delta^2 + k_s\left(m-1-k_s\right)2\alpha\delta + \left(m-1-k_s\right)\left(m-2-k_s\right)\alpha^2.$$

Thus, the difference is

$$v\left(S_{k_s,m-1-k_s}\right) - v(K)$$
$$\geq \left(m-1-k_s-k_w\right)\left(2\alpha - c\right) + (k_s(k_s-1) - m(m-1) + 2k_s + 2k_w)\delta^2$$
$$+k_s\left(m-1-k_s\right)2\alpha\delta + \left(m-1-k_s\right)\left(m-2-k_s\right)\alpha^2$$
$$= a\left(2\alpha - c\right) + b\delta^2 + d\alpha\delta + e\alpha^2,$$

where a, b, d and e denote the coefficients in the last expression. Note that $a \leq 0$, while b, d and e are ≥ 0. As $2\alpha - c < 2\alpha^2$, by replacing $2\alpha - c$ by $2\alpha^2$ in the last expression and taking into account that $\alpha \leq \delta$ we have

$$\begin{aligned} v\left(S_{k_s,m-1-k_s}\right) - v(K) &\geq a2\alpha^2 + b\delta^2 + d\alpha\delta + e\alpha^2 \\ &\geq a2\alpha^2 + b\alpha^2 + d\alpha^2 + e\alpha^2 = (2a + b + d + e)\alpha^2 \end{aligned}$$

Therefore, if $2a + b + d + e \geq 0$ the proof is concluded in the 1^{st} case, and summing up these coefficients we have $2a + b + d + e = 0$.

2^{nd} *case*: $A = k_s(k_s - 1)$. In this case $k_s(k_s - 1)/2$ is the maximal number of non-directly linked pairs that can receive δ^2 from each other. Now

$$B = \min\{2k_s(m - 1 - k_s), m(m - 1) - 2k_s - 2k_w - k_s(k_s - 1)\}.$$

Thus, we again have two cases:

Case 2.1: $B = m(m - 1) - 2k_s - 2k_w - k_s(k_s - 1)$. In this case $C = 0$, and

$$\begin{aligned} v(K) \leq\ & k_s\left(2\delta - 2c\right) + k_w\left(2\alpha - c\right) + k_s(k_s - 1)\delta^2 \\ & +(m(m - 1) - 2k_s - 2k_w - k_s(k_s - 1))\alpha\delta. \end{aligned}$$

Thus, subtracting this value from that of a star with k_s strong links and $m - 1 - k_s$ weak links the difference is

$$\begin{aligned} v\left(S_{k_s,m-1-k_s}\right) - v(K) &\geq \left(m - 1 - k_s - k_w\right)\left(2\alpha - c\right) \\ &+(2k_s(m - 1 - k_s) - (m(m - 1) - 2k_s - 2k_w - k_s(k_s - 1))\alpha\delta \\ &+ \left(m - 1 - k_s\right)\left(m - 2 - k_s\right)\alpha^2 = a\left(2\alpha - c\right) + b\alpha\delta + d\alpha^2, \end{aligned}$$

and proceeding just as in the first case we similarly conclude that $v\left(S_{k_s,m-1-k_s}\right) - v(K) \geq 0$.

Case 2.2: $B = 2k_s(m - 1 - k_s)$. In this case

$$C = m(m - 1) - 2k_s - 2k_w - k_s(k_s - 1) - 2k_s(m - 1 - k_s), \text{ and}$$

$$\begin{aligned} v\left(S_{k_s,m-1-k_s}\right) - v(K) &\geq \left(m - 1 - k_s - k_w\right)\left(2\alpha - c\right) \\ &+(k_s + k_w - (m - 1))2\alpha^2 = a\left(2\alpha - c\right) + b\alpha^2, \end{aligned}$$

and proceeding again as before we conclude that $v\left(S_{k_s,m-1-k_s}\right) - v(K) \geq 0$. ■

Lemma 4 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$, *and* $c < 2\left(\delta - \alpha\right)$, *then:* (*i*) *in an non-empty efficient network all links are strong;* (*ii*) *if in addition* $c > \delta - \delta^2$, *then a weak component is dominated by a star of strong links.*

Proof. (i) Let g be a nonempty efficient network, and assume $ij \in g$ and $ji \notin g$, then the contribution of i's (j's) unit of value to j's (i's) payoff is α, otherwise ij would be superfluous, but then, as $c < 2(\delta - \alpha)$, by making ij double the sum of the payoffs of i and j would increase, and no other player's payoff would decrease, which contradicts g's efficiency.

(ii) Let K be a weak component with no superfluous links. By (i), all its links must be strong. But then by Lemma 1 it is dominated by a star of strong links (note that $c > 2(\alpha - \delta^2)$ follows easily from $c < 2(\delta - \alpha)$ and $c > \delta - \delta^2$, and Lemma 1 can be applied.) ∎

Lemmas 1, 2, 3 and 4 establish that, for different configurations of values of the parameters, any component is dominated by a star, possibly mixed. The following lemma shows that mixed stars are always dominated by stars containing only one type of link.

Lemma 5 *If the payoff function is given by* (6) *with* $0 \le \alpha \le \delta < 1$*, a star containing both strong and weak links is strictly dominated either by a star with the same number of links all of which are strong or by a star with the same number of links all of which are weak.*

Proof. Let S_{k_s,k_w} be a star connecting m nodes with $k_s > 0$ strong links and $k_w = m - 1 - k_s > 0$ weak links. Its value is given by

$$v(S_{k_s,k_w}) = k_s(2\delta - 2c) + k_w(2\alpha - c) + k_s(k_s - 1)\delta^2 + 2k_sk_w\alpha\delta + k_w(k_w - 1)\alpha^2.$$

By making double a weak link, S_{k_s+1,k_w-1} results, and

$$v(S_{k_s+1,k_w-1}) = (k_s + 1)(2\delta - 2c) + (k_w - 1)(2\alpha - c) \\ + (k_s + 1)k_s\delta^2 + 2(k_s + 1)(k_w - 1)\alpha\delta + (k_w - 1)(k_w - 2)\alpha^2.$$

Thus, as $k_w = m - 1 - k_s$, $v(S_{k_s+1,k_w-1}) - v(S_{k_s,k_w}) =$

$$(2\delta - 2c) - (2\alpha - c) + 2(m - 2)\alpha(\delta - \alpha) + 2k_s(\delta - \alpha)^2. \tag{7}$$

Note that if this number is > 0, the greater k_s is the greater this number will be, and consequently the value of a star of strong links connecting m nodes is greater than that of S_{k_s,k_w}.

By making a double link weak, S_{k_s-1,k_w+1} results, and

$$v(S_{k_s-1,k_w+1}) = (k_s - 1)(2\delta - 2c) + (k_w + 1)(2\alpha - c) \\ + (k_s - 1)(k_s - 2)\delta^2 + 2(k_s - 1)(k_w + 1)\alpha\delta + (k_w + 1)k_w\alpha^2.$$

Thus, as $k_w = m - 1 - k_s$, $v(S_{k_s-1,k_w+1}) - v(S_{k_s,k_w}) =$

$$-(2\delta - 2c) + (2\alpha - c) + 2\left(\delta^2 - m\alpha\delta + (m - 1)\alpha^2\right) - 2k_s(\delta - \alpha)^2. \tag{8}$$

If this number is > 0, the smaller k_s is the greater this number will be and consequently the value of a star of weak links is greater than that of S_{k_s,k_w}.

