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Jens L. Hougaard, Juan D. Moreno-Ternero, and Lars P. Østerdal

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Departamento de Teoría e Historia Económica
Facultad de Ciencias Económicas y Empresariales
Universidad de Málaga
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Jens Leth Hougaard[†] **Juan D. Moreno-Ternero**[‡]

Lars Peter Østerdal[§]

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Abstract

We analyze a general model of rationing in which agents have baselines, in addition to claims against the (insufficient) endowment of the good to be allocated. Many real-life problems fit this extended model (e.g., bankruptcy with prioritized claims, resource allocation in the public health care sector, water distribution in drought periods). We introduce (and characterize) a natural class of allocation methods for such problems. Any method within the class is associated with a rule in the standard rationing model, and we show that if the latter obeys some properties reflecting principles of impartiality, priority and solidarity, the former obeys them too.

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[†]Corresponding author. Institute of Food and Resource Economics, University of Copenhagen, Rolighedsvej 25, DK-1958 Frederiksberg C., Denmark, phone +45 35 33 68 14, email: jlh@foi.dk

[‡]Universidad de Málaga, Universidad Pablo de Olavide, and CORE, Université catholique de Louvain.

[§]Department of Economics, University of Copenhagen.

1 Introduction

The problem of dividing when there is not enough is one of the oldest problems in the history of economic thought. Problems of this sort (and possible solutions for them) are already documented in ancient sources, but their formalization is much more recent. O'Neill (1982) was indeed the first to introduce a simple model to analyze the problem in which a group of individuals have conflicting claims over an insufficient amount of a (perfectly divisible) good.[1] Such a model, which will be referred here as the *standard rationing model*, is able to accommodate many real-life situations, such as the division of an estate that is insufficient to cover all the debts incurred by the deceased, the collection of a given tax from taxpayers, the allocation of equities in privatized firms, the distribution of commodities in a fixed-price setting, sharing the cost of a public facility, etc. It fails, however, to accommodate more complex rationing situations, such as those described next, in which not only claims, but also individual rights, needs, or other objective entitlements, play an important role in the rationing process.

One obvious example comes from actual bankruptcy laws, in which typically some claims are prioritized. More precisely, bankruptcy codes normally list all claims that should be treated identically as various categories and assigns to them lexicographic priorities (e.g., Kaminski, 2006). Typically, there exists a category of *secured claims* (involving, for instance, unpaid salaries) receiving the highest priority, which implies that those claims are fully honored (if possible) before allocating the remaining part of the liquidation value among other categories. One could interpret then that agents with secured claims have a *baseline* (i.e., a *right* or an *entitlement*) to be considered in the allocation process.

Another instance refers to the case of resource allocation in the public health care sector. For instance, in the context of allocation of scarce health care resources, it is often argued that patients' *needs* should be given priority in treatment (e.g., Daniels, 1981; Doyal and Gough, 1991; Wiggins, 1998). Although the precise meaning of *needs* in this context is debated, there seems to be consensus that it is related to avoidance of serious harm and, hence, that it is different from a mere want or desire, which seem to be more in line with the concept of claim in our context.[2] On a more practical level, hospital department budgets are typically determined according to a production target (baseline activity).[3] By the end of each year, the actual

[1] Another important early contribution dealing with this same model is Aumann and Maschler (1985). The reader is referred to Moulin (2002) or Thomson (2003, 2006) for recent surveys of the sizable related literature.

[2] See, for instance, Hasman et al., (2006) and Hope et al., (2010).

[3] The reader is referred to Chalkley and Malcomson (2000) for a general discussion of health care budgeting

number of services delivered is recorded (the claim). On the basis of these measures, and the overall health care budget, which typically will not cover the full claim but will often exceed the budgeted activities, the final department funding is settled by allocating residual funds according to residual claims.

Other examples refer to protocols for the reduction of pollution (such as the reduction of greenhouse gas emissions), in which, typically, each party has a specific preferred emissions level (claim) and historical emission allowances (baselines). Somewhat related are the case of water distribution in drought periods (where past consumption can be considered as a baseline) or the so-called *river-sharing problem* (e.g., Ambec and Sprumont, 2002; Ansink and Weikard, 2010) which models international agreements for sharing water resources of a river. More precisely, in the river-sharing problem, a set of agents is located along a river and the river picks up volume along its course. Each agent extracts water from the river for consumption and/or production. Thus, each agent has an endowment (baseline) and a claim to river water.

The aim of this paper is to explore a more general model of rationing able to accommodate all the above situations. The extended model of rationing we analyze here enriches the standard model described above by assuming the existence of a *baselines* profile, aimed to complement the claims profile of a rationing problem.

We take first a *direct approach* to analyze this new model.[4] That is, we single out a natural class of baseline rationing rules which aims to encompass the real-life rationing situations mentioned above. In short, rules within this class tentatively allocate each agent with their baselines and then adjust this tentative allocation by using a standard rationing rule to distribute the remaining surplus, or deficit, relative to the initial endowment. We focus on the study of the robustness of the class, by showing that the rules within the class inherit some important basic properties from the associated standard rationing rules reflecting principles of impartiality, priority and solidarity.

We then take an *axiomatic approach* and study the implications of new axioms reflecting ethical or operational principles in this general context. More precisely, we provide an axiomatic characterization for the class of baseline rationing rules just described.[5]

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procedures.

[4]The terminology is borrowed from Thomson (2006).

[5]There is yet a third approach to rationing problems that we do not consider here: the so-called *game theoretic approach*, which consists in modeling rationing problems as a transferable utility game and aims at identifying the likely outcome of such a game as the solution of the rationing problem. This approach has been taken, for instance, by Pulido et al., (2002, 2008) to analyze what they call *bankruptcy situations with*

The rest of the paper is organized as follows. In Section 2, we describe the basic framework of the standard rationing model, as well as the new one to address more general (baseline) rationing problems. In Section 3, we present our family of baseline rationing rules and study its robustness. In Section 4, we derive the family axiomatically. In Section 5, we complement the previous study by analyzing an alternative, albeit related, form of baseline rationing. We conclude in Section 6 with some further insights. For a smooth passage, we defer all the proofs and provide them in the appendix.

2 Model and basic concepts

2.1 The benchmark framework

We study rationing problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of N. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i \in N}$ the claims profile.[6] A *standard rationing problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the set of rationing problems with population N and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *standard rationing rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem. Each rule R has a dual rule R^* defined as $R^*(N, c, E) = c - R(N, c, C - E)$, for all $(N, c, E) \in \mathcal{D}$.

Some classical rules are the *constrained equal awards* rule, which distributes the endowment equally among all agents, subject to no agent receiving more than what she claims; the *constrained equal losses* rule, which imposes that losses are as equal as possible, subject to no one receiving a negative amount; and the *proportional* rule, which yields awards proportionally to claims.[7]

Rules are typically evaluated in terms of the properties (axioms) they satisfy. The literature

references, which is a specific case of our model.

[6]For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^n$, let $z_M \equiv (z_i)_{i \in M}$.

[7]The reader is referred to Moulin (2002) or Thomson (2003, 2006) for their formal definitions as well as for further details about them.

has provided a wide variety of axioms for rules reflecting ethical or operational principles (e.g., Thomson, 2003; 2006). Here we shall concentrate on those formalizing the principles of impartiality, priority, and solidarity, which have a long tradition in the theory of justice (e.g., Moreno-Ternero and Roemer, 2006).[8]

Impartiality refers to the fact that ethically irrelevant information is excluded from the allocation process. In this context, it is modeled by the axiom of *Equal Treatment of Equals*, which requires allotting equal amounts to those agents with equal claims. Formally, a rule R satisfies equal treatment of equals if, for all $(N, c, E) \in \mathcal{D}$, and all $i, j \in N$, we have $R_i(N, c, E) = R_j(N, c, E)$, whenever $c_i = c_j$.

The principle of *priority* requires imposing a positive discrimination (albeit only to a certain extent) towards worse-off individuals. In this context, priority is modeled by the axiom of *Order Preservation*, which says that agents with larger claims receive larger awards but face larger losses too. That is, $c_i \geq c_j$ implies that $R_i(N, c, E) \geq R_j(N, c, E)$ and $c_i - R_i(N, c, E) \geq c_j - R_j(N, c, E)$, for all $(N, c, E) \in \mathcal{D}$, all $i, j \in N$. If only the first condition is satisfied then the axiom is referred to as *Order Preservation in Gains.* If, on the other hand, only the second condition is satisfied then the axiom is referred to as *Order Preservation in Losses.*

The principle of *solidarity*, with a long tradition in the axiomatic literature, can be modeled in various related ways. *Resource Monotonicity* says that when there is more to be divided, other things being equal, nobody should lose. Formally, a rule R is resource monotonic if, for each $(N, c, E) \in \mathcal{D}$ and $(N, c, E') \in \mathcal{D}$ such that $E \leq E'$, then $R(N, c, E) \leq R(N, c, E')$. *Claims Monotonicity* says that if an agent's claim increases, ceteris paribus, she should receive at least as much as she did initially. Formally, a rule R is claims monotonic if, for all $(N, c, E) \in \mathcal{D}$ and all $i \in N$, $c_i \leq c'_i$ implies $R_i(N, (c_i, c_{N\setminus\{i\}}), E) \leq R_i(N, (c'_i, c_{N\setminus\{i\}}), E)$. A related property says that if an agent's claim and the endowment increase by the same amount, the agent's award should increase by at most that amount. Formally, a rule satisfies *Linked Monotonicity* if, for all $(N, c, E) \in \mathcal{D}$ and $i \in N$, $R_i(N, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) \leq R_i(N, c, E) + \varepsilon$.

Population monotonicity is a relevant solidarity property in the context of a variable population. It says that if new claimants arrive, each claimant initially present should receive at most as much as she did initially. Equivalently, if some claimants leave but there still is not enough to honor all of the remaining claims, each remaining claimant should receive at least as much as she

[8] In what follows, we only consider properties that are either *punctual* (i.e., applying to a rule for each problem separately, point by point) or *relational* (i.e., linking the recommendations made by the rule for a finite set of different problems that are related in a certain way).

did initially. Formally, R is population monotonic if for all $(N,c,E) \in \mathcal{D}$ and $(N',c',E) \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then $R_i(N',c',E) \leq R_i(N,c,E)$, for all $i \in N$. A related property says that if new claimants arrive and the endowment increases by the sum of their claims, then each claimant initially present should receive at least as much as she did initially. Formally, a rule R satisfies *Resource-and-Population Monotonicity* if for all $(N,c,E) \in \mathcal{D}$ and $(N',c',E) \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then $R_i(N,c,E) \leq R_i\left(N',c',E+\sum_{N'\setminus N} c'_j\right)$, for all $i \in N$.

The next axiom also amounts to simultaneous changes in the endowment and the population. It says that the arrival of new agents should affect all the incumbent agents in the same direction. In other words, agents cannot benefit from a change (either in the available wealth or in the number of agents) if someone else suffers from it. Formally, a rule R satisfies *Resource-and-Population Uniformity* if for all $(N,c,E) \in \mathcal{D}$ and $(N',c',E') \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then, either $R_i(N',c',E') \leq R_i(N,c,E)$, for all $i \in N$, or $R_i(N',c',E') \geq R_i(N,c,E)$, for all $i \in N$. This axiom implies resource monotonicity. As a matter of fact, it also satisfies the following axiom that relates the solution of a given problem to the solutions of the subproblems that appear when we consider a subgroup of agents as a new population and the amounts gathered in the original problem as the available endowment. *Consistency* requires that the application of the rule to each subproblem produces precisely the allocation that the subgroup obtained in the original problem.[9] More formally: A rule R is consistent if, for all $(N,c,E) \in \mathcal{D}$, all $M \subset N$, and all $i \in M$, we have $R_i(N,c,E) = R_i(M,c_M,E_M)$, where $E_M = \sum_{i\in M} R_i(N,c,E)$. It turns out that consistency and resource monotonicity together imply resource-and-population uniformity.

To conclude with this section, let us mention that, for any given property $\mathcal{P}$, $\mathcal{P}^*$ is the *dual property of* $\mathcal{P}$ if for each rule R, R satisfies $\mathcal{P}$ if and only if its dual rule $\mathcal{P}^*$ satisfies $\mathcal{P}^*$. A property is said to be self-dual if it coincides with its dual. Equal treatment of equals, order preservation, consistency, and resource monotonicity are self-dual properties. Claims monotonicity and linked monotonicity, population monotonicity and resource-and-population monotonicity, and order preservation in gains and order preservation in losses are pairs of dual properties (e.g., Thomson, 2006).

[9]See Thomson (1996) for an excellent survey of the many applications that have been made on the idea of consistency.

2.2 The extended framework

We now enrich the model to account for individual baselines that will be part of the rationing process. An *extended rationing problem* (or problem with baselines) will be a tuple consisting of a population $N \in \mathcal{N}$, a *baselines* profile $b \in \mathbb{R}^n_+$, a claims profile $c \in \mathbb{R}^n_+$, and an endowment $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. We denote by $\mathcal{E}^N$ the set of extended problems with population N and $\mathcal{E} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$. For each extended problem $(N, b, c, E) \in \mathcal{E}$, let $\widetilde{b}$ denote the corresponding truncated baseline vector, i.e., $\widetilde{b} = \{\widetilde{b}_i\}_{i \in N}$, where $\widetilde{b}_i = \min\{b_i, c_i\}$, for all $i \in N$. For ease of notation, let $\widetilde{B} = \sum_{i \in N} \widetilde{b}_i$.

Given an extended problem $(N, b, c, E) \in \mathcal{E}^N$, an (extended) *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. An *extended rationing rule* on $\mathcal{E}$, $S \colon \mathcal{E} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each extended problem $(N, b, c, E) \in \mathcal{E}$ an (extended) allocation $x = S(N, b, c, E)$ for the problem.

3 The direct approach to baseline rationing

3.1 Baseline rationing rules

This paper will focus on a natural way of defining extended rationing rules from standard rationing rules. In words, extended rules will be constructed such that agents are first allocated their truncated baselines, and then the resulting deficit or surplus is further allocated using a standard rationing rule to the resulting standard problem after embedding baselines into claims. More specifically, a potential deficit is allocated according to the amounts already received by the agents while a potential surplus is allocated according to the gap between their claims and what has already been allocated to them. We shall refer to the extended rules, so constructed, under the term *baseline rationing rules.*

Formally,

$$\widetilde{R}(N, b, c, E) = \begin{cases} \widetilde{b} - R(N, \widetilde{b}, \widetilde{B} - E) & \text{if } E \leq \widetilde{B} \\ \widetilde{b} + R(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } E \geq \widetilde{B} \end{cases} \quad (1)$$

We shall refer to $\widetilde{R}$ as the baseline rationing rule induced by (standard rationing rule) R.

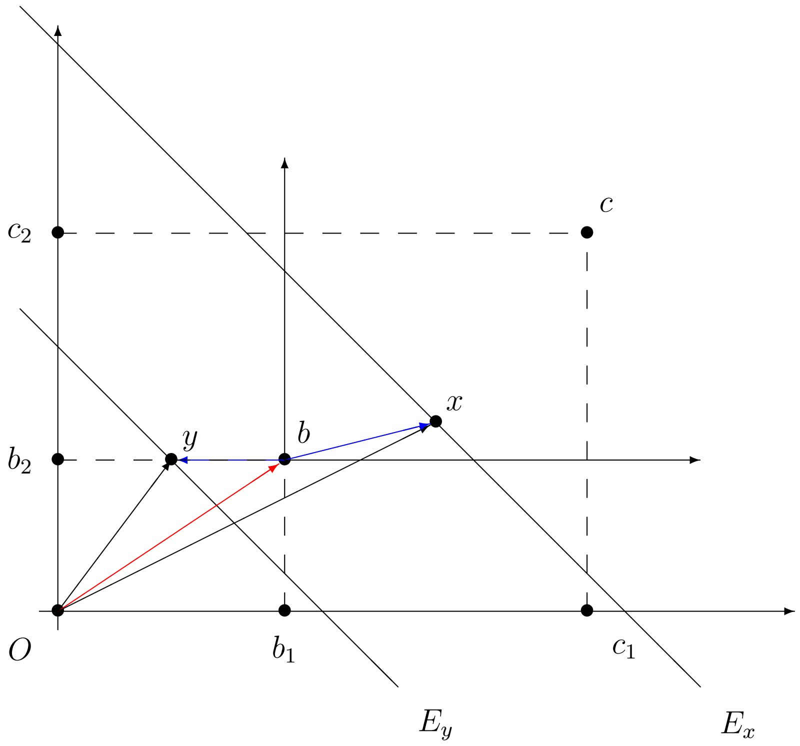

Figure 1: Baseline rationing rules in the two-claimant case. This figure illustrates how baseline rationing rules behave for $N = \{1, 2\}$, and $b, c \in \mathbb{R}^N_+$, with $c_i > b_i$, for $i = 1, 2$. If the endowment is $E_x > b_1 + b_2$, then the proposed solution x can be decomposed as $b + (x - b)$ where $x - b$ is to be interpreted as the solution for the standard rationing problem arising after adjusting claims (and endowment) down by the baselines, which implies that b is the new origin, i.e., $x = b + R(N, c - b, E_x - b_1 - b_2)$. If, however, $E_y < b_1 + b_2$ then the proposed solution y can be decomposed as $b - (b - y)$ where $b - y$ is to be interpreted as the solution for the standard rationing problem arising after replacing claims by baselines and the endowment by the difference between the aggregate baseline and the original endowment, i.e., $b - y = R(N, b, b_1 + b_2 - E_y)$.

Note that, if $b = 0$, then $\widetilde{R} \equiv R$. More interestingly, note that, for any standard rationing rule R, and any extended problem (N, b, c, E), the induced baseline rationing rule results in an allocation x satisfying

$$x_i \leq \widetilde{b}_i \text{ for all } i \in N \text{ if and only if } E \leq \widetilde{B},$$

$$x_i \geq \widetilde{b}_i \text{ for all } i \in N \text{ if and only if } E \geq \widetilde{B}.$$

In other words, baseline rationing rules impose a rationing of the same sort for each individual and the whole society according to the profile of baselines.

It is not difficult to show that the following expression is equivalent to (1).

$$\widetilde{R}(N, b, c, E) = \begin{cases} R^*(N, \widetilde{b}, E) & \text{if } \; E \leq \widetilde{B} \\ \widetilde{b} + R(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } \; E \geq \widetilde{B} \end{cases} \tag{2}$$

It follows from (2) that if each individual baseline is exactly one half of each individual claim, then the baseline rationing rule induced by the constrained equal losses rule described above would solve the extended problem as the so-called *Talmud* rule (e.g., Aumann and Maschler, 1985) would solve the original standard problem. Similarly, the rule induced by the constrained equal awards rule would solve the extended problem as the so-called *Reverse Talmud* rule (e.g., Chun et al., 2001) would solve the original standard problem. If instead of one half, baselines are any other fixed proportion of claims, $\theta \in (0,1)$, then the rule induced by the constrained equal losses rule would solve the extended problem as the corresponding member of the so-called TAL-family of rules (e.g., Moreno-Ternero and Villar, 2006) would solve the original standard problem, whereas the rule induced by the constrained equal awards rule would solve the extended problem as the corresponding member of the so-called Reverse TAL-family (e.g., van den Brink et al., 2008) would solve the original standard problem. Thus, the family of baseline rationing rules presented here can provide rationale for a wide variety of existing standard rationing rules.

3.2 Robustness to baseline rationing

As mentioned above, baseline rationing rules associate to an existing standard rule an extended rule. A natural question that arises from that is whether the extended rules so constructed inherit the properties of the original standard rules. We shall say that a standard axiom is *robust* to baseline rationing if whenever a standard rule R satisfies it, then the induced baseline rationing rule $\widetilde{R}$ satisfies the corresponding *extended* version of the axiom.[10] Note that, as mentioned above, if $b=0$, then $\widetilde{R} \equiv R$. Thus, saying that "a property is robust to baseline rationing" is indeed equivalent to saying that "a standard rationing rule R satisfies a property if and only if $\widetilde{R}$ satisfies the extended version of such property".

Our first result says that many of the well-known axioms in the benchmark framework are not robust to baseline rationing.

Theorem 1 *If a property is not self-dual then it is not robust to baseline rationing.*

Contrary to what one might have guessed from the statement of Theorem 1, not all self-dual properties are robust to baseline rationing either. An obvious counterexample is equal treatment of equals, which will not be satisfied by an induced baseline rationing rule if two agents with equal claims

[10] For ease of exposition, we skip the straightforward definitions of the extended versions of each axiom introduced above. Just as an illustration, we say, for instance, that an extended rule S satisfies claims monotonicity if for each $(N,b,c,E), (N,b,c',E) \in \mathcal{E}$ such that $c_i \leq c'_i$ for some $i \in N$, and $c'_{N\setminus\{i\}} \equiv c_{N\setminus\{i\}}$, we have that $S_i(N,b,c,E) \leq S_i(N,b,c',E)$. As for variable-population axioms, we say, for instance, that an extended rule S satisfies consistency if for each $(N,b,c,E) \in \mathcal{E}$ and $M \subset N$, $\widetilde{R}(M,b_M,c_M,\sum_{i\in M} x_i) = x_M$, where $x = \widetilde{R}(N,b,c,E)$.

may have different baselines. Nevertheless, it turns out that some other important self-dual properties are indeed robust.