It only remains to show that the value necessarily increases by either making a weak link double or making a strong one weak, that is, either (7) or (8) is greater than 0. Write $X = (2\delta - 2c) - (2\alpha - c)$, $Y = 2(m-2)\alpha(\delta-\alpha) + 2k_s(\delta-\alpha)^2$ and $Y' = 2\left(\delta^2 - m\alpha\delta + (m-1)\alpha^2\right) - 2k_s(\delta-\alpha)^2$. Thus we prove that necessarily either

$$v\left(S_{k_s+1,k_w-1}\right) - v\left(S_{k_s,k_w}\right) = X + Y > 0 \quad \text{or} \quad V\left(S_{k_s-1,k_w+1}\right) - v\left(S_{k_s,k_w}\right) = -X + Y' > 0.$$

Assume $X + Y \leq 0$, i.e. $X \leq -Y$, then we prove that $-X + Y' > 0$, i.e. $X < Y'$. For this it suffices to show that $-Y < Y'$, i.e. $Y + Y' > 0$. In fact we have $Y + Y' = 2(\delta-\alpha)^2 > 0$. ■

Above $c = \delta - \delta^2$ and $c = 2\left(\alpha - \delta^2\right)$ the preceding lemmas show the domination of stars, either of weak links or of strong links, for all the configurations of values of the parameters *except* the region considered in the next two lemmas, where two sorts of "hybrid" structure, somewhere between stars of strong links and weak-complete (see Figure 3)[14], serve as a term of comparison.

Figure 3: "Hybrid" structures

Lemma 6 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$ *and*

$$\max\{2(\delta-\alpha), 2\left(\alpha - \delta^2\right)\} < c < 2\left(\alpha - \alpha^2\right),$$

then a weak component containing m *nodes and fewer than* $m-1$ *strong links is dominated by a network consisting of a star with the same number of strong links and*: (*i*) *if* $c > 2(\alpha - \alpha\delta)$, *with the rest of the nodes along with the center of the star forming a complete subnetwork of weak links*; (*ii*) *if* $c < 2(\alpha - \alpha\delta)$, *any other pairs, except those of peripheral nodes of the star, are connected by weak links*; (*iii*) *in particular, in both cases, if the component contains no strong links it is dominated by the weak-complete graph.*

Proof. (*i*) Let K be a weak component, containing m nodes, $k_s < m-1$ strong links and $k_w \geq m - 1 - k_s > 0$ weak links. Without loss of generality, it can be assumed that no link is superfluous. Thus we have

$$v(K) \leq k_s(2\delta - 2c) + k_w(2\alpha - c) + A\delta^2 + B\alpha\delta,$$

[14] A strong link between two nodes is represented by a thick line connecting them, while a weak link is represented by a thin line between them that only touches the node that supports it.

where

$$A = \min\{k_s(k_s - 1), m(m-1) - 2k_s - 2k_w\}.$$

Let $g^*_{k_s}$ be a m-node network consisting of a star with k_s strong links and the rest of the nodes along with the center of the star forming a complete subnetwork of weak links (see Figure 3-a). Then

$$v(g^*_{k_s}) = k_s\,(2\delta - 2c) + k'_w\,(2\alpha - c) + A'\delta^2 + B'\alpha\delta, \tag{9}$$

where

$$k'_w = (m - k_s)(m - k_s - 1)/2, \quad A' = k_s(k_s - 1) \quad \text{and} \quad B' = 2k_s(m - k_s - 1). \tag{10}$$

Now, depending on the value of A, we have two cases:

1^{st} *case*: $A = m(m-1) - 2k_s - 2k_w$. In this case $B = 0$ and $A \leq A'$. Then we have

$$v(g^*_{k_s}) - v(K) \geq (k'_w - k_w)\,(2\alpha - c) + (A' - A)\delta^2 + B'\alpha\delta.$$

As $k'_w + A'/2 + B'/2 = k_w + A/2$ and $A \leq A'$, $k'_w \leq k_w$. And as $2\alpha - c < 2\alpha\delta$ and $k'_w - k_w + B'/2 = A/2 - A'/2$, and $2\alpha\delta < 2\delta^2$ we have:

$$\begin{aligned} v(g^*_{k_s}) - v(K) &\geq (A' - A)\delta^2 + (k'_w - k_w + B'/2)2\alpha\delta \\ &\geq (k'_w - k_w + B'/2 + A'/2 - A/2)2\alpha\delta = 0. \end{aligned}$$

2^{nd} *case*: $A = k_s(k_s - 1)$. Then

$$B = \min\{2k_s(m - 1 - k_s), m(m-1) - 2k_s - 2k_w - k_s(k_s - 1)\}.$$

In both cases $B' \geq B \geq 0$, and $k_w \geq k'_w \geq 0$, with $k'_w + B'/2 = k_w + B/2$. And as $c < 2\,(\alpha - \alpha\delta)$, we have

$$\begin{aligned} v(g^*_{k_s}) - v(K) &\geq (k'_w - k_w)\,(2\alpha - c) + (B' - B)2\alpha\delta \\ &\geq (k'_w - k_w)(2\alpha - c - 2\alpha\delta) \geq 0. \end{aligned}$$

(*ii*) Let K be a weak component as in (*i*). Let $g^{**}_{k_s}$ be a m-node network consisting of a star with k_s strong links and any other pairs of nodes, except those of peripheral nodes of the star, connected by weak links (see Figure 3-b). Thus

$$v(g^{**}_{k_s}) = k_s\,(2\delta - 2c) + k'_w\,(2\alpha - c) + A'\delta^2, \tag{11}$$

where

$$A' = k_s(k_s - 1) \quad \text{and} \quad k'_w = m(m-1)/2 - k_s - k_s(k_s - 1)/2. \tag{12}$$

Two cases must be considered depending on the value of A:

1^{st} *case*: $A = m(m-1) - 2k_s - 2k_w$. In this case $B = 0$ and $A \leq A'$. Thus we have

$$v(K) \leq k_s\,(2\delta - 2c) + k_w\,(2\alpha - c) + A\delta^2,$$

and consequently

$$v(g_{k_s}^{**}) - v(K) \geq (k'_w - k_w)(2\alpha - c) + (A' - A)\delta^2.$$

As $k'_w + A'/2 = k_w + A/2$, that is, $k'_w - k_w = A/2 - A'/2 \leq 0$, and we have

$$v(g_{k_s}^{**}) - v(K) \geq (k'_w - k_w)(2\alpha - c - 2\delta^2) \geq 0.$$

2^{nd} *case*: $A = k_s(k_s - 1)$. Thus

$$B = \min\{2k_s(m - 1 - k_s), m(m-1) - 2k_s - 2k_w - k_s(k_s - 1)\}.$$

In this case $B \geq 0$, and as $k_w + B/2 = k'_w$ and $c < 2(\alpha - \alpha\delta)$, i.e. $2\alpha\delta < 2\alpha - c$, we have:

$$\begin{aligned} v(K) &\leq k_s(2\delta - 2c) + k_w(2\alpha - c) + k_s(k_s - 1)\delta^2 + B\alpha\delta \\ &\leq k_s(2\delta - 2c) + (k_w + B/2)(2\alpha - c) + k_s(k_s - 1)\delta^2. \end{aligned}$$

Thus

$$v(g_{k_s}^{**}) - v(K) \geq (k'_w - (k_w + B/2))(2\alpha - c) = 0.$$

(iii) Just note that in both cases, (i) and (ii), the component is assumed to have fewer than $m-1$ strong links, which includes the case of no strong links. But note that the structure proved to dominate the component, i.e. g_0^* or g_0^{**}, is a weak-complete network. ■

Lemma 7 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$ *and*

$$\max\{2(\delta - \alpha), 2(\alpha - \delta^2)\} < c < 2(\alpha - \alpha^2),$$

then a weak component of a network is dominated either by a weak-complete subnetwork or by a star of strong links with the same number of nodes.