Theorem 2 *The following statements hold:*

- *Resource monotonicity is robust to baseline rationing.*
- *Consistency is robust to baseline rationing.*
- *Resource-and-population uniformity is robust to baseline rationing.*

Our next step is to explore whether pairs of dual properties are robust to baseline rationing. As before, an obvious counterexample arises: order preservation, which is equivalent to the pair formed by order preservation in gains and order preservation in losses, will not be satisfied by an induced baseline rationing rule if baselines may not be ordered as claims. Nevertheless, some other pairs are indeed robust to baseline rationing, either by themselves or assisted by an additional robust property, as shown in the next results.[11]

Theorem 3 *The pair formed by claims monotonicity and linked monotonicity is robust to baseline rationing.*

In words, Theorem 3 says that if a standard rationing rule R satisfies claims monotonicity and linked monotonicity, then the induced baseline rationing rule $\widetilde{R}$ satisfies the corresponding two extended properties.

Theorem 4 *The pair formed by population monotonicity and resource-and-population monotonicity is robust to baseline rationing, if assisted by resource monotonicity.*

In other words, Theorem 4 says that if a standard rationing rule R satisfies resource monotonicity, population monotonicity, and resource-and-population monotonicity then the induced baseline rationing rule $\widetilde{R}$ satisfies the corresponding three extended properties.

To conclude with this section, we focus on equal treatment of equals and order preservation which, as mentioned above, are not robust to baseline rationing. It turns out that they are (*partially*) robust, provided we impose additional (mild) conditions on baselines. More precisely, we say that baselines and claims are *uniformly impartial* if whenever $c_i = c_j$ then $b_i = b_j$. We say that *baselines are ordered like claims* if whenever $c_i \leq c_j$ then $b_i \leq b_j$. Finally, we say that *claim-baseline differences are ordered like claims* if whenever $c_i \leq c_j$ then $c_i - b_i \leq c_j - b_j$.

[11]The terminology is borrowed from Hokari and Thomson (2008).

Proposition 1 *The following statements hold:*

- *If baselines and claims are uniformly impartial then equal treatment of equals is robust to baseline rationing.*
- *If baselines are ordered like claims and R is order preserving then $\widetilde{R}$ is order preserving in gains.*
- *If baselines and claim-baseline differences are ordered like claims then order preservation is robust to baseline rationing.*

4 The axiomatic approach to baseline rationing

In the previous section, we introduced a natural class of rules for the extended setting of rationing with baselines and focussed on how this class performs with respect to some basic axioms from the standard rationing model. In this section, we take a different approach to baseline rationing. We now provide some (new) axioms conveying natural ways of taking baselines into account, while designing the rationing scheme, and study their implications. As we shall show, a combination of these axioms will lead to a characterization of the family of baseline rationing rules presented above.

Our first axiom requires baselines to be disregarded to the extent that they are above claims.[12] Formally, an extended rule S satisfies **baseline truncation** if, for each $(N, b, c, E) \in \mathcal{E}$, $S(N, b, c, E) = S(N, \widetilde{b}, c, E)$. The rationale for this idea is that as no agent can get more than her claim, as stated in the definition of (extended) rules, baselines above that level should be considered irrelevant.

The second axiom requires to disregard the amount of a claim exceeding its corresponding baseline, whenever all truncated baselines cannot be covered. Formally, an extended rule S satisfies **truncation of excessive claims** if, for each $(N, b, c, E) \in \mathcal{E}$ such that $E \leq \widetilde{B}$, $S(N, b, c, E) = S(N, b, \widetilde{c}, E)$, where $\widetilde{c}_j = \min\{c_j, b_j\}$, for all $j \in N$. The rationale for this idea is somewhat related to the rationale for the previous one. Namely, as not all baselines can be honored, no agent will achieve more than her baseline and, thus, the portion of their claims above their baselines should be considered irrelevant.

The third axiom is somewhat polar to the previous one as it refers to a situation where all truncated baselines can be covered. It states that, in such a case, if an individual's claim and baseline are increased by an amount k_i, and so does the endowment, then such increase in the endowment should go to that individual while others remain unaffected. Formally, an extended rule S satisfies **baseline invariance** if for each $(N, b, c, E) \in \mathcal{E}$ such that $E \geq \widetilde{B}$, and $k \in \mathbb{R}^n_+$ such that $k_j \leq \min\{c_j, b_j\}$, for all $j \in N$, then $S(N, b, c, E) = k + S(N, b-k, c-k, E - \sum_{i \in N} k_i)$. In particular, the axiom says that, for the cases in which all truncated baselines can be covered, the rationing problem can be solved in

[12]This property is reminiscent of the so-called independence of irrelevant claims axiom introduced by Dagan (1996) for the standard rationing model.

two stages; the first one amounts to grant all agents a fixed amount, and the second one amounts to solve the resulting problem after adjusting down baselines, claims and endowment.[13]

Finally, we consider a fourth axiom dealing with the two polar cases of *non-informative* baselines, and inspired by the notion of self-duality from the standard model of rationing. It states that a rule should allocate awards for a problem with null baselines in the same way as it allocates losses for the corresponding problem in which baselines are equal to claims. Formally, an extended rule S satisfies **polar baseline self-duality** if, for each $(N, c, E) \in \mathcal{D}$, $S(N, 0, c, E) = c - S(N, c, c, C - E)$.[14]

As the next theorem shows, these four axioms together characterize our family of baseline rationing rules introduced above.

Theorem 5 *An extended rationing rule satisfies Baseline Truncation, Truncation of Excessive Claims, Baseline Invariance and Polar Baseline Self-Duality if and only if it is a baseline rationing rule.*

As shown in the appendix, Theorem 5 is tight. It turns out that the first three axioms of its statement characterize the family of extended rules arising from using (possibly) different standard rules when $\widetilde{B} \geq E$ or $\widetilde{B} \leq E$. More precisely, let us define the family of *generalized baseline rationing rules* by

$$\widetilde{RT}\,(N, b, c, E) = \begin{cases} \widetilde{b} - R(N, \widetilde{b}, \widetilde{B} - E) & \text{if } \ E \leq \widetilde{B} \\ \widetilde{b} + T(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } \ E > \widetilde{B} \end{cases},$$

where R and T are standard (rationing) rules. Then, we have the following:

Proposition 2 *An extended rationing rule satisfies Baseline Truncation, Truncation of Excessive Claims and Baseline Invariance if and only if it is a generalized baseline rationing rule.*

5 Further insights on baseline rationing

We start this section presenting an alternative to the axiom of polar baseline self-duality, in order to deal with the two polar cases of non-informative baselines. More precisely, the axiom of **polar baseline equivalence** states that a problem with null baselines and the corresponding problem in which baselines are equal to claims should be allocated identically. Formally, an extended rule S

[13]This is somehow reminiscent of the composition from minimal rights axiom in the standard rationing model (e.g., Dagan, 1996).

[14]Note that the standard definition of self-duality does not make sense in the present case as there is no clear definition of a loss when both baselines and claims are in play. That is why we restrict the notion only to the two polar cases of *non-informative* baselines.

satisfies **polar baseline equivalence** if, for each $(N, c, E) \in \mathcal{D}$, $S(N, 0, c, E) = S(N, c, c, E)$. The rationale for this axiom is that non-informative baselines should be treated identically.

The following extended rules satisfy this axiom.

$$\widehat{R}\,(N, b, c, E) = \begin{cases} R(N, \widetilde{b}, E) & \text{if } \; E \leq \widetilde{B} \\ \widetilde{b} + R(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } \; E \geq \widetilde{B} \end{cases} \tag{3}$$

The rules so constructed follow closely the spirit underlying two important properties in the axiomatic approach to the standard rationing problem, known as *composition up* and *composition down*, which pertain to the way rules react to tentative allocations of wrong estimations of the endowment (e.g., Moulin, 2000; Thomson, 2003). For this reason, we shall refer to these rules as *composition rationing rules*.

It follows from (3) that if each individual baseline is exactly one half of each individual claim, then the composition rationing rule induced by the constrained equal losses rule described above would solve the extended problem as the so-called *Piniles* rule (e.g., Thomson, 2003) would solve the original standard problem. Similarly, the rule induced by the constrained equal awards rule would solve the extended problem as the dual of the Piniles rule would solve the original standard problem.

In turns out that the composition rationing rules are the only generalized baseline rationing rules satisfying the axiom of polar baseline equivalence. Formally,

Proposition 3 *An extended rationing rule satisfies Baseline Truncation, Truncation of Excessive Claims, Baseline Invariance and Polar Baseline Equivalence if and only if it is a composition rationing rule.*

We shall say that a standard axiom is robust to composition rationing if whenever a standard rule R satisfies it, then the induced baseline rationing rule $\widehat{R}$ satisfies the corresponding extended version of the axiom. It turns out that much of the above analysis for baseline rationing rules can be extended to composition rationing rules, as summarized in the following result.

Proposition 4 *The following statements hold:*

- *Resource monotonicity, consistency, and resource-and-population uniformity are robust to composition rationing.*
- *The pair formed by claims monotonicity and linked monotonicity is robust to composition rationing.*
- *The pair formed by population monotonicity and resource-and-population monotonicity is robust to composition rationing, if assisted by resource monotonicity.*

- *If baselines and claims are uniformly impartial then equal treatment of equals is robust to composition rationing.*

- *If baselines are ordered like claims and R is order preserving then $\widehat{R}$ is order preserving in gains.*

- *If baselines and claim-baseline differences are ordered like claims then order preservation is robust to composition rationing.*

6 Final remarks

We have explored in this paper an extended framework to analyze general rationing problems in which agents have claims over the (insufficient) endowment, but also relevant baselines for the allocation process. An individual baseline can be interpreted as an objective entitlement (as in budgeting situations), a right (as in the case of unpaid salaries in bankruptcy situations), or as a measure of needs (as in health care prioritization). We have introduced a natural family of (extended) rationing rules and argued that this family encompasses a series of real-life rationing situations. It is somewhat surprising that with little or no structure on the arbitrarily chosen baseline profiles this family still proves relatively robust in preserving a series of well known and desirable properties from the standard rationing model.

Our contribution is somehow reminiscent of a route previously explored for cooperative models of bargaining. A variety of studies has extended Nash's original bargaining model by means of specifying an additional reference point to the disagreement point, which plays a role in the bargaining solution. Such a reference point can be interpreted as a status quo, as a first step towards the final compromise, or simply as a vector of claims (e.g., Gupta and Livne, 1988; Thomson, 1994). It is the obvious counterpart to the baselines profile in our setting.

Our model is also related to another extension of the standard rationing model that has been recently considered to account for *multi-issue rationing problems*, i.e., rationing problems in which claims refer to different issues (e.g., Kaminski, 2006; Ju et al., 2007; Moreno-Ternero, 2009). The multi-issue rationing model departs from the standard model by assuming vectors of claims (rather than single claims), each indicating the individual claim for a given issue. Thus, our model could be seen as a specific instance of the multi-issue rationing model in which only two issues are considered. The distinguishing feature, however, that we endorse here is to provide an asymmetric role to the two issues, considering baselines as "rights" that might eventually arise in a first-step tentative allocation, and claims as means to finally settle them.

We believe that our work can also shed some light in the search of a dynamic rationale for some classical rationing rules. More precisely, imagine we consider a sequence of rationing problems (involving the same group of agents), at different periods of time, whose period-wise solutions might not only be determined by the data of the rationing problem at such period, but also by the solutions in previous periods. A plausible way to start approaching this issue would be by assuming that, at each period, the corresponding rationing problem is enriched by an index summarizing the amounts each agent obtained in the previous period. If so, we would just be providing an alternative interpretation for the baselines profile that we consider in our model. Needless to say that it would be interesting to go beyond this point and, ultimately, to provide a dynamic rationale for some classical rationing rules, as recently done by Fleurbaey and Roemer (2010) for the three canonical axiomatic bargaining solutions.

Finally, it is worth mentioning that an alternative model of baseline rationing, in which baselines are assumed to be an endogenous component of a standard rationing problem, can also be considered. The analysis of such endogenous baselines, and their connections to the study of operators for the space of (standard) rationing rules (e.g., Thomson and Yeh, 2008), as well as to the concept of lower bounds in rationing problems (e.g., Moreno-Ternero and Villar, 2004) is undertaken in a companion paper (Hougaard et al., 2011).

7 Appendix: Proofs of the results

7.1 Proof of Theorem 1

Let $\mathcal{P}$ be a property and $\mathcal{P}^*$ be its dual. Let R be a rule satisfying $\mathcal{P}$, but not $\mathcal{P}^*$. Then, R^*, the dual rule of R, satisfies $\mathcal{P}^*$ but not $\mathcal{P}$.

If $\mathcal{P}$ is "punctual" then there exists a problem $(N, c, E) \in \mathcal{D}$ for which R^* violates $\mathcal{P}$. We then consider the corresponding extended problem $(N, b, c, E) \in \mathcal{E}$ in which $b = c$. It then follows that $\widetilde{R}(N, b, c, E) = R^*(N, c, E)$ and hence we conclude that $\widetilde{R}$ violates P.

If $\mathcal{P}$ is "relational" a similar argument can be applied. For ease of exposition, we assume that $\mathcal{P}$ only involves a finite collection of problems. Formally, if R^* violates $\mathcal{P}$ then there exists a collection of problems $\{(N^j, c^j, E^j)\}_{j=1,\ldots,k} \subset \mathcal{D}$ for which $\mathcal{P}$ is violated. Now, consider the corresponding extended problems $\{(N^j, b^j, c^j, E^j)\}_{j=1,\ldots,k} \subset \mathcal{E}$ where, for each $j = 1, \ldots, k$,

$$b_i^j = \begin{cases} \max_{l=1,\ldots,k}\{c_i^l\}, & \text{if } \ i \in \bigcap_{l=1,\ldots,k} N^l \\ c_i^j & \text{if } \ i \in N^j \setminus \bigcap_{l=1\ldots k} N^l \end{cases}$$

It is straightforward to show that, for each $j = 1, \ldots, k$, $\widetilde{R}(N^j, b^j, c^j, E^j) = R^*(N^j, c^j, E^j)$, from where it follows that $\widetilde{R}$ violates $\mathcal{P}$. ■

7.2 Proof of Theorem 2

Resource monotonicity. Let R be a rule satisfying resource monotonicity. Let $(N,c,E),(N,c,E') \in \mathcal{D}$ be two problems such that $E < E'$. Let $b \in \mathbb{R}^n$ be a baseline profile and let $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$, and $\widetilde{B} = \sum_{j\in N}\min\{b_j, c_j\}$. Finally, let $i \in N$ be a given agent. The aim is to show that $\widetilde{R}_i(N,b,c,E) \leq \widetilde{R}_i(N,b,c,E')$. To do so, we distinguish three cases.

Case 1. $E < E' \leq \widetilde{B}$.

In this case, $\widetilde{R}_i(N,b,c,E) = R^*_i(N,\widetilde{b},E)$ and $\widetilde{R}_i(N,b,c,E') = R^*_i(N,\widetilde{b},E')$. Now, as resource monotonicity is a self-dual property, it follows that R^* satisfies resource monotonicity too and hence $R^*_i(N,\widetilde{b},E) \leq R^*_i(N,\widetilde{b},E')$, as desired.

Case 2. $\widetilde{B} \leq E < E'$.

In this case,

$$\widetilde{R}_i(N,b,c,E) = \widetilde{b}_i + R_i(N,c-\widetilde{b},E-\widetilde{B}) \leq \widetilde{b}_i + R_i(N,c-\widetilde{b},E'-\widetilde{B}) = \widetilde{R}_i(N,b,c,E'),$$

where the inequality follows from the fact that R satisfies resource monotonicity.

Case 3. $E < \widetilde{B} < E'$.

In this case, the definition of baseline rationing guarantees that $\widetilde{R}_i(N,b,c,E) \leq \widetilde{b}_i \leq \widetilde{R}_i(N,b,c,E')$.

Consistency. Let R be a rule satisfying consistency. Let $(N,c,E) \in \mathcal{D}$ and $b \in \mathbb{R}^n_+$. Let $x = \widetilde{R}(N,b,c,E)$. The aim is to show that, for any $M \subset N$,

$$\widetilde{R}(M, b_M, c_M, \sum_{i\in M} x_i) = x_M.$$

Fix $M \subset N$ and let $E' = \sum_{i\in M} x_i$ and $\widetilde{B}' = \sum_{j\in M}\min\{b_j,c_j\}$. Thus, $E \leq \widetilde{B}$ if and only if $E' \leq \widetilde{B}'$. We then distinguish two cases.

Case 1. $E \leq \widetilde{B}$.

In this case, $x_i = \widetilde{b}_i - R_i(N,\widetilde{b},\widetilde{B}-E)$ for all $i \in N$, and thus $\widetilde{B}' - E' = \sum_{i\in M} R_i(N,\widetilde{b},\widetilde{B}-E)$. Therefore, $\widetilde{R}_i(M,b_M,c_M,E') = \widetilde{b}_i - R_i(M,\widetilde{b}_M,\widetilde{B}'-E')$ for all $i \in M$. Now, as R is consistent, it follows that $R_i(N,\widetilde{b},\widetilde{B}-E) = R_i(M,\widetilde{b}_M,\widetilde{B}'-E')$, for all $i \in M$, which concludes the proof of this case.

Case 2. $E \geq \widetilde{B}$.

In this case, $x_i = \widetilde{b}_i + R_i(N,c-\widetilde{b},E-\widetilde{B})$ for all $i \in N$, and thus $\widetilde{B}' - E' = \sum_{i\in M} R_i(N,c-\widetilde{b},E-\widetilde{B})$. Therefore, $\widetilde{R}_i(M,b_M,c_M,E') = \widetilde{b}_i + R_i(M,c_M-\widetilde{b}_M,E'-\widetilde{B}')$ for all $i \in M$. Now, as R is consistent, it follows that $R_i(N,c-\widetilde{b},E-\widetilde{B}) = R_i(M,c_M-\widetilde{b}_M,E'-\widetilde{B}')$, for all $i \in M$, which concludes the proof of this case.

Resource-and-population uniformity follows from the first two statements and the relationship among the axioms described in Section 2. ■

7.3 Proof of Theorem 3

Let R be a rule satisfying claims monotonicity and linked monotonicity. Our aim is to show that $\widetilde{R}$ satisfies the extended versions of the two properties.

Claims monotonicity. Let $(N,c,E),(N,c',E) \in \mathcal{D}$ be two problems such that, for some $i \in N$, $c_i \leq c'_i$, whereas $c_{N\setminus\{i\}} \equiv c'_{N\setminus\{i\}}$. Let $b \in \mathbb{R}^n_+$ be a baseline profile and let $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$, $\widetilde{b}'_j = \min\{b_j, c'_j\}$ for all $j \in N$, $\widetilde{B} = \sum_{j \in N} \widetilde{b}_j$, and $\widetilde{B}' = \sum_{j\in N} \widetilde{b}'_j$.[15] The aim is to show that

$$\widetilde{R}_i(N,b,c,E) \leq \widetilde{R}_i(N,b,c',E). \tag{4}$$

We distinguish several cases:

Case 1. $E \leq \widetilde{B}$.

In this case, $\widetilde{R}_i(N,b,c,E) = R^*_i(N,\widetilde{b},E)$ and $\widetilde{R}_i(N,b,c',E) = R^*_i(N,\widetilde{b}',E) = R^*_i(N,(\widetilde{b}_{N\setminus\{i\}},\widetilde{b}'_i),E)$. As R satisfies linked monotonicity, it follows that R^* satisfies claims monotonicity, from where we obtain (4).

Case 2. $E \geq \widetilde{B}'$.

In this case, $\widetilde{R}_i(N,b,c,E) = \widetilde{b}_i + R_i(N,c-\widetilde{b},E-\widetilde{B})$, and

$$\widetilde{R}_i(N,b,c',E) = \widetilde{b}'_i + R_i(N,c'-\widetilde{b}',E-\widetilde{B}') = \widetilde{b}'_i + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},c'_i-\widetilde{b}'_i),E-\widetilde{B}-\widetilde{b}'_i+\widetilde{b}_i).$$

Let $\varepsilon = \widetilde{b}'_i - \widetilde{b}_i \geq 0$. Then, (4) is equivalent to

$$R_i(N,c-\widetilde{b},E-\widetilde{B}) \leq \varepsilon + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},c'_i-\widetilde{b}'_i),E-\widetilde{B}-\varepsilon). \tag{5}$$

We then distinguish three subcases.

Case 2.1. $b_i > c'_i$.

Here, $\widetilde{b}_i = c_i < c'_i = \widetilde{b}'_i$ (and thus $\varepsilon = c'_i - c_i$). Then, (5) becomes

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}) \leq \varepsilon + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}-\varepsilon).$$

Now, by claims monotonicity (of R),

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}) \leq R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},\varepsilon),E-\widetilde{B}).$$

And, by linked monotonicity (of R),

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},\varepsilon),E-\widetilde{B}) \leq \varepsilon + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}-\varepsilon),$$

which concludes the proof in this case.