Proof. In view of the preceding lemma, in this region a component with no strong links is dominated by a weak-complete network with the same number of nodes. If a component with m nodes contains at least $m-1$ strong links, Lemma 1 establishes that it is dominated by a star with $m-1$ strong links. If it contains some strong links, but fewer than $m-1$, Lemma 6 shows that it is dominated by one of two types of structure with the same number k_s of strong links, either $g_{k_s}^*$ or $g_{k_s}^{**}$. We now prove that such structures are dominated either by a weak-complete subnetwork or by star of strong links with the same number of nodes. Consider first the case when $c > 2(\alpha - \alpha\delta)$. In this case, the dominant structure is $g_{k_s}^*$. Thus $v(g_{k_s}^*)$ is given by (9), with (10). Thus, comparing this value with that of $v(g_{k_s+1}^*)$ and $v(g_{k_s-1}^*)$, we have

$$v(g_{k_s+1}^*) - v(g_{k_s}^*) = X + (2\alpha - c) - 4\alpha\delta,$$

$$v(g^*_{k_s-1}) - v(g^*_{k_s}) = -X + 2\delta^2,$$

where $X = 2\delta - 2c - (m - k_s)(2\alpha - c) + k_s 2\delta^2 + (m - 2k_s)2\alpha\delta$. We prove that one of these differences is necessarily positive. Assume $-X + 2\delta^2 \leq 0$, that is, $X \geq 2\delta^2$. Thus

$$v(g^*_{k_s+1}) - v(g^*_{k_s}) \geq 2\delta^2 + 2\alpha - c - 4\alpha\delta,$$

which is > 0 if $c < 2\delta^2 + 2\alpha - 4\alpha\delta$. To see that this is so, note that $\alpha < \delta$, thus $(\alpha - \delta)^2 > 0$, i.e. $\alpha^2 + \delta^2 - 2\alpha\delta > 0$. Then, as $c < 2\alpha - 2\alpha^2$

$$c < 2\alpha - 2\alpha^2 < 2\alpha - 2(2\alpha\delta - \delta^2) < 2\delta^2 + 2\alpha - 4\alpha\delta.$$

Therefore one of the two differences must be positive. In other words $g^*_{k_s}$ is dominated either by $g^*_{k_s-1}$ or by $g^*_{k_s+1}$. This entails that $g^*_{k_s}$ is dominated by one of the extreme cases: g^*_0 or g^*_m, i.e. a m-node weak-complete network or star of strong links.

Consider now the case, $c < 2(\alpha - \alpha\delta)$. In this case, the dominant structure is $g^{**}_{k_s}$. Thus $v(g^{**}_{k_s})$ is given by (11) with (12). Thus, comparing this value with that of $v(g^{**}_{k_s+1})$ and $v(g^{**}_{k_s-1})$, we have

$$v(g^{**}_{k_s+1}) - v(g^{**}_{k_s}) = X - (2\alpha - c),$$

$$v(g^{**}_{k_s-1}) - v(g^{**}_{k_s}) = -X + 2\delta^2,$$

where $X = 2\delta - 2c - k_s(2\alpha - c) + k_s 2\delta^2$. But then one of these differences is necessarily positive. Assume $-X + 2\delta^2 \leq 0$, that is, $X \geq 2\delta^2$. Thus

$$v(g^{**}_{k_s+1}) - v(g^{**}_{k_s}) \geq 2\delta^2 - 2\alpha + c > 0,$$

given that $c > 2\alpha - 2\delta^2$ is one of the inequalities specifying the region under consideration. Thus $g^{**}_{k_s}$ is dominated either by $g^{**}_{k_s-1}$ or by $g^{**}_{k_s+1}$. And consequently $g^{**}_{k_s}$ is dominated by either g^{**}_0 or g^{**}_m, i.e. an m-node weak-complete network or a star of strong links. ■

The following result, pulling together the partial results established in the preceding lemmas, characterizes efficiency for the transitional model.

Proposition 3 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$, *then the unique efficient profile is*:
(i) The strong-complete graph if $c < \min\{\delta - \delta^2, 2(\delta - \alpha)\}$ *(Region I in Figures* 4-7*).*
(ii) The weak-complete graph if

$$2(\delta - \alpha) < c < 2\left(\alpha - \alpha^2\right)$$

and $c(n-4) < 2n\alpha - 4\delta - 2(n-2)\delta^2$ *(Region I' in Figures* 4-7*).*
(iii) All-encompassing stars of strong links if

$$\delta - \delta^2 < c < \max\{2(\delta - \alpha) + (n-2)\left(\delta^2 - \alpha^2\right), \delta + (n-2)\delta^2/2\}.$$

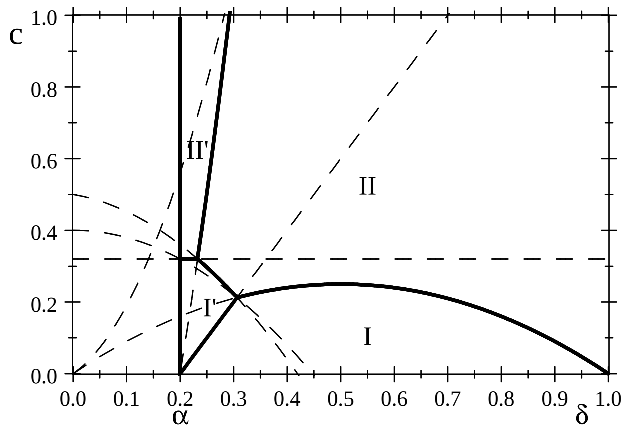

Figure 4: Efficiency: $\alpha = 0.2$, $n = 20$

and $c\,(n-4) > 2n\alpha - 4\delta - 2\,(n-2)\,\delta^2$ (*Region II in Figures* 4-7).
(*iv*) *All-encompassing stars of weak links if*

$$\max\{2\,(\delta-\alpha) + (n-2)\left(\delta^2-\alpha^2\right), 2\left(\alpha-\alpha^2\right)\} < c < 2\alpha + (n-2)\,\alpha^2$$

(*Region II' in Figures* 4-7).
(*v*) *The empty network if*

$$c > \max\{2\alpha + (n-2)\,\alpha^2, \delta + (n-2)\,\delta^2/2\}$$

(*Region III in Figure* 6).

Proof. (*i*) As $c < \delta$, an efficient network is non-empty, and, as $c < 2\,(\delta-\alpha)$, by Lemma 4-(*i*), in a non-empty efficient network all links are strong. Let g then be a network where all links are strong and assume nodes i and j are not connected. As $c < \delta - \delta^2$, i.e. $\delta^2 < \delta - c$, both i and j improve their payoffs if the strong link $\overline{ij}$ forms, and the other players' payoffs do not decrease. Therefore, the unique efficient network is the strong-complete one.