Case 2.2. $c'_i \geq b_i \geq c_i$.

[15] Note that $\widetilde{b}'_{N\setminus\{i\}} \equiv \widetilde{b}_{N\setminus\{i\}}$, $\widetilde{b}'_i \geq \widetilde{b}_i$ and thus, $\widetilde{B}' \geq \widetilde{B}$.

Here, $\widetilde{b}_i = c_i \leq b_i = \widetilde{b}'_i$ (and thus $\varepsilon = b_i - c_i$). Then, (5) becomes

$$R_i(N, ((c - \widetilde{b})_{N\setminus\{i\}}, 0), E - \widetilde{B}) \leq b_i - c_i + R_i(N, ((c - \widetilde{b})_{N\setminus\{i\}}, c'_i - b_i), E - \widetilde{B} - b_i + c_i).$$

Now, by claims monotonicity (of R),

$$R_i(N, ((c - \widetilde{b})_{N\setminus\{i\}}, 0), E - \widetilde{B}) \leq R_i(N, ((c - \widetilde{b})_{N\setminus\{i\}}, c'_i - c_i), E - \widetilde{B}).$$

And, by linked monotonicity (of R),

$$R_i(N, ((c - \widetilde{b})_{N\setminus\{i\}}, c'_i - c_i), E - \widetilde{B}) \leq \varepsilon + R_i(N, ((c - \widetilde{b})_{N\setminus\{i\}}, c'_i - c_i - \varepsilon), E - \widetilde{B} - \varepsilon),$$

which concludes the proof in this case.

Case 2.3. $c_i > b_i$.

Here, $\widetilde{b}_i = b_i = \widetilde{b}'_i$ (and thus $\varepsilon = 0$), from where (5) trivially follows as a consequence of the fact that R satisfies claims monotonicity.

Case 3. $\widetilde{B} < E < \widetilde{B}'$. In this case, $\widetilde{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$, and $\widetilde{R}_i(N, b, c', E) = R^*_i(N, \widetilde{b}', E) = \widetilde{b}'_i - R_i(N, (\widetilde{b}_{N\setminus\{i\}}, \widetilde{b}'_i), \widetilde{B}' - E)$. Thus, in order to prove (4), it suffices to show that

$$R_i(N, (\widetilde{b}_{N\setminus\{i\}}, \widetilde{b}'_i), \widetilde{B}' - E) + R_i(N, c - \widetilde{b}, E - \widetilde{B}) \leq \widetilde{b}'_i - \widetilde{b}_i \tag{6}$$

Note that Case 3 implies that $\widetilde{b}_i = c_i$ (otherwise, $\widetilde{b}_i = b_i$ and hence $\widetilde{B} = \widetilde{B}'$). Thus, by boundedness, $R_i(N, c - \widetilde{b}, E - \widetilde{B}) = 0$. Also, by balance and boundedness, $R_i(N, (\widetilde{b}_{N\setminus\{i\}}), \widetilde{b}'_i, \widetilde{B}' - E) \leq \widetilde{B}' - E = \widetilde{B} - E + \widetilde{b}'_i - \widetilde{b}_i \leq \widetilde{b}'_i - \widetilde{b}_i$, from where (6) follows.

Linked monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $i \in N$. Let $b \in \mathbb{R}^n_+$ be a baseline profile and let $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$, and $\widetilde{B} = \sum_{j\in N} \min\{b_j, c_j\}$. Let $\varepsilon > 0$, $\widetilde{b}'_i = \min\{b_i, c_i + \varepsilon\}$ and $\widetilde{B}' = \widetilde{B} + \widetilde{b}'_i - \widetilde{b}_i$. Then, $\widetilde{b}'_i \leq \widetilde{b}_i + \varepsilon$ and $\widetilde{B} \leq \widetilde{B}' \leq \widetilde{B} + \varepsilon$. The aim is to show that

$$\widetilde{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) \leq \widetilde{R}_i(N, b, c, E) + \varepsilon \tag{7}$$

We distinguish several cases:

Case 1. $E \leq \widetilde{B}' - \varepsilon$.

In this case, $\widetilde{R}_i(N, b, c, E) = R^*_i(N, \widetilde{b}, E)$ and $\widetilde{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = R^*_i(N, \widetilde{b}', E + \varepsilon) = R^*_i(N, (\widetilde{b}'_i, \widetilde{b}_{-i}), E + \varepsilon)$. By claims monotonicity (of R^*), $R^*_i(N, \widetilde{b}', E + \varepsilon) \leq R^*_i(N, (\widetilde{b}_i + \varepsilon, \widetilde{b}_{-i}), E + \varepsilon)$. By linked monotonicity (of R^*), $R^*_i(N, (\widetilde{b}_i + \varepsilon, \widetilde{b}_{-i}), E + \varepsilon) \leq R^*_i(N, \widetilde{b}, E) + \varepsilon$, from where (7) follows.

Case 2. $E \geq \widetilde{B}$.

In this case, $\widetilde{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$ and $\widetilde{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = \widetilde{b}'_i + R_i(N, (c_i + \varepsilon, c_{N\setminus\{i\}}) - (\widetilde{b}'_i, \widetilde{b}_{-i}), E + \varepsilon - \widetilde{B}')$. As $(E - \varepsilon - \widetilde{B}') - (E - \widetilde{B}) = \varepsilon - (\widetilde{b}'_i - \widetilde{b}_i)$ and $(c_i + \varepsilon - \widetilde{b}'_i) - (c_i - \widetilde{b}_i) = \varepsilon - (\widetilde{b}'_i - \widetilde{b}_i)$, (7) follows from linked monotonicity (of R).

Case 3. $\widetilde{B}' - \varepsilon < E < \widetilde{B}$.

In this case, $\widetilde{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = \widetilde{b}'_i + R_i(N, (c_i + \varepsilon - \widetilde{b}'_i, (c - \widetilde{b})_{-i}), E + \varepsilon - \widetilde{B}')$, and $\widetilde{R}_i(N, b, c, E) = \widetilde{b}'_i - R_i(N, \widetilde{b}, \widetilde{B} - E)$. Thus, (7) becomes

$$\varepsilon - \widetilde{b}'_i + \widetilde{b}_i \geq R_i(N, \widetilde{b}, \widetilde{B} - E) + R_i(N, (c_i + \varepsilon - \widetilde{b}'_i, (c - \widetilde{b})_{-i}), E + \varepsilon - \widetilde{B}') \tag{8}$$

Now, by balance and boundedness, the right hand side of (8) is bounded above by $\widetilde{B} - E + E + \varepsilon - \widetilde{B}'$, which is precisely the left hand side of (8).

7.4 Proof of Theorem 4

Let R be a rule satisfying resource monotonicity, population monotonicity and resource-and-population monotonicity. By Theorem 2, $\widetilde{R}$ satisfies the extended property of resource monotonicity. Our aim is to show that $\widetilde{R}$ also satisfies the extended properties of population monotonicity and resource-and-population monotonicity.

Population monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E) \in \mathcal{D}$ be such that $N \subseteq N'$ and $c'_N = c$. Let $b \in \mathbb{R}^n_+$ and $b' \in \mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N = b$, and let $\widetilde{b}'_j = \min\{b'_j, c'_j\}$ for all $j \in N'$, and $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$. In particular, $\widetilde{b}'_j = \widetilde{b}_j$ for all $j \in N$. Finally, let $\widetilde{B} = \sum_{j\in N} \widetilde{b}_j$, and $\widetilde{B}' = \sum_{j\in N'} \widetilde{b}'_j$. The aim is to show that

$$\widetilde{R}_i(N', b', c', E) \leq \widetilde{R}_i(N, b, c, E), \tag{9}$$

for each $i \in N$.

We distinguish several cases:

Case 1. $E \leq \widetilde{B}$.

In this case, $\widetilde{R}_i(N', b', c', E) = R^*_i(N', \widetilde{b}', E)$ and $\widetilde{R}_i(N, b, c, E) = R^*_i(N, \widetilde{b}, E)$. As R satisfies resource-and-population monotonicity, it follows that R^* satisfies population monotonicity, from where we obtain (9).

Case 2. $E \geq \widetilde{B}'$.

In this case, $\widetilde{R}_i(N', b', c', E) = \widetilde{b}'_i + R_i(N', c' - \widetilde{b}', E - \widetilde{B}')$ and $\widetilde{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$. As R satisfies resource monotonicity and population monotonicity, (9) follows.

Case 3. $\widetilde{B} < E < \widetilde{B}'$.

In this case, the definition of baseline rationing guarantees that $\widetilde{R}_i(N', b', c', E) \leq \widetilde{b}'_i = \widetilde{b}_i \leq \widetilde{R}_i(N, b, c, E)$.

Resource-and-population monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E') \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$. Let $b \in \mathbb{R}^n_+$ and $b' \in \mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N = b$, and let

$\widetilde{b}'_j = \min\{b'_j, c'_j\}$ for all $j \in N'$, and $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$. In particular, $\widetilde{b}'_j = \widetilde{b}_j$ for all $j \in N$. Finally, let $\widetilde{B} = \sum_{j\in N} \widetilde{b}_j$, and $\widetilde{B}' = \sum_{j\in N'} \widetilde{b}'_j$. The aim is to show that, for each $i \in N$,

$$\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_i(N', b', c', E'), \tag{10}$$

where $E' = E + \sum_{N'\setminus N} c'_j$.

We distinguish several cases:

Case 1. $E \leq \widetilde{B} - \sum_{N'\setminus N}\left(c'_j - \min\{b'_j, c'_j\}\right)$.

In this case, $E \leq \widetilde{B}$ and $E' \leq \widetilde{B}'$ and, therefore, $\widetilde{R}_i(N', b', c', E') = R^*_i(N', \widetilde{b}', E')$ and $\widetilde{R}_i(N, b, c, E) = R^*_i(N, \widetilde{b}, E)$. By resource monotonicity and population monotonicity of R^*, (10) follows.

Case 2. $\widetilde{B} - \sum_{N'\setminus N}\left(c'_j - \min\{b'_j, c'_j\}\right) \leq E \leq \widetilde{B}$.

In this case, $E \leq \widetilde{B}$ whereas $E' \geq \widetilde{B}'$ and, hence, the definition of baseline rationing guarantees that $\widetilde{R}_i(N', b', c', E') \geq \widetilde{b}'_i = \widetilde{b}_i \geq \widetilde{R}_i(N, b, c, E)$, as desired.

Case 3. $E \geq \widetilde{B}$.

In this case, $E' \geq \widetilde{B}'$ and, hence, $\widetilde{R}_i(N', b', c', E') = \widetilde{b}'_i + R_i(N', c' - \widetilde{b}', E' - \widetilde{B}')$ and $\widetilde{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$. It is straightforward to show that $E' - \widetilde{B}' = E - \widetilde{B} + \sum_{N'\setminus N}\left(c'_j - \min\{b'_j, c'_j\}\right)$. Thus, (10) follows from the fact that R satisfies resource -and-population monotonicity. ■

7.5 Proof of Proposition 1

We concentrate on the proof of the second statement, as the other two are straightforward. Let R be a rule satisfying order preservation and let (N, b, c, E) be an extended problem for which baselines are ordered like claims. Let $i, j \in N$ be such that $c_i \leq c_j$ (and hence $b_i \leq b_j$). We aim to show that

$$\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_j(N, b, c, E)$$

To do so, we distinguish two cases.

Case 1. $E \leq \widetilde{B}$.

In this case, $\widetilde{R}_i(N, b, c, E) = R^*_i(N, \widetilde{b}, E)$ and $\widetilde{R}_j(N, b, c, E) = R^*_j(N, \widetilde{b}, E)$. Now, as order preservation is a self-dual property, it follows that R^*, the dual rule of R, is order preserving too. As $b_i \leq b_j$ and $c_i \leq c_j$, it follows that $\widetilde{b}_i \leq \widetilde{b}_j$. Altogether, we have that $\widetilde{R}_i(N, b, c, E) \leq \widetilde{R}_j(N, b, c, E)$, as desired.

Case 2. $E \geq \widetilde{B}$.

In this case, $\widetilde{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$ and $\widetilde{R}_j(N, b, c, E) = \widetilde{b}_j + R_j(N, c - \widetilde{b}, E - \widetilde{B})$. Note that, as mentioned above, $\widetilde{b}_i \leq \widetilde{b}_j$. Thus, if $c_i - \widetilde{b}_i \leq c_j - \widetilde{b}_j$, the desired inequality would trivially follow from the fact that R satisfies order preservation. If, on the contrary, $c_i - \widetilde{b}_i \geq c_j - \widetilde{b}_j$ the fact that R^* satisfies order preservation guarantees that $R^*_i(N, c - \widetilde{b}, C - E) \geq R^*_j(N, c - \widetilde{b}, C - E)$. Thus,

$$\widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B}) = c_i - R^*_i(N, c - \widetilde{b}, C - E) \leq c_j - R^*_j(N, c - \widetilde{b}, C - E) = \widetilde{b}_j + R_j(N, c - \widetilde{b}, E - \widetilde{B}),$$

as desired. ■

7.6 Proof of Theorem 5

It is straightforward to see that, for any standard rationing rule R, $\widetilde{R}$ satisfies the four axioms. Thus, we focus on the converse implication. Let S be an extended rule satisfying the four axioms from the list and let (N, b, c, E) be a given extended problem. We distinguish two cases.

Case 1. $E \geq \widetilde{B}$.

In this case, by baseline invariance, $S(N, b, c, E) = \widetilde{b} + S(N, b - \widetilde{b}, c - \widetilde{b}, E - \widetilde{B})$. Note that, if $b_i \leq c_i$ then $b_i - \widetilde{b}_i = 0 < c_i - \widetilde{b}_i = c_i - b_i$, whereas if $b_i \geq c_i$ then $b_i - \widetilde{b}_i = b_i - c_i > 0 = c_i - \widetilde{b}_i$. Thus, by baseline truncation, $S(N, b - \widetilde{b}, c - \widetilde{b}, E - \widetilde{B}) = S(N, 0, c - \widetilde{b}, E - \widetilde{B})$.

Case 2. $E \leq \widetilde{B}$.

In this case, by baseline truncation, $S(N, b, c, E) = S(N, \widetilde{b}, c, E)$. And, by truncation of excessive claims, $S(N, \widetilde{b}, c, E) = S(N, \widetilde{b}, \widetilde{b}, E)$. Finally, by polar baseline self-duality, $S(N, \widetilde{b}, \widetilde{b}, E) = \widetilde{b} - S(N, 0, \widetilde{b}, \widetilde{B} - E)$.

To summarize,

$$S(N, b, c, E) = \begin{cases} \widetilde{b} - S(N, 0, \widetilde{b}, \widetilde{B} - E) & \text{if } E \leq \widetilde{B} \\ \widetilde{b} + S(N, 0, c - \widetilde{b}, E - \widetilde{B}) & \text{if } E > \widetilde{B} \end{cases}. \tag{11}$$

Let $R : \mathcal{D} \rightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$ be such that, for any $(N, c, E) \in \mathcal{D}$,

$$R(N, c, E) = S(N, 0, c, E).$$

In other words, R assigns to each standard rationing problem the solution that S yields for the corresponding extended problem in which baselines are null. Hence, (11) becomes

$$S(N, b, c, E) = \begin{cases} \widetilde{b} - R(N, \widetilde{b}, \widetilde{B} - E) & \text{if } E \leq \widetilde{B} \\ \widetilde{b} + R(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } E > \widetilde{B} \end{cases},$$

which implies that $S \equiv \widetilde{R}$. ■

We conclude showing the tightness of the result. In what follows, let A denote the (standard) constrained equal awards rule and P denote the (standard) proportional rule.[16]

- Let S be defined by

$$S(N, c, b, E) = \begin{cases} P(N, \widetilde{b}, E), & \text{if } E \leq \widetilde{B} \\ \widetilde{b} + P(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } E \geq \widetilde{B} \quad \text{and } c_i \geq b_i \text{ for all } i \in N \\ \widetilde{b} + A(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } E \geq \widetilde{B} \quad \text{and } c_i < b_i \text{ for some } i \in N, \end{cases}$$

[16]Formally, for all $(N, c, E) \in \mathcal{D}$, $A(N, c, E) = (\min\{c_i, \lambda\})_{i \in N}$ where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$, whereas $P(N, c, E) = \frac{E}{C} \cdot c$.

It is straightforward to show that S is a well-defined rule that satisfies Truncation of Excessive Claims, Baseline Invariance, Polar Baseline Self-Duality and Polar Baseline Equivalence, but not Baseline Truncation.

- Let S be defined by

$$S(N,b,c,E)=\begin{cases} P(N,c,E) & \text{if } E\leq \widetilde{B} \\ \widetilde{b}+P(N,c-\widetilde{b},E-\widetilde{B}) & \text{if } E\geq \widetilde{B} \end{cases}$$

It is straightforward to show that S is a well-defined rule that satisfies Baseline Truncation, Baseline Invariance, Polar Baseline Self-Duality and Polar Baseline Equivalence, but not Truncation of Excessive Claims.

- Let S be defined by

$$S(N,b,c,E)=\begin{cases} \widetilde{b}-P(N,\widetilde{b},\widetilde{B}-E) & \text{if } E\leq \widetilde{B} \\ P(N,c,E) & \text{if } E\geq \widetilde{B} \end{cases}$$

It is straightforward to show that S is a well-defined rule that satisfies Baseline Truncation, Truncation of Excessive Claims, Polar Baseline Self-Duality and Polar Baseline Equivalence, but not Baseline Invariance.

- Let S be defined by

$$S(N,b,c,E)=\begin{cases} \widetilde{b}-A(N,\widetilde{b},\widetilde{B}-E) & \text{if } E\leq \widetilde{B} \\ \widetilde{b}+P(N,c-\widetilde{b},E-\widetilde{B}) & \text{if } E\geq \widetilde{B} \end{cases}$$

It is straightforward to show that S is a well-defined rule that satisfies Baseline Truncation, Truncation of Excessive Claims and Baseline Invariance, but neither Polar Baseline Self-Duality nor Polar Baseline Equivalence.

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Appendix that is not part of the submission for publication

To save space, we have included in this appendix, which is not for publication, formal proofs of some statements made in the body of the paper. More precisely, we present the complete proofs of the propositions appearing in Sections 4 and 5. As it can be seen, most of these proofs simply mimic their counterparts for the results of Sections 3 and 4 presented above.

Proof of Proposition 2

It is straightforward to show that for any standard rationing rules R and T, $\widetilde{RT}$ satisfies the three axioms. Thus, we focus on the converse implication. Let S be an extended rule satisfying the three axioms from the list and let (N, b, c, E) be a given extended problem. We distinguish two cases.

Case 1. $E \geq \widetilde{B}$.

In this case, by baseline invariance, $S(N, b, c, E) = \widetilde{b} + S(N, b - \widetilde{b}, c - \widetilde{b}, E - \widetilde{B})$. Note that, if $b_i \leq c_i$ then $b_i - \widetilde{b}_i = 0 < c_i - \widetilde{b}_i = c_i - b_i$, whereas if $b_i \geq c_i$ then $b_i - \widetilde{b}_i = b_i - c_i > 0 = c_i - \widetilde{b}_i$. Thus, by baseline truncation, $S(N, b - \widetilde{b}, c - \widetilde{b}, E - \widetilde{B}) = S(N, 0, c - \widetilde{b}, E - \widetilde{B})$.

Case 2. $E \leq \widetilde{B}$.

In this case, by baseline truncation, $S(N, b, c, E) = S(N, \widetilde{b}, c, E)$. And, by truncation of excessive claims, $S(N, \widetilde{b}, c, E) = S(N, \widetilde{b}, \widetilde{b}, E)$.

To summarize,

$$S\left(N, b, c, E\right) = \begin{cases} S(N, \widetilde{b}, \widetilde{b}, \widetilde{B} - E) & \text{if } \ E \leq \widetilde{B} \\ \widetilde{b} + S(N, 0, c - \widetilde{b}, E - \widetilde{B}) & \text{if } \ E > \widetilde{B} \end{cases}. \tag{12}$$

Let $R^*, T : \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$ be such that, for any $(N, c, E) \in \mathcal{D}$,

$$R^* \left(N, c, E\right) = S(N, c, c, E),$$

and

$$T \left(N, c, E\right) = S(N, 0, c, E).$$

In other words, R^* assigns to each standard rationing problem the solution that S yields for the corresponding extended problem in which baselines are equal to claims, whereas T assigns to each standard rationing problem the solution that S yields for the corresponding extended problem in which baselines are null. It is straightforward to see that R^* and T are well-defined (standard) rationing rules. Thus, if R denotes the dual rule of R^*, (12) becomes

$$S\left(N, b, c, E\right) = \begin{cases} \widetilde{b} - R(N, \widetilde{b}, \widetilde{B} - E) & \text{if } \ E \leq \widetilde{B} \\ \widetilde{b} + T(N, c - \widetilde{b}, E - \widetilde{B}) & \text{if } \ E > \widetilde{B} \end{cases},$$

which implies that $S \equiv \widetilde{RT}$. ■

Proof of Proposition 3

By Proposition 2, we know that an extended rationing rule satisfies Baseline Truncation, Truncation of Excessive Claims and Baseline Invariance if and only if it is a generalized baseline rationing rule. Thus, it just remains to show that composition rationing rules are the only generalized baseline rationing rules that satisfy Polar Baseline Equivalence. Now, a generalized baseline rationing rule $\widetilde{RT}$ satisfies Polar Baseline Equivalence if and only if $c - R(N, c, C - E) = T(N, c, E)$, for any $(N, c, E) \in \mathcal{D}$. Thus, $R \equiv T^*$, which implies that $\widetilde{RT} \equiv \widehat{R}$, as desired. ■

Proof of Proposition 4

- Resource monotonicity, consistency and resource-and-population uniformity are robust to composition rationing.