(*ii*) Consider first the subregion where $c < 2\left(\alpha - \delta^2\right)$. Let g be a network where two nodes, i and j, are not directly connected. Thus i (j) receives at most δ^2 from j's (i's) unit of value. As $2\delta^2 < 2\alpha - c$, the sum of the payoffs of i and j increases if a weak link between them forms, and the other players' payoffs do not decrease. Thus, if $c < 2\left(\alpha-\delta^2\right)$ an efficient network must be complete. Note that α must be greater than δ^2. Now as $2\,(\delta-\alpha) < c$, if a strong link $\overline{ij}$ in a *complete* network is replaced by a weak link, then the sum of the payoffs of i and j increases, and the other players' payoffs do not decrease. Therefore, if $2\,(\delta-\alpha) < c < 2\left(\alpha-\delta^2\right)$ then the unique efficient profile is the weak-complete graph. The rest of the region remains to be examined, i.e. where $c \geq 2\left(\alpha-\delta^2\right)$. But this is a subset of the range of values of the

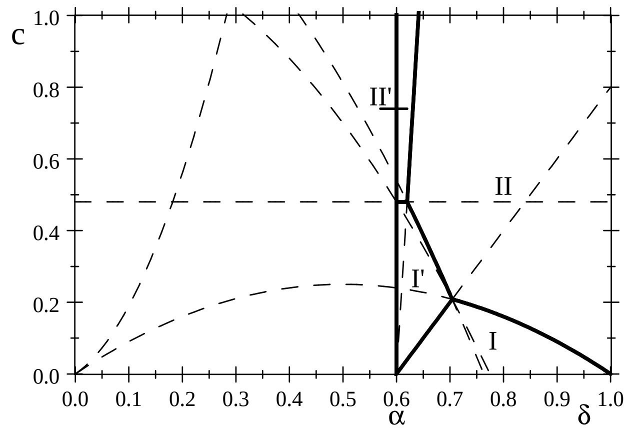

Figure 5: Efficiency: $\alpha = 0.6$, $n = 20$

parameters considered in Lemmas 6 and 7, where any component is dominated either by a weak-complete subnetwork or by a star of strong links with the same number of nodes. As $c < 2(\alpha - \alpha^2) < 2\alpha$, an efficient network must be connected, therefore any network is dominated either by a weak-complete network or by an all-encompassing star of strong links. Finally, it can be checked immediately that the former dominates the latter if and only if $c(n-4) \leq 2n\alpha - 4\delta - 2(n-2)\delta^2$, strictly if the inequality is strict, while both structures are equally efficient in case of equality.

(iii) By Lemma 1, any component with at least $m-1$ strong links is dominated by a star of strong links. It remains to be checked that this is also the case if it has fewer than $m-1$ strong links. As seen in Lemma 4, in this region, when $c < 2(\delta - \alpha)$, a weak component is dominated by a star of strong links, therefore the statement is proven in this case. Now consider the case $c \geq 2(\delta - \alpha)$. If $c > 2\alpha$, Lemma 2 ensures that any component is dominated by a mixed star with the same number of strong links, and by Lemma 3 the same holds if $2(\alpha - \alpha^2) < c < 2\alpha$. By Lemma 5 mixed stars are dominated either by stars of weak links or by stars of strong links, so this conclusion applies to the subset of the region under consideration where $c > 2(\alpha - \alpha^2)$. The subset where $2(\delta - \alpha) \leq c < 2(\alpha - \alpha^2)$ remains to be discussed, where Lemmas 6 and 7 apply and ensure that any component is dominated either by a weak-complete subnetwork or by a star of strong links with the same number of nodes. If $c < 2\alpha$, an efficient network must be connected, therefore in this region any network is dominated either by a weak-complete network or by an all-encompassing star of strong links. But the latter is dominated by the former if and only if $c(n-4) \leq 2n\alpha - 4\delta - 2(n-2)\delta^2$, strictly if the inequality is strict, while both structures are equally efficient in case of equality. Now if $c \geq 2\alpha$, Lemmas 2 and 5, ensure that any component of an efficient network must be a star of either weak links or strong links. As the value of a component of an efficient network must be non-negative, it is immediate to check that the value

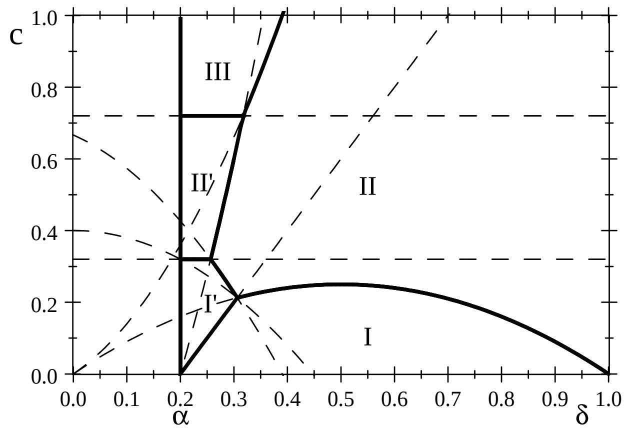

Figure 6: Efficiency: $\alpha = 0.2$, $n = 10$

of a star with $m_1 + m_2$ nodes is greater than the sum of the values of two stars with m_1 and m_2 nodes each. In short, it is proved that throughout the region a component is dominated by a star of strong or of weak links. It then follows immediately that the former dominates the latter if and only if $c \leq 2(\delta - \alpha) + (n-2)\left(\delta^2 - \alpha^2\right)$, strictly if the inequality is strict, while both structures are equally efficient in case of equality. Thus, in the whole region the only non-empty efficient network is the all-encompassing star of strong links. Finally, the all-encompassing star of strong links yields a non-negative value if and only if $c < \delta + (n-2)\,\delta^2/2$.

(iv) By the same argument used in *(iii)*, Lemmas 1-5 ensure that in this region any network is dominated by an all-encompassing star of weak links or by one of strong links. As stated before, the former dominates the latter if and only if $c \geq 2(\delta - \alpha) + (n-2)\left(\delta^2 - \alpha^2\right)$, strictly if the inequality is strict, while both structures are equally efficient in case of equality. Thus, in the whole region the only efficient non-empty network is the all-encompassing star of weak links. Finally, the all-encompassing star of weak links yields non-negative value if and only if $c < 2\alpha + (n-2)\,\alpha^2$.

(v) This follows from the discussion in *(iii)* and *(iv)*. ■

Remarks: (i) Figures 4-7 summarize Proposition 3 relative to efficiency. The images correspond to the cases $\alpha = 0.2$ and $\alpha = 0.6$, with $n = 20$ and $n = 10$. Note that, as $0 \leq \alpha \leq \delta < 1$, only the part where $0.2 \leq \delta < 1$ in Figures 4 and 6 ($0.6 \leq \delta < 1$ in Figures 5 and 7) is meaningful. The *strong*-complete network is the only efficient graph in Region I: below the straight line $c = 2(\delta - \alpha)$ and the parabola $c = \delta - \delta^2$. The only efficient networks in Region I' are *weak*-complete: above the line $c = 2(\delta - \alpha)$, and below the horizontal line $c = 2(\alpha - \alpha^2)$ and the curve $c(n-4) = 2n\alpha - 4\delta - 2(n-2)\,\delta^2$ (a parabola). All-encompassing stars of *strong* links are the only efficient graphs in Region II: above the last parabola and $c = \delta - \delta^2$, and below two parabolas: $c = 2(\delta - \alpha) + (n-2)\left(\delta^2 - \alpha^2\right)$ and $c = \delta + (n-2)\,\delta^2/2$

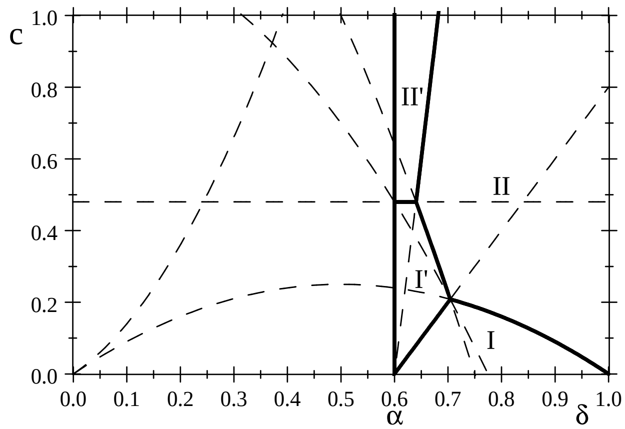

Figure 7: Efficiency: $\alpha = 0.6$, $n = 10$

(the part of the boundary corresponding to the latter is only visible in Figure 6 since only the part of the pictures for $c \leq 1$ is represented in the figures). All-encompassing stars of *weak* links are the only efficient graphs in Region II': above the horizontal line $c = 2(\alpha - \alpha^2)$ and the parabola $c = 2(\delta - \alpha) + (n-2)(\delta^2 - \alpha^2)$, and below the horizontal line $c = 2\alpha + (n-2)\alpha^2$ (the part of the boundary corresponding to the latter is only visible in Figure 6). Finally, in Region III the only efficient graph is the empty network: above $c = 2\alpha + (n-2)\alpha^2$ and $c = \delta + (n-2)\delta^2/2$ (only visible in Figure 6).