Resource monotonicity. Let R be a rule satisfying resource monotonicity. Let $(N, c, E), (N, c, E') \in \mathcal{D}$ be two problems such that $E < E'$. Let $b \in \mathbb{R}^n$ be a baseline profile and let $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$, and $\widetilde{B} = \sum_{j \in N} \min\{b_j, c_j\}$. Finally, let $i \in N$ be a given agent. The aim is to show that $\widehat{R}_i(N, b, c, E) \leq \widehat{R}_i(N, b, c, E')$. To do so, we distinguish three cases.

Case 1. $E < E' \leq \widetilde{B}$.

In this case, $\widehat{R}_i(N, b, c, E) = R_i(N, \widetilde{b}, E)$ and $\widehat{R}_i(N, b, c, E') = R_i(N, \widetilde{b}, E')$. Now, as R satisfies resource monotonicity, it follows that $R_i(N, \widetilde{b}, E) \leq R_i(N, \widetilde{b}, E')$, as desired.

Case 2. $\widetilde{B} \leq E < E'$.

In this case,

$$\widehat{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B}) \leq \widetilde{b}_i + R_i(N, c - \widetilde{b}, E' - \widetilde{B}) = \widehat{R}_i(N, b, c, E'),$$

where the inequality follows from the fact that R satisfies resource monotonicity.

Case 3. $E < \widetilde{B} < E'$.

In this case, the definition of baseline rationing guarantees that $\widehat{R}_i(N, b, c, E) \leq \widetilde{b}_i \leq \widehat{R}_i(N, b, c, E')$.

Consistency. Let R be a rule satisfying consistency. Let $(N, c, E) \in \mathcal{D}$ and $b \in \mathbb{R}^n_+$. Let $x = \widehat{R}(N, b, c, E)$. The aim is to show that, for any $M \subset N$,

$$\widehat{R}(M, b_M, c_M, \sum_{i \in M} x_i) = x_M.$$

Fix $M \subset N$ and let $E' = \sum_{j \in M} x_j$ and $\widetilde{B}' = \sum_{j \in M} \min\{b_j, c_j\}$. Then, $E \leq \widetilde{B}$ if and only if $E' \leq \widetilde{B}'$. We then distinguish two cases.

Case 1. $E \leq \widetilde{B}$.

In this case, $x_i = R_i(N, \widetilde{b}, E)$ for all $i \in N$, and thus $E' = \sum_{i \in M} R_i(N, \widetilde{b}, E)$. Therefore, $\widehat{R}_i(M, b_M, c_M, E') = R_i(M, \widetilde{b}_M, E')$ for all $i \in M$. Now, as R is consistent, it follows that $R_i(N, \widetilde{b}, E) = R_i(M, \widetilde{b}_M, E')$, for all $i \in M$, which concludes the proof of this case.

Case 2. $E \geq \widetilde{B}$.

In this case, $x_i = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$ for all $i \in N$, and thus $\widetilde{B}' - E' = \sum_{i \in M} R_i(N, c - \widetilde{b}, E - \widetilde{B})$. Therefore, $\widehat{R}_i(M, b_M, c_M, E') = \widetilde{b}_i + R_i(M, c_M - \widetilde{b}_M, E' - \widetilde{B}')$ for all $i \in M$. Now, as R is consistent, it follows that $R_i(N, c - \widetilde{b}, E - \widetilde{B}) = R_i(M, c_M - \widetilde{b}_M, E' - \widetilde{B}')$, for all $i \in M$, which concludes the proof of this case.

Resource-and-population uniformity follows directly from the above statements on resource monotonicity and consistency.

- The pair formed by claims monotonicity and linked monotonicity is robust to composition rationing.

Let R be a rule satisfying claims monotonicity and linked monotonicity. Our aim is to show that $\widehat{R}$ satisfies the extended versions of the two properties.

Claims monotonicity. Let $(N, c, E), (N, c', E) \in \mathcal{D}$ be two problems such that, for some $i \in N$, $c_i \leq c'_i$, whereas $c_{N \setminus \{i\}} \equiv c'_{N \setminus \{i\}}$. Let $b \in \mathbb{R}^n_+$ be a baseline profile and let $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$, $\widetilde{b}'_j = \min\{b_j, c'_j\}$ for all $j \in N$, $\widetilde{B} = \sum_{j \in N} \widetilde{b}_j$, and $\widetilde{B}' = \sum_{j \in N} \widetilde{b}'_j$.[17] The aim is to show that

$$\widehat{R}_i(N, b, c, E) \leq \widehat{R}_i(N, b, c', E). \tag{13}$$

We distinguish several cases:

Case 1. $E \leq \widetilde{B}$.

In this case, $\widehat{R}_i(N, b, c, E) = R_i(N, \widetilde{b}, E)$ and $\widehat{R}_i(N, b, c', E) = R_i(N, \widetilde{b}', E) = R_i(N, (\widetilde{b}_{N \setminus \{i\}}, \widetilde{b}'_i), E)$. As R satisfies claims monotonicity, we obtain (13).

Case 2. $E \geq \widetilde{B}'$.

In this case, $\widehat{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$, and

$$\widehat{R}_i(N, b, c', E) = \widetilde{b}'_i + R_i(N, c' - \widetilde{b}', E - \widetilde{B}') = \widetilde{b}'_i + R_i(N, ((c - \widetilde{b})_{N \setminus \{i\}}, c'_i - \widetilde{b}'_i), E - \widetilde{B} - \widetilde{b}'_i + \widetilde{b}_i).$$

Let $\varepsilon = \widetilde{b}'_i - \widetilde{b}_i \geq 0$. Then, (13) is equivalent to

$$R_i(N, c - \widetilde{b}, E - \widetilde{B}) \leq \varepsilon + R_i(N, ((c - \widetilde{b})_{N \setminus \{i\}}, c'_i - \widetilde{b}'_i), E - \widetilde{B} - \varepsilon). \tag{14}$$

We then distinguish three subcases.

Case 2.1. $b_i > c'_i$.

Here, $\widetilde{b}_i = c_i < c'_i = \widetilde{b}'_i$ (and thus $\varepsilon = c'_i - c_i$). Then, (14) becomes

$$R_i(N, ((c - \widetilde{b})_{N \setminus \{i\}}, 0), E - \widetilde{B}) \leq \varepsilon + R_i(N, ((c - \widetilde{b})_{N \setminus \{i\}}, 0), E - \widetilde{B} - \varepsilon).$$

[17]Note that $\widetilde{b}'_{N \setminus \{i\}} \equiv \widetilde{b}_{N \setminus \{i\}}$, $\widetilde{b}'_i \geq \widetilde{b}_i$ and thus, $\widetilde{B}' \geq \widetilde{B}$.

Now, by claims monotonicity,

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}) \leq R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},\varepsilon),E-\widetilde{B}).$$

And, by linked monotonicity,

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},\varepsilon),E-\widetilde{B}) \leq \varepsilon + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}-\varepsilon),$$

which concludes the proof in this case.

Case 2.2. $c_i' \geq b_i \geq c_i$.

Here, $\widetilde{b}_i = c_i \leq b_i = \widetilde{b}_i'$ (and thus $\varepsilon = b_i - c_i$). Then, (14) becomes

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}) \leq b_i - c_i + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},c_i'-b_i),E-\widetilde{B}-b_i+c_i).$$

Now, by claims monotonicity,

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},0),E-\widetilde{B}) \leq R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},c_i'-c_i),E-\widetilde{B}).$$

And, by linked monotonicity,

$$R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},c_i'-c_i),E-\widetilde{B}) \leq \varepsilon + R_i(N,((c-\widetilde{b})_{N\setminus\{i\}},c_i'-c_i-\varepsilon),E-\widetilde{B}-\varepsilon),$$

which concludes the proof in this case.

Case 2.3. $c_i > b_i$.

Here, $\widetilde{b}_i = b_i = \widetilde{b}_i'$ (and thus $\varepsilon = 0$), from where (14) trivially follows as a consequence of the fact that R satisfies claims monotonicity.

Case 3. $\widetilde{B} < E < \widetilde{B}'$. In this case, $\widehat{R}_i(N,b,c,E) = \widetilde{b}_i + R_i(N,c-\widetilde{b},E-\widetilde{B})$, and $\widehat{R}_i(N,b,c',E) = R_i(N,\widetilde{b}',E) = \widetilde{b}_i' - R_i^*(N,(\widetilde{b}_{N\setminus\{i\}},\widetilde{b}_i'),\widetilde{B}'-E)$. Thus, in order to prove (4), it suffices to show that

$$R_i^*(N,(\widetilde{b}_{N\setminus\{i\}},\widetilde{b}_i'),\widetilde{B}'-E) + R_i(N,c-\widetilde{b},E-\widetilde{B}) \leq \widetilde{b}_i' - \widetilde{b}_i \tag{15}$$

Note that Case 3 implies that $\widetilde{b}_i = c_i$ (otherwise, $\widetilde{b}_i = b_i$ and hence $\widetilde{B} = \widetilde{B}'$). Thus, by boundedness, $R_i(N,c-\widetilde{b},E-\widetilde{B}) = 0$. Also, by balance and boundedness, $R_i^*(N,(\widetilde{b}_{N\setminus\{i\}}),\widetilde{b}_i',\widetilde{B}'-E) \leq \widetilde{B}'-E = \widetilde{B}-E+\widetilde{b}_i'-\widetilde{b}_i \leq \widetilde{b}_i'-\widetilde{b}_i$, from where (15) follows.