(ii) All inequalities in Proposition 3 are strict to preserve uniqueness, but on the boundaries separating any two regions *both* structures are efficient.

(iii) Observe that as α decreases towards 0 the image of these regions approaches the "map" in Figure 1, corresponding to Proposition 1 (i.e. Proposition 1 of Jackson and Wolinsky (1996)), namely Regions I and II in Figures 4-7 expand towards Regions I and II in Figure 1, while regions where "weak" structures are efficient shrink and finally collapse when $\alpha = 0$. In fact, Proposition 3 applied to case $\alpha = 0$ yields Proposition 1. That is, setting $\alpha = 0$ in (i), (iii) and (v) in Proposition 3, yields (i), (ii) and (iii) in Proposition 1, respectively.

(iv) As α "moves rightward", ranging from 0 to 1, the vertical line $\delta = \alpha$ *is* Bala and Goyal's two-way flow model, with $\delta = \alpha$ being the fraction of a unit of information at one node that reaches another one through a link. Thus, as this line sweeps the rectangle, the boundary points separating Regions I', II' and III on the vertical line $\delta = \alpha$, follow the curves $c = 2(\alpha - \alpha^2)$, and $c = 2\alpha + (n-2)\alpha^2$, which depict Figure 2 exactly. In fact, Proposition 3 applied to case $\alpha = \delta$ yields Proposition 2 (i.e. Proposition 5.5 of Bala and Goyal (2000a)). That is, setting $\alpha = \delta$ in (ii), (iv) and (v) in Proposition 3, yields (i), (ii) and (iii) in Proposition 2, respectively.

5.2 Stability

We now study the stability of the efficient structures established in Proposition 3. Pairwise stable architectures are not characterized in Jackson and Wolinsky (1996), and nor are Nash stable networks in Bala and Goyal (2000a). The following results relative to pairwise stability in Jackson and Wolinsky's connections model and to Nash and strict Nash architectures in Bala and Goyal's two-way flow model with decay are proved in those seminal papers assuming homogeneity in costs and values across players. Their statements are adapted to the terminology used here. In Jackson and Wolinsky's model all links are strong, while in Bala and Goyal's all links are weak, but who supports them may affect the stability of an architecture (the same occurs in our model). A *center-sponsored* (*periphery-sponsored*, *mixed-sponsored*) star is a star of weak links where the center supports all links (no link, some but not all links).

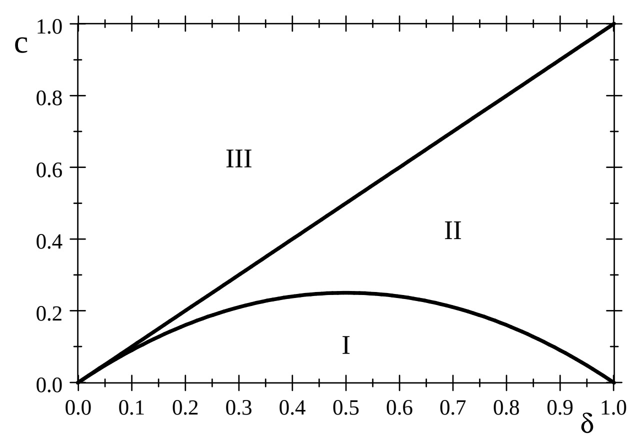

Figure 8: Pairwise stability: $\alpha = 0$ (Jackson and Wolinsky)

Proposition 4 (*Proposition* 2, *Jackson and Wolinsky,* 1996) *In Jackson and Wolinsky's connections model*:
(*i*) *A pairwise stable network has at most one nontrivial strong component.*
(*ii*) *If* $0 < c < \delta - \delta^2$, *then the unique pairwise stable network is the strong-complete graph* (*Region I in Figure* 8).
(*iii*) *If* $\delta - \delta^2 < c < \delta$, *then an all-encompassing star of strong links is pairwise stable* (*Region II in Figure* 8), *but not necessarily the unique pairwise stable graph* (*e.g. if* $n = 4$ *and* $\delta - \delta^3 < c < \delta$ *a line of strong links is also stable, and if* $c < \delta - \delta^3$, *then a wheel of strong links is also stable*).
(*iv*) *If* $\delta < c$, *then in a nonempty pairwise stable network no player is peripheral* (*Region III in Figure* 8).

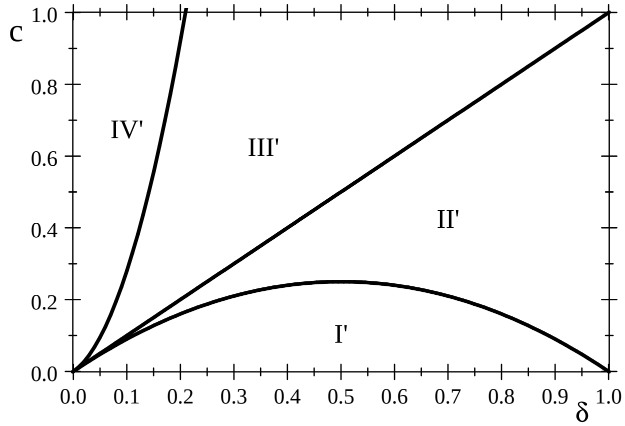

Figure 9: Strict Nash stability: $\alpha = \delta$ (Bala and Goyal), $n = 20$

Proposition 5 (*Proposition* 5.3*, Bala and Goyal,* 2000*a*) *In Bala and Goyal's two-way flow model with decay:*
(*i*) *A strict Nash network is either weakly connected or empty.*
(*ii*) *If* $0 < c < \delta - \delta^2$*, then the unique strict Nash network is the weak-complete graph* (*Region I' in Figure* 9).
(*iii*) *If* $\delta - \delta^2 < c < \delta$*, then all-encompassing stars of weak links are strict Nash* (*Region II' in Figure* 9).
(*iv*) *If* $\delta < c < \delta + (n-2)\delta^2$*, then all-encompassing periphery-sponsored stars, and only them among all-encompassing stars, are strict Nash* (*Region III' in Figure* 9).
(*v*) *If* $\delta < c$*, then the empty network is strict Nash* (*Region IV' in Figure* 9).

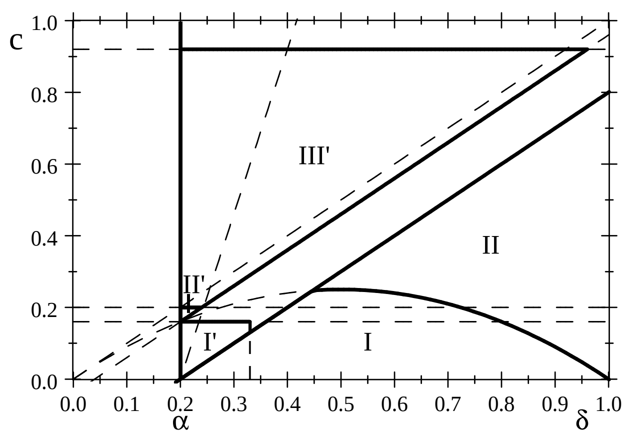

Figure 10: Pairwise stability: $\alpha = 0.2$, $n = 20$

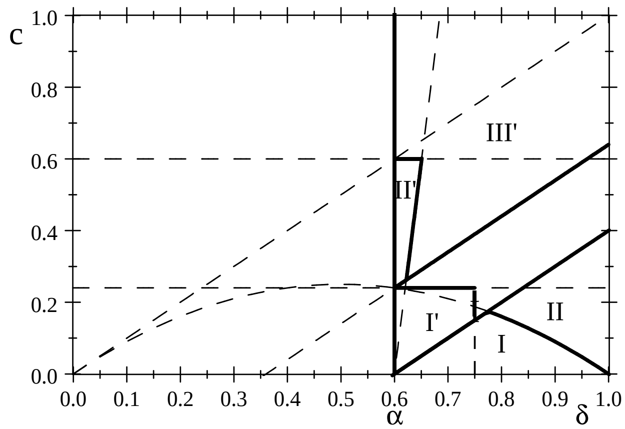

Figure 11: Pairwise stability: $\alpha = 0.6$, $n = 20$

In contrast with the seminal models, in which each requires a different notion of stability, in the transitional model both a strictly noncooperative approach and one allowing for pairwise agreements make sense, so the question of stability is addressed from both points of view. The following two propositions establish the transition between the preceding results. Proposition 6 deals with pairwise stability and Proposition 7 with Nash stability.

Proposition 6 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$, *we have*:
(*i*) *A pairwise stable network has at most one non-trivial weak component* (*which is strong if* $\alpha = 0$), *and has at most one non-trivial strong component.*
(*ii*) *If* $0 < c < \min\{\delta - \delta^2, \delta - \alpha\}$, *then the strong-complete graph is the unique pairwise stable network* (*Region I in Figures* 10-11).
(*iii*) *If* $\delta - \alpha < c < \alpha - \alpha^2$ *and* $\delta < 2\alpha/(1+\alpha)$, *then weak-complete graphs are the unique pairwise stable networks* (*Region I' in Figures* 10-11).
(*iv*) *If* $\delta - \delta^2 < c < \delta - \alpha$, *then all-encompassing stars of strong links are pairwise stable* (*Region II in Figures* 10-11).
(*v*) *If* $\delta - \alpha^2 < c < \alpha + (n-2)\alpha^2$, *then all-encompassing periphery-sponsored stars of weak links are pairwise stable* (*Regions II' and III' in Figures* 10-11).
(*vi*) *If* $\max\{(\delta - \alpha)(1 + (n-2)\alpha), \delta - \alpha^2\} < c < \alpha$, *then all-encompassing stars of weak links* (*periphery-sponsored, center-sponsored or mixed-sponsored*) *are pairwise stable* (*Region II' in Figures* 10-11).
(*vii*) *If* $c > \delta - \alpha$, *then in a pairwise stable network a peripheral player cannot be connected by a strong link. If* $c > \alpha$, *then in a pairwise stable network a peripheral player cannot be sponsored by a weak link. If* $c > \delta$, *then the empty network is pairwise stable.*

Proof. (*i*) Let g be a pairwise stable network. Assume g has more than one non-trivial weak component. Let $ij \in g$ and $kl \in g$ be two links in different weak components. If both are strong, i.e. $\overline{ij} \in g$ and $\overline{kl} \in g$, it easily follows that both j and k benefit by creating a strong link $\overline{jk}$. If $\alpha > 0$ and one of them, say ij, is weak, i.e. $ji \notin g$, then k will benefit by creating a weak link with j. In both cases there is a contradiction with pairwise stability.

(*ii*) Let g be the strong-complete network. A player i has no incentive to withdraw support to a double link $\overline{ij}$ if and only if $\delta - c$ is greater than or equal to α and to δ^2. In other words, it must be $\delta - c \geq \max\{\alpha, \delta^2\}$, which is equivalent to $c \leq \min\{\delta - \delta^2, \delta - \alpha\}$. Now assume this condition holds strictly: any two players not connected by a link must then benefit by forming a strong link, and any player benefits by making a weak link supported by another player double, so only the strong-complete network is pairwise stable.

(*iii*) Let g be a weak-complete network. For a weak link $ij \in g$, player i has no incentive to withdraw support for it if and only if $\alpha - c \geq \alpha^2$. On the other hand, j has no incentive to double this link if and only if $\alpha \geq \delta - c$. Thus we have two necessary conditions for pairwise stability: $\delta - \alpha \leq c \leq \alpha - \alpha^2$. Finally, i has no incentive to switch its support from ij to another, say ik, thus making double the existing weak link ki, if and only if $2\alpha \geq \delta + \alpha\delta$, i.e. $\delta \leq 2\alpha/(1+\alpha)$. Now assume all these conditions hold strictly. Then, as $c < \alpha - \alpha^2 < \alpha$, either of the two players in any pair not connected by a link would benefit by creating a weak link. Thus g must be complete, and as $c > \delta - \alpha$ no strong link can exist. Therefore g is weak-complete. Finally, $\delta < 2\alpha/(1+\alpha)$ guarantees that weak-complete networks alone are pairwise stable.

(*iv*) Let g be an all-encompassing star of strong links. The player with most incentive to withdraw support for a link is the center, who has no incentive to do so if $\delta - c \geq \alpha$, i.e. $c \leq \delta - \alpha$. No two peripheral nodes are interested in forming a strong link if $c \geq \delta - \delta^2$.

(*v*) Let g be an all-encompassing periphery-sponsored star. No peripheral node has an incentive to sever its weak link if $\alpha + (n-2)\alpha^2 - c \geq 0$. No pair of peripheral nodes are interested in forming a strong link if $\alpha + (n-2)\alpha^2 - c \geq \delta + \alpha + (n-3)\alpha^2 - 2c \geq 0$, that is, if $c \geq \delta - \alpha^2$. Note that this implies $c \geq \delta - \alpha$, and consequently the center has no incentive to double any link. Therefore, g is pairwise stable if and only if $\delta - \alpha^2 \leq c \leq \alpha + (n-2)\alpha^2$.

(*vi*) Note that $\delta - \alpha^2 \leq c \leq \alpha$ implies $\delta - \alpha^2 \leq c \leq \alpha + (n-2)\alpha^2$, therefore, as proven in (*v*), an all-encompassing periphery-sponsored star is pairwise stable. Now let g be an all-encompassing center-sponsored or mixed-sponsored star of weak links. The center has no incentive to sever any link if $c \leq \alpha$. No peripheral node whose link is supported by the center has an incentive to double it if $\alpha + (n-2)\alpha^2 \geq \delta + (n-2)\delta\alpha - c$, i.e. if $c \geq (\delta - \alpha)(1 + (n-2)\alpha)$. Finally, no pair of peripheral nodes are interested in forming a strong link (in this respect the situation is entirely similar to (*v*)) if $c \geq \delta - \alpha^2$.