Linked monotonicity. Let $(N,c,E) \in \mathcal{D}$ and $i \in N$. Let $b \in \mathbb{R}^n_+$ be a baseline profile and let $\widetilde{b}_j = \min\{b_j,c_j\}$ for all $j \in N$, and $\widetilde{B} = \sum_{j\in N}\min\{b_j,c_j\}$. Let $\varepsilon > 0$, $\widetilde{b}_i' = \min\{b_i,c_i+\varepsilon\}$ and $\widetilde{B}' = \widetilde{B} + \widetilde{b}_i' - \widetilde{b}_i$. Then, $\widetilde{b}_i' \leq \widetilde{b}_i + \varepsilon$ and $\widetilde{B} \leq \widetilde{B}' \leq \widetilde{B} + \varepsilon$. The aim is to show that

$$\widehat{R}_i(N,b,(c_i+\varepsilon,c_{N\setminus\{i\}}),E+\varepsilon) \leq \widehat{R}_i(N,b,c,E) + \varepsilon \tag{16}$$

We distinguish several cases:

Case 1. $E \leq \widetilde{B}' - \varepsilon$.

In this case, $\widehat{R}_i(N, b, c, E) = R_i(N, \widetilde{b}, E)$ and $\widehat{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = R_i(N, \widetilde{b}', E + \varepsilon) = R_i(N, (\widetilde{b}'_i, \widetilde{b}_{-i}), E + \varepsilon)$. By claims monotonicity, $R_i(N, \widetilde{b}', E+\varepsilon) \leq R_i(N, (\widetilde{b}_i + \varepsilon, \widetilde{b}_{-i}), E+\varepsilon)$. By linked monotonicity, $R_i(N, (\widetilde{b}_i + \varepsilon, \widetilde{b}_{-i}), E + \varepsilon) \leq R_i(N, \widetilde{b}, E) + \varepsilon$, from where (16) follows.

Case 2. $E \geq \widetilde{B}$.

In this case, $\widehat{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$ and $\widehat{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = \widetilde{b}'_i + R_i(N, (c_i + \varepsilon, c_{N\setminus\{i\}}) - (\widetilde{b}'_i, \widetilde{b}_{-i}), E + \varepsilon - \widetilde{B}')$. As $\widetilde{B}' = \widetilde{B} + \widetilde{b}'_i - \widetilde{b}_i$ and $c_i + \varepsilon - \widetilde{b}'_i = c_i - \widetilde{b}_i + \varepsilon - \widetilde{b}'_i + \widetilde{b}_i$, (16) follows from linked monotonicity.

Case 3. $\widetilde{B}' - \varepsilon < E < \widetilde{B}$.

In this case, $\widehat{R}_i(N, b, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) = \widetilde{b}'_i + R_i(N, (c_i + \varepsilon - \widetilde{b}'_i, (c - \widetilde{b})_{-i}), E + \varepsilon - \widetilde{B}')$, and $\widehat{R}_i(N, b, c, E) = \widetilde{b}'_i - R^*_i(N, \widetilde{b}, \widetilde{B} - E)$. Thus, (16) becomes

$$\varepsilon - \widetilde{b}'_i + \widetilde{b}_i \geq R^*_i(N, \widetilde{b}, \widetilde{B} - E) + R_i(N, (c_i + \varepsilon - \widetilde{b}'_i, (c - \widetilde{b})_{-i}), E + \varepsilon - \widetilde{B}') \tag{17}$$

Now, by balance and boundedness, the right hand side of (17) is bounded above by $\widetilde{B} - E + E + \varepsilon - \widetilde{B}'$, which is precisely the left hand side of (17).

- The pair formed by population monotonicity and resource-and-population monotonicity is robust to composition rationing, if assisted by resource monotonicity.

Let R be a rule satisfying resource monotonicity, population monotonicity and resource-and-population monotonicity. By the above statement, $\widehat{R}$ satisfies the extended property of resource monotonicity. Our aim is to show that $\widehat{R}$ also satisfies the extended properties of population monotonicity and resource-and-population monotonicity.

Population monotonicity. Let $(N, c, E) \in \mathcal{D}$ and $(N', c', E) \in \mathcal{D}$ be such that $N \subseteq N'$ and $c'_N = c$. Let $b \in \mathbb{R}^n_+$ and $b' \in \mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N = b$, and let $\widetilde{b}'_j = \min\{b'_j, c'_j\}$ for all $j \in N'$, and $\widetilde{b}_j = \min\{b_j, c_j\}$ for all $j \in N$. In particular, $\widetilde{b}'_j = \widetilde{b}_j$ for all $j \in N$. Finally, let $\widetilde{B} = \sum_{j\in N} \widetilde{b}_j$, and $\widetilde{B}' = \sum_{j\in N'} \widetilde{b}'_j$. The aim is to show that

$$\widehat{R}_i(N', b', c', E) \leq \widehat{R}_i(N, b, c, E), \tag{18}$$

for each $i \in N$.

We distinguish several cases:

Case 1. $E \leq \widetilde{B}$.

In this case, $\widehat{R}_i(N', b', c', E) = R_i(N', \widetilde{b}', E)$ and $\widehat{R}_i(N, b, c, E) = R_i(N, \widetilde{b}, E)$. As R satisfies population monotonicity, (18) follows.

Case 2. $E \geq \widetilde{B}'$.

In this case, $\widehat{R}_i(N',b',c',E)=\widetilde{b}'_i+R_i(N',c'-\widetilde{b}',E-\widetilde{B}')$ and $\widehat{R}_i(N,b,c,E)=\widetilde{b}_i+R_i(N,c-\widetilde{b},E-\widetilde{B})$. As R satisfies resource monotonicity and population monotonicity, (18) follows.

Case 3. $\widetilde{B}<E<\widetilde{B}'$.

In this case, the definition of composition rationing guarantees that $\widehat{R}_i(N',b',c',E)\leq\widetilde{b}'_i=\widetilde{b}_i\leq\widehat{R}_i(N,b,c,E)$.

Resource-and-population monotonicity. Let $(N,c,E)\in\mathcal{D}$ and $(N',c',E')\in\mathcal{D}$ such that $N\subseteq N'$ and $c'_N=c$. Let $b\in\mathbb{R}^n_+$ and $b'\in\mathbb{R}^{n'}_+$ be two baseline profiles such that $b'_N=b$, and let $\widetilde{b}'_j=\min\{b'_j,c'_j\}$ for all $j\in N'$, and $\widetilde{b}_j=\min\{b_j,c_j\}$ for all $j\in N$. In particular, $\widetilde{b}'_j=\widetilde{b}_j$ for all $j\in N$. Finally, let $\widetilde{B}=\sum_{j\in N}\widetilde{b}_j$, and $\widetilde{B}'=\sum_{j\in N'}\widetilde{b}'_j$. The aim is to show that, for each $i\in N$,

$$\widehat{R}_i(N,b,c,E)\leq\widehat{R}_i(N',b',c',E'), \tag{19}$$

where $E'=E+\sum_{N'\setminus N}c'_j$.

We distinguish several cases:

Case 1. $E\leq\widetilde{B}-\sum_{N'\setminus N}\left(c'_j-\min\{b'_j,c'_j\}\right)$.

In this case, $E\leq\widetilde{B}$ and $E'\leq\widetilde{B}'$ and, therefore, $\widehat{R}_i(N',b',c',E')=R_i(N',\widetilde{b}',E')$ and $\widehat{R}_i(N,b,c,E)=R_i(N,\widetilde{b},E)$. By resource monotonicity and population monotonicity, (19) follows.

Case 2. $\widetilde{B}-\sum_{N'\setminus N}\left(c'_j-\min\{b'_j,c'_j\}\right)\leq E\leq\widetilde{B}$.

In this case, $E\leq\widetilde{B}$ whereas $E'\geq\widetilde{B}'$ and, hence, the definition of composition rationing guarantees that $\widehat{R}_i(N',b',c',E')\geq\widetilde{b}'_i=\widetilde{b}_i\geq\widehat{R}_i(N,b,c,E)$, as desired.

Case 3. $E\geq\widetilde{B}$.

In this case, $E'\geq\widetilde{B}'$ and, hence, $\widehat{R}_i(N',b',c',E')=\widetilde{b}'_i+R_i(N',c'-\widetilde{b}',E'-\widetilde{B}')$ and $\widehat{R}_i(N,b,c,E)=\widetilde{b}_i+R_i(N,c-\widetilde{b},E-\widetilde{B})$. It is straightforward to show that $E'-\widetilde{B}'=E-\widetilde{B}+\sum_{N'\setminus N}\left(c'_j-\min\{b'_j,c'_j\}\right)$. Thus, (19) follows from the fact that R satisfies resource-and-population monotonicity. ■

- If baselines and claims are uniformly impartial then equal treatment of equals is robust to composition rationing.

We omit the proof of this straightforward statement.

- If baselines are ordered like claims and R is order preserving then $\widehat{R}$ is order preserving in gains.

Let R be a rule satisfying order preservation and let (N,b,c,E) be an extended problem for which baselines are ordered like claims. Let $i,j\in N$ be such that $c_i\leq c_j$ (and hence $b_i\leq b_j$). We aim to show that

$$\widehat{R}_i(N,b,c,E)\leq\widehat{R}_j(N,b,c,E)$$

To do so, we distinguish two cases.

Case 1. $E \leq \widetilde{B}$.

In this case, $\widehat{R}_i(N, b, c, E) = R_i(N, \widetilde{b}, E)$ and $\widehat{R}_j(N, b, c, E) = R_j(N, \widetilde{b}, E)$. As R is order preserving (in gains) and baselines are ordered like claims, the desired inequality follows.

Case 2. $E \geq \widetilde{B}$.

In this case, $\widehat{R}_i(N, b, c, E) = \widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B})$ and $\widehat{R}_j(N, b, c, E) = \widetilde{b}_j + R_j(N, c - \widetilde{b}, E - \widetilde{B})$. Note that, as mentioned above, $\widetilde{b}_i \leq \widetilde{b}_j$. Thus, if $c_i - \widetilde{b}_i \leq c_j - \widetilde{b}_j$, the desired inequality would trivially follow from the fact that R satisfies order preservation. If, on the contrary, $c_i - \widetilde{b}_i \geq c_j - \widetilde{b}_j$ the fact that R^* satisfies order preservation guarantees that $R_i^*(N, c - \widetilde{b}, C - E) \geq R_j^*(N, c - \widetilde{b}, C - E)$. Thus,

$$\widetilde{b}_i + R_i(N, c - \widetilde{b}, E - \widetilde{B}) = c_i - R_i^*(N, c - \widetilde{b}, C - E) \leq c_j - R_j^*(N, c - \widetilde{b}, C - E) = \widetilde{b}_j + R_j(N, c - \widetilde{b}, E - \widetilde{B}),$$

as desired. ■

- If baselines and claim-baseline differences are ordered like claims then order preservation is robust to composition rationing.

We omit the proof of this straightforward statement.