(*vii*) This is straightforward. ■

Remarks: (i) Figures 10 and 11 summarize Proposition 6 for $n = 20$, and $\alpha = 0.2$ and $\alpha = 0.6$, respectively, depicting the regions where the different architectures are pairwise stable. The strong-complete network is the only pairwise stable architecture in Region I: below the line $c = \delta - \alpha$ and the parabola $c = \delta - \delta^2$. The weak-complete networks are the only pairwise stable architectures in Region I': above the line $c = \delta - \alpha$, below $c = \alpha - \alpha^2$, and to the left of the vertical line $\delta = 2\alpha / (1+\alpha)$. All-encompassing stars of strong links are pairwise stable in Region II: below the line $c = \delta - \alpha$ and above the parabola $c = \delta - \delta^2$. Periphery-sponsored stars are pairwise stable in Regions II' and III': above the line $\delta = \delta - \alpha^2$ and below the horizontal line $c = \alpha + (n-2)\alpha^2$ (note that this last constraint is only visible in Figure 10 because for $\alpha = 0.6$ this upper bound is greater than 1). Other stars of weak links, i.e. center-sponsored and mixed-sponsored stars, are pairwise stable in a relatively small subset of this region, namely in Region II': below $c = \alpha$, above the lines $c = \delta - \alpha^2$ and $c = (\delta - \alpha)(1 + (n-2)\alpha)$.

(ii) Observe that as α decreases towards 0 the image of these regions approaches the "map" of Figure 8, corresponding to Proposition 4 (i.e. Proposition 2 of Jackson and Wolinsky (1996)), namely Regions I and II in Figures 10 and 11 expand approaching Regions I and II in Figure 8, while regions where "weak" structures are pairwise stable shrink and finally collapse when $\alpha = 0$. In fact, Proposition 6 applied to case $\alpha = 0$ yields Proposition 4. That is, by setting $\alpha = 0$ in (i), (ii), (iv) and (vii) in Proposition 6, yields (i), (ii), (iii) and (iv) in Proposition 4, respectively.

(iii) The architectures studied in Proposition 6 are not the only ones which are pairwise stable. For example, for $n = 4$: a wheel of strong links is pairwise stable if $\delta - \delta^2 < c < \delta - \alpha$; a line of strong links if $\delta - \delta^3 < c < \delta - \alpha$; any wheel of weak links if $\delta - \alpha^2 < c < \alpha - \alpha^3$ and $\delta < \alpha(2+\alpha) / (1 + \alpha + \alpha^2)$ (this last condition only applies if it is possible for a node to switch its support from one weak link to making double another existing weak link); a line of weak links whose peripheral nodes are sponsored if $\delta + \delta\alpha - \alpha^2 - \alpha^3 < c < \alpha$ and $\delta < \alpha(2+\alpha) / (1+\alpha)$.

(iv) Note that above the line $c = \delta - \alpha^2$ all pairwise stable structures considered are formed exclusively by weak links, while below line $c = \delta - \alpha$ they consist of strong links. As soon as $\alpha > 0$ a gap opens between lines $c = \delta - \alpha$ and $c = \delta - \alpha^2$. The following straightforward corollary emerges relative to this gap:

Corollary 1 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$, $\delta - \alpha < c < \delta - \alpha^2$, *and* $\alpha < c$, *a non-empty pairwise stable network necessarily contains cycles.*

Proof. Assume g is a pairwise stable network. By Proposition 6-(i), g is weakly connected. If $\delta - \alpha < c$, no peripheral player can be connected by a strong link, nor sponsored by a weak link if $\alpha < c$. Therefore a peripheral node can only be connected by a weak link supported by itself. But only one such peripheral node can exist, because if there were more, as $c < \delta - \alpha^2$, it would then be profitable for any pair of them to form a strong link. Consequently, under these conditions g cannot be a weakly connected graph with no cycles. ■

We now address noncooperative stability for the transitional model.

Proposition 7 *If the payoff function is given by* (6) *with* $0 \leq \alpha \leq \delta < 1$, *we have:*

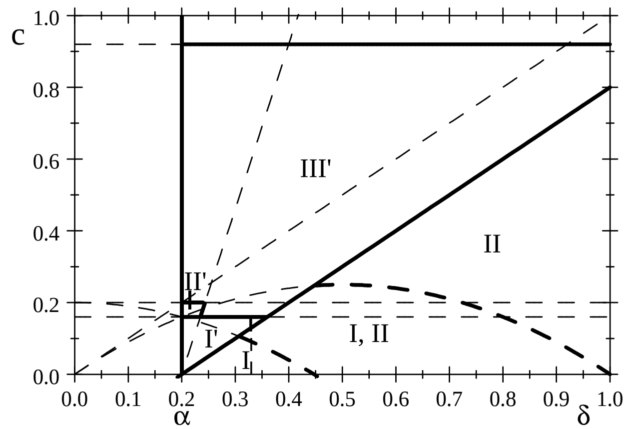

Figure 12: Nash stability: $\alpha = 0.2$, $n = 20$

(i) A Nash network is either weakly connected or all its components are strongly connected.
(ii) If $0 < c \leq \min\{\delta - \delta^2, \delta - \alpha\}$ *then the strong-complete network is Nash stable (strict Nash if the inequalities hold strictly) (Region I in Figures* 12-13*).*
(iii) If $\delta - \alpha \leq c \leq \alpha - \alpha^2$ *and* $\delta \leq 2\alpha/(1+\alpha)$ *then weak-complete networks are Nash (strict Nash if the inequalities hold strictly) (Region I' in Figures* 12-13*).*
(iv) If $\alpha - \delta^2 \leq c \leq \delta - \alpha$*, then all-encompassing stars of strong links are Nash stable (strict Nash if the inequalities hold strictly) (Region II in Figures* 12-13*).*
(v) If $c \geq \delta - \alpha$*, and* $\alpha - \alpha^2 \leq c \leq \alpha + (n-2)\alpha^2$*, then all-encompassing periphery-sponsored stars are Nash stable (strict Nash if the inequalities hold strictly) (Regions II' and III' in Figures* 12-13*).*
(vi) If $\max\{(\delta - \alpha)(1 + (n-2)\alpha), \alpha - \alpha^2\} \leq c \leq \alpha$*, then all-encompassing stars of weak links (periphery-sponsored, center-sponsored or mixed-sponsored) are Nash stable (strict Nash if the inequalities hold strictly) (Region II' in Figures* 12-13*).*
(vii) If $c > \delta - \alpha$*, then in a Nash stable network a peripheral player cannot be connected by a strong link. If* $c > \alpha$*, then in a Nash stable network a peripheral player cannot be sponsored by a weak link. If* $c \geq \alpha$*, then the empty network is Nash stable (strict Nash if the inequality holds strictly).*

Proof. *(i)* Let g be a Nash network. Assume g has more than one non-trivial weak component. If any of them is not strongly connected it contains at least one weak link, say ij, i.e. $ij \in g$ and $ji \notin g$, but then any node in a different weak component will benefit by creating a weak link with j, which contradicts that g is a Nash network.

(ii) Let g be the strong-complete network. A player i has no incentive to withdraw support for a double link $\overline{ij}$ (or a set of them), if and only if $\delta - c$ is greater than or equal to α and to δ^2. In other words, if $\delta - c \geq \max\{\alpha, \delta^2\}$, which is equivalent to

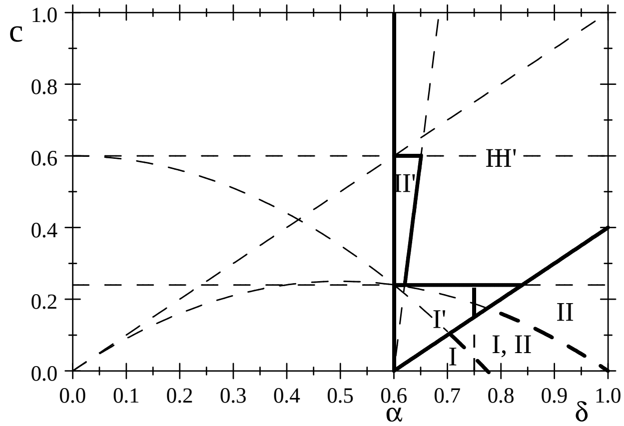

Figure 13: Nash stability: $\alpha = 0.6$, $n = 20$

$c \leq \min\{\delta - \delta^2, \delta - \alpha\}$. Now assume these conditions hold strictly, then the network described is strict Nash.

(*iii*) Let g be a weak-complete network. Proceeding as in part (*iii*) of Proposition 6, it can be concluded that under these conditions no player has an incentive to withdraw support for any number of weak links, or switch their support to double any others, or to double any weak ones. Thus, if all these conditions hold (strictly) g is a Nash (strict Nash) network.

(*iv*) Let g be an all-encompassing star of strong links. The center has no incentive to withdraw support for a link (or a set of them) if $\delta - c \geq \alpha$. No peripheral node is interested in forming a weak link with another (or a set of them) if $c \geq \alpha - \delta^2$. If these conditions hold strictly, then g is strict Nash.

(*v*) Let g be an all-encompassing periphery-sponsored star. No peripheral node has an incentive to sever its link if $\alpha + (n-2)\alpha^2 - c \geq 0$. If $c \geq \delta - \alpha$, the center has no incentive to double a link (or a set of them). If $c \geq \alpha - \alpha^2$ no peripheral node has an incentive to form a weak link with another (or a set of them). Therefore, if all three conditions hold (strictly) g is a Nash (strict Nash) network. In (*vi*) we show that for other stars to be Nash $c \leq \alpha$ is required.

(*vi*) Let g be an all-encompassing star. If it is center-sponsored or a mixed-sponsored star, the center has no incentive to sever a link (or a set of them) if $c \leq \alpha$ (which does not apply if the star is periphery-sponsored) and no peripheral node whose link is supported by the center has an incentive to double it if $\alpha + (n-2)\alpha^2 \geq \delta + (n-2)\delta\alpha - c$, i.e. if $c \geq (\delta - \alpha)(1 + (n-2)\alpha)$. Finally, no peripheral node is interested in forming a weak link with another (or a set of them) if $c \geq \alpha - \alpha^2$. If these conditions hold strictly, then g is strict Nash.

(*vii*) This is straightforward. ■

Remarks: (i) Figures 12 and 13 summarize Proposition 7 for $n = 20$, and $\alpha = 0.2$ and $\alpha = 0.6$, respectively. The strong-complete network is Nash stable in Region I: below the line $c = \delta - \alpha$ and the parabola $c = \delta - \delta^2$. This region *overlaps* with the region where all-encompassing stars of strong links are Nash, namely Region II: below the line $c = \delta - \alpha$ and above the parabola $c = \alpha - \delta^2$. The weak-complete networks are Nash in Region I' (the same region where they are pairwise stable): above the line $c = \delta - \alpha$, and below $c = \alpha - \alpha^2$, and to the left of the vertical line $\delta = 2\alpha / (1 + \alpha)$. Periphery-sponsored stars are Nash in Regions II' and III': above lines $c = \delta - \alpha$ and $c = \alpha - \alpha^2$, and below the horizontal line $c = \alpha + (n-2)\alpha^2$ (only visible in Figure 12). Finally, other stars of weak links, i.e. center-sponsored and mixed-sponsored stars, are pairwise stable in a relatively small subset of this region, namely in Region II': between the horizontal lines $c = \alpha$ and $c = \alpha - \alpha^2$ and above $c = (\delta - \alpha)(1 + (n-2)\alpha)$.

(ii) Observe that as α "moves rightward", ranging from 0 to 1, the vertical line $\delta = \alpha$ *is* Bala and Goyal's two-way flow model with $\delta = \alpha$ being the fraction of the unit of information at one node that reaches another one through a link. Thus, as this line sweeps the rectangle, the boundary points of Regions I', II' and III' on the vertical line $\delta = \alpha$, follow the curves $c = \alpha$, $c = \alpha - \alpha^2$, and $c = \alpha + (n-2)\alpha^2$, which depict Figure 9 exactly. In fact, Proposition 7 applied to case $\alpha = \delta$ yields Proposition 5 (i.e. Proposition 5.3 of Bala and Goyal (2000a)). That is, setting $\alpha = \delta$ in (i), (iii), (v) and (vi) in Proposition 7, yields (i), (ii), (iii) and (iv) in Proposition 5, respectively.

(iii) A comparison with the results for pairwise stability (Proposition 6) shows the following. The region where the strong-complete network is stable in either sense is the same, and the same goes for weak-complete networks (Regions I and I' in Figures 10-11 and 12-13). However these regions are different for stars of strong links and stars of weak links (periphery-sponsored or not). In both cases the region where such structures are pairwise stable is a subset of the region where they are Nash stable, due to the possibility of pairwise coordination to form new strong links, which destabilizes some Nash stable networks. But note that if attention is constrained to Bala and Goyal setting, i.e. to the line where $\alpha = \delta$, weak-complete networks, stars of weak links and periphery-sponsored stars of weak links are stable in either sense in the same regions.

(iv) Again, as with pairwise stability, above the line $c = \delta - \alpha^2$ all Nash structures considered are formed exclusively by weak links, while now below this line they consist of strong links only.

(v) The architectures studied in Proposition 7 are not the only ones which are Nash stable. For example, for $n = 4$: a wheel of strong links is Nash stable if $\alpha - \delta^2 \leq c \leq \delta - \alpha$; a line of strong links if $\alpha - \delta^3 \leq c \leq \delta - \alpha$; a wheel of weak links if $\delta - \alpha^2 < c < \alpha - \alpha^3$ and $\delta < \alpha(2 + \alpha) / (1 + \alpha + \alpha^2)$ (this last condition only applies if it is possible for a node to switch its support from one weak link to double another existing weak one); a line of weak links whose peripheral nodes are sponsored if $\max\{\alpha - \alpha^3, \delta + \delta\alpha + \delta\alpha^2 - \alpha^2 - \alpha^3 - \alpha\} \leq c \leq \alpha$.

6 Conclusion

In this paper we introduce a model which bridges the gap between the two basic models of strategic network formation and incorporates them as extreme cases: Jackson and Wolinsky (1996) bilateral connections model and Bala and Goyal (2000a) unilateral connections two-way flow model. This richer hybrid model, provides a common setting for them and makes it possible to transition from one to the other.

The efficient architectures are fully-characterized for all possible values of the parameters and the results relative to efficiency in both seminal papers extended. One noteworthy result is that only the structures which are efficient in the seminal models emerge as efficient in this transitional model.

The strictly noncooperative approach and the approach based on pairwise stability both make sense and are applied in this setting. Jackson and Wolinsky (1996) pairwise stability results for their connections model and those of Bala and Goyal (2000a) noncooperative stability results for their two-way flow model are extended to the more general model.

The point of view provided by this continuum of models, bridging the gap between the two seminal models, shows the perfect compatibility and robustness of both in the sense that the transition from one to the other is smooth in all respects.

Some lines of further research are the following. A similar transitional model between Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000a) one-way flow model with decay, or between Bala and Goyal's one-way flow and two-way flow models with decay[15] remains to be explored. Some of the extensions of the benchmark models in the literature can also be tested in this mixed model.

